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On the global attractor of delay differential equations with unimodal feedback not satisfying the negative Schwarzian derivative condition

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. We study the size of the global attractor for a delay differential equation with unimodal feedback. We are interested in extending and complementing a dichotomy result by Liz and Röst, which assumed that the Schwarzian derivative of the nonlinear feedback is negative in a certain interval. Using recent stability results for difference equations, we obtain a stability dichotomy for the original delay differential equation in the situation wherein the Schwarzian derivative of the nonlinear term may change sign. We illustrate the applicability of our results with several examples.

Keywords: delay differential equations, difference equations, global attractor.

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1 Introduction

The nonlinear delay differential equation

$$x'(t) = -\mu\big(x(t) - f(x(t-\tau))\big), \quad t > 0, \tag{1.1}$$

with $\mu, \tau > 0$ and $f\colon I \subset \mathbb{R} \to I$, has been widely studied in the literature because of its multiple applications in, for example, biology, physics or economics [1,2,14,17]. In the case of f being monotone, the dynamics are well understood, see [8,9,18] and references therein. In particular, it is known that chaotic dynamics cannot occur [15]. The natural generalization of

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the previous case, in which f changes monotonicity once, is more complicated and may lead to chaotic behaviour [10].

In this paper, f is assumed to be unimodal. More specifically, we impose that the following condition holds for f.

(U) $f\colon (a,b) \subset \mathbb{R} \to (a,b)$ is differentiable, with $-\infty \le a < b \le +\infty$; satisfies that there is a unique x_* such that $f'(x) > 0$ if $a \le x < x_*$, $f'(x_*) = 0$, and $f'(x) < 0$ if $x_* < x < b$; and that there exists $K \in (x_*, b)$ such that $f(K) = K$, $f(x) > x$ for $x \in (a,K)$, and $f(x) < x$ for $x \in (K,b)$.

Notice that if condition **(U)** holds, then K is the unique *fixed point* for the map f, i.e. $f(K) = K$, and therefore the constant function $x(t) = K$ is a positive equilibrium of the delay equation (1.1). Moreover, we empshasise that assuming that the fixed point K belongs to (x_*, b) is not restrictive for our purpose of studying the asymptotic behaviour of equation (1.1), since if K belongs to the interval (a, x_*), then all the solutions of the delay equation are known to converge to K; see, for example [16].

Whenever condition **(U)** holds, we denote the image by f of the point where the maximum of f is attained and the image by f of this maximum by β and α, respectively, that is,

$$\beta := f(x_*) \quad \text{and} \quad \alpha := f(\beta). \tag{1.2}$$

With the notation in (1.2), we introduce an additional assumption on f.

(L) Condition **(U)** holds and $f(f(x_*)) > x_*$.

A well-known approach for investigating equation (1.1) comprises studying the behaviour of the related difference equation

$$x_{n+1} = f(x_n), \quad x_0 \in (a,b), \tag{1.3}$$

see, for example, [7, 13]. Using that approach and taking advantage of the properties of unimodal maps it is possible to show that if **(L)** holds, then for any solution $x(t)$ of (1.1) with initial condition in $C([-\tau,0],(a,b))$ there exists $t_0 > 0$ such that $x(t) \in [\alpha, \beta]$ for $t \ge t_0$; and we informally say that the interval $[\alpha, \beta]$ contains the global attractor of the equation (1.1). Thus, if **(L)** holds, then the interval $[\alpha, \beta]$ contains the global attractor of (1.1) independently of the delay τ. Moreover, complicated dynamics cannot occur for equation (1.1) since the ω-limit set of any solution is the positive equilibrium $\{K\}$ or a periodic orbit. We refer the reader to [16] for a proof of these results in the particular case of $(a,b) = (0,+\infty)$.

The interval $[\alpha, \beta]$ might not be the sharpest, that is, it might have a proper subinterval which contains the global attractor of (1.1). Therefore, an interesting problem stated in [16] is to try to estimate this sharpest attracting interval—or even better to calculate it—when condition **(L)** holds. Here, we deal with such a problem.

In [11], Liz and Röst consider the same problem and showed that when f satisfies **(L)** and has negative Schwarzian derivative, then the sharpest interval containing the attractor of equation (1.1) can be determined and the following dichotomy result holds.

Theorem 1.1 (Theorem 6 in [11])**.** *Assume that condition **(L)** holds and, further, that f satisfies the following condition.*

(S) *f is three times differentiable and $(Sf)(x) < 0$ on the interval $[\alpha, \beta]$, where Sf denotes the Schwarzian derivative of f, defined by*

$$(Sf)(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 .$$

Then exactly one of the following holds:

1. *$f'(K) \geq -1$ and the global attractor of (1.1) for all values of the delay τ is $\{K\}$.*
2. *$f'(K) < -1$ and the sharpest invariant and attracting interval containing the global attractor of (1.1) for all values of the delay τ is $[\bar{\alpha}, \bar{\beta}]$, where $\{\bar{\alpha}, \bar{\beta}\}$ is the unique nontrivial 2-cycle (i.e., $\bar{\alpha} = f(\bar{\beta})$ and $\bar{\beta} = f(\bar{\alpha})$) of the map f in $[\alpha, \beta]$.*

Remark 1.2. We note that Theorem 1.1 as stated above is, in fact, a slightly generalized version of Theorem 6 in [11], which follows from the ideas in [11]. Specifically, the result by Liz and Röst is stated for the particular case in which $a = 0$ and $b = +\infty$. Moreover, their condition (U) imposes $f''(x) > 0$ on $(0, x_*)$, but this is just to guarantee that f has a unique positive fixed point. We note that under the conditions in [11], $f\colon [0, +\infty) \to [0, +\infty)$ is continuous and satisfies $f(0) = 0$ and $f(x) > 0$ for $x > 0$, hence, the restriction to the open interval $(0, +\infty)$ that we consider in Theorem 1.1 is well defined. Finally, note that the initial condition in [11] is a nonzero and nonnegative real function on $[-\tau, 0]$ and consequently all the solutions are strictly positive for $t > 0$, as remarked there. Hence, there is no loss of generality in assuming the initial condition to be strictly positive as we do here.

As the authors of [11] highlighted, the function f appearing in important examples of equation (1.1), including the Mackey–Glass and Nicholson's blowflies models [6, 12], does have negative Schwarzian derivative. Nevertheless, it is not hard to find situations where **(S)** does not hold and, therefore, Theorem 1.1 is not applicable. Hence, it is interesting to look for results extending and complementing Theorem 1.1.

In order to obtain such results, without the assumption that f has negative Schwarzian derivative, we instead take advantage of a consequence of **(L)**, namely, that $f_{|(\alpha,\beta)}$ is strictly decreasing. In this case, the recent method presented in [4, 5] for studying the dynamics of difference equations is applicable, and we employ it to establish a dichotomy result for (1.1) by studying (1.3). Proposition 2.6 is the key technical ingredient for the difference equations we consider, and our main result is Theorem 3.2. The latter result has the same conclusions of Theorem 1.1, but different hypotheses.

Interestingly, the proof of Proposition 2.6 uses the second inequality in the Hermite–Hadamard inequality for a strictly convex function $h\colon [a, b] \to \mathbb{R}$,

$$h\left(\frac{a+b}{2}\right) < \frac{1}{b-a}\int_a^b h(x)dx < \frac{h(a)+h(b)}{2}, \tag{1.4}$$

to show that certain function, intimately linked to the dynamics of the difference equation, is strictly increasing. Whereas, for guaranteeing that such a function has a strict global minimum—enough for obtaining the first conclusion in Theorem 1.1—one needs to invoke the first inequality in (1.4).

In recognition of the current special volume, Professor Webb is a world-expert on the use of topological tools in the study of nonlinear problems. Indirectly, fixed point index theory applied to the study of differential equations plays a role in this paper. Indeed, this is one of the tools used by Mallet-Paret and Nussbaum in [13] to prove the existence of slowly oscillating periodic solutions. The properties of those slowly oscillating periodic solutions underpin [11, Proposition 5], which we invoke in the proof of our main result.

The rest of the paper is organized as follows. The next section contains the preliminaries: notation and some stability results for difference equations. Section 3 contains our main results. Finally, last section of the paper includes some examples to illustrate these main results and compare them with Theorem 1.1.

2 Preliminaries

2.1 Notation

As usual $\mathbb{N}$ and $\mathbb{R}$ denote the positive integers (natural numbers) and real numbers, respectively. Furthermore, $\mathbb{R}_+ := \{r \in \mathbb{R} \,:\, r \geq 0\}$.

Let $I \subset \mathbb{R}$ be an interval (bounded or unbounded) and f a continuous map from I to itself. We denote

$$f^{(0)} := \mathrm{id}, \quad f^{(n+1)} = f \circ f^{(n)}, \quad n \geq 1,\ n \in \mathbb{N},$$

with id denoting the identity map; i.e., $\mathrm{id}(x) = x$ for all $x \in I$.

We let $\mathcal{C}^n(J, I)$ denote the space of functions $\xi \colon J \to I$ with n continuous derivatives and, to simplify the notation, $\mathcal{C}^n((a,b),\mathbb{R})$ is denoted by $\mathcal{C}^n(a,b) := \mathcal{C}^n((a,b),\mathbb{R})$ when no confusion is possible. We do not explicitly indicate the domains of the identity and constant functions. They are assumed to be the largest sets for which the corresponding expressions make sense.

We say that $x(t;\xi)$ is a solution of the differential equation (1.1) with $f \in \mathcal{C}(\mathbb{R})$ and initial condition $\xi \in \mathcal{C}([-\tau,0],\mathbb{R})$ if $x(\cdot;\xi) \in \mathcal{C}([-\tau,+\infty),\mathbb{R})$, $x(\cdot\,;\xi)_{|(0,+\infty)} \in \mathcal{C}^1(0,+\infty)$, it satisfies the differential equation (1.1) for $t > 0$ and $x(s;\xi) = \xi(s)$ for $s \in [-\tau, 0]$. The method of steps, e.g., see [17], shows that there exists a unique solution of the differential equation (1.1) for any $f \in \mathcal{C}(\mathbb{R})$ and initial condition $\xi \in \mathcal{C}([-\tau,0],\mathbb{R})$. Moreover, if $f \in \mathcal{C}(I, I)$ and $\xi \in \mathcal{C}([-\tau,0], I)$, then the invariance principle (see [7, Theorem 2.1]) guarantees that the unique solution of (1.1) satisfies $x(t;\xi) \in I$ for all $t \in [-\tau,+\infty)$.

2.2 Stability of difference equations

In this section, we study stability properties of the difference equation

$$y_{n+1} = y_n + g(y_n), \quad y_0 \in \mathrm{dom}\ g\,, \tag{2.1}$$

with $g \in \mathfrak{G}$, where

$$\mathfrak{G} := \bigcup_{-\infty \leq a < b \leq \infty} \mathfrak{G}(a,b)\,,$$

and

$$\mathfrak{G}(a,b) := \left\{ g \in \mathcal{C}^1(a,b) \,:\, a < \mathrm{id} + g < b,\ g' < 0,\ 0 \in g\big((a,b)\big) \right\}.$$

Here $g\big((a,b)\big)$ denotes the image of (a,b) under g. It is clear that for each $g \in \mathfrak{G}_*$ the difference equation (2.1) is well defined and there exists a unique $y_g \in (a,b)$ such that $g(y_g) = 0$. In particular, y_g is a unique equilbrium of (2.1). We use the usual definitions of stability, local asymptotic stability and global asymptotic stability for the equilibrium y_g of the difference equation (2.1). From now on, G.A.S. stands for globally asymptotically stable and L.A.S. for locally asymptotically stable. We state what we understand by a global repeller.

Definition 2.1. We say that y_g is a *a global repeller* for the difference equation (2.1) if the sequence $((\mathrm{id}+g)^{(n)}(y))_n$ has no accumulation points in (a, b) for any $y \in (a, b) \setminus y_g$.

Next, we define a function to study the stability properties of the equilibrium of (2.1), which was introduced in [5].

Definition 2.2. For each $g \in \mathfrak{G}$, set $b_g := \min\{-\inf g, \sup g\}$. The function $\sigma_g\colon (-b_g, b_g) \to (0, +\infty)$ is defined by

$$\sigma_g(u) = \begin{cases} \frac{g^{-1}(-u)-g^{-1}(u)}{u} & \text{if } u \neq 0\,, \\ \frac{-2}{g'(y_g)} & \text{if } u = 0\,. \end{cases}$$

The following remark will be very useful in this section.

Remark 2.3. Since $(g^{-1})'$ is continuous, σ_g satisfies

$$\sigma_g(u) = \frac{1}{u}\int_{-u}^{u} -(g^{-1})'(s)ds \quad \forall\, u \in (0, b_g).$$

The function σ_g is continuous, even and positive. Moreover, $y \in \operatorname{dom} g \setminus y_g$ satisfies $(\mathrm{id}+g)^{(2)}(y) = y$ if, and only if, $u = g^{-1}(y) \in \operatorname{dom} \sigma_g$ satisfies $\sigma(u) = \sigma(-u) = 1$, see [5]. In other words, the nontrivial period-2 solutions of (2.1) correspond to the symmetric intersections of the graph of σ_g with the the graph of the constant function with value 1.

Our next result shows that the stability properties of y_g are intimately linked to the relative position of the function σ_g with respect to the constant function with value 1.

Theorem 2.4. *Let $g \in \mathfrak{G}$. The following statements hold for the unique equilibrium y_g of (2.1):*

a) y_g is L.A.S. if $\sigma_g(0) > 1$, and it is unstable if $\sigma_g(0) < 1$.

b) y_g is G.A.S. if, and only if, $\sigma_g(u) > 1$ for all $u \in (-b_g, b_g) \setminus \{0\}$.

c) If $\sigma_g(u) \geq 1$ for all u in a neighbourhood of $u = 0$, then y_g is stable.

d) If $\sigma_g(u) < 1$ for all u in a punctured neighbourhood of $u = 0$, then y_g is unstable.

e) y_g is a global repeller if, and only if, $\sigma_g(u) < 1$ for all $u \in (-b_g, b_g) \setminus \{0\}$.

f) If $\sigma_g(u) > 1$ for all u in a punctured neighbourhood of $u = 0$, then y_g is L.A.S.

Proof. The proof of statements a)–d) can be found in [5]. Similar ideas can be used to prove statements e) and f). Indeed, the reader just needs to reverse the inequalities in the proof of b) and to invoke [5, Proposition 4.d] to obtain the proof of e); whereas reversing the inequalities in d) and invoking [5, Proposition 3.a] gives the proof of f). □

Our next result illustrates how Theorem 2.4 can be used to obtain sufficient conditions for the (in)stability of the equilibrium y_g of the difference equation (2.1).

Proposition 2.5. *Let $g \in \mathfrak{G}$. The following statements hold.*

a) If $g'(y) < -2$ for all $y \in (a, b)\backslash\{y_g\}$, then y_g is a global repeller for (2.1).

b) If $g'(y) > -2$ for all $y \in (a, b)\backslash\{y_g\}$, then y_g is G.A.S. for (2.1).

Proof. To prove statement a), we argue that

$$\sigma_g(u) < 1 \quad \forall\, u \in (0, b_g)\,, \tag{2.2}$$

and invoke statement e) of Theorem 2.4, combined with the property that σ_g is an even function.

For which purpose, recalling that

$$\big(g^{-1}\big)'(u) = \frac{1}{g'(g^{-1}(u))} \quad \forall\, u \in (-b_g, b_g)\,,$$

our hypothesis on g in statement a) implies that

$$-\big(g^{-1}\big)'(u) < \frac{1}{2} \quad \forall\, u \in (-b_g, b_g)\backslash\{0\}\,.$$

Therefore, recalling Remark 2.3,

$$\sigma_g(u) = \frac{1}{u}\int_{-u}^{u} -\big(g^{-1}\big)'(s)ds < 1 \quad \forall\, u \in (0, b_g)\,,$$

and so (2.2) holds.

The proof of statement b) is similar, and argue that

$$\sigma_g(u) > 1 \quad \forall\, u \in (0, b_g)\,,$$

which, when combined with statement b) of Theorem 2.4, proves the claim. ☐

Proposition 2.6. *Let $g \in \mathfrak{G}$. The following statements hold.*

a) If $\big(g^{-1}\big)'$ is strictly convex or strictly concave, then the difference equation (2.1) *has at most one nontrivial period-2 solution.*

b) If $\big(g^{-1}\big)'$ is strictly convex and $g'(y_g) \le -2$, then y_g is a global repeller for (2.1)*.*

c) If $\big(g^{-1}\big)'$ is strictly concave and $g'(y_g) \ge -2$, then y_g is G.A.S. for (2.1)*.*

Noting that

$$(g^{-1})'''(u) = \frac{3(g''(y))^2 - g'(y)g'''(y)}{(g'(y))^5} \quad \forall\, u = g(y),\ y \in (a, b)\,,$$

a sufficient condition for strict convexity (concavity) of $\big(g^{-1}\big)'$ in the case that $g \in \mathcal{C}^3(\mathrm{dom}\ g)$ is that

$$3(g'')^2 - g'g''' \tag{2.3}$$

is negative (positive).

Proof of Proposition 2.6. We claim that if g^{-1} is strictly convex (concave), then the function σ_g is strictly decreasing (increasing) on the interval $(0, b_g)$. Assuming this, strict monotonicity of σ_g implies that there is at most one solution of $1 = \sigma_g(u)$ in $(0, b_g)$, and so invoking the properties of σ_g recalled after Definition 2.2, we conclude that (2.1) has at most one nontrivial period-2 solution, proving statement a).

Thus, if $\left(g^{-1}\right)'$ is strictly convex, then Remark 2.3 and an application of the second Hermite–Hadamard inequality in (1.4) yields

$$\begin{aligned}u\frac{d}{du}\sigma_g(u) &= u\frac{d}{du}\left(\frac{-1}{u}\int_{-u}^{u}(g^{-1})'(s)ds\right)\\ &= \frac{1}{u}\int_{-u}^{u}(g^{-1})'(s)ds - \left((g^{-1})'(-u)+(g^{-1})'(u)\right) < 0 \quad \forall\, u\in(0,b_g),\end{aligned}$$

that is, $\sigma_g' < 0$ and so σ_g is strictly decreasing on $(0, b_g)$.

Analogously, if $(g^{-1})'$ is strictly concave, then $-(g^{-1})'$ is strictly convex and so

$$\begin{aligned}-u\frac{d}{du}\sigma_g(u) &= u\frac{d}{du}\left(\frac{-1}{u}\int_{-u}^{u}-(g^{-1})'(s)ds\right)\\ &= \frac{1}{u}\int_{-u}^{u}-(g^{-1})'(s)ds - \left(-(g^{-1})'(-u)+-(g^{-1})'(u)\right) < 0 \quad \forall\, u\in(0,b_g).\end{aligned}$$

that is, $\sigma_g' > 0$. We conclude that σ_g is strictly increasing on $(0, b_g)$. The proof of statement a) is complete.

Under the hypotheses in statement b), that (2.2) holds is clear upon noting that $\sigma_g(0) = -2/g'(y_g) \leq 1$ and that we have just shown in proof of statement a) that σ_g is strictly increasing on the interval $(0, b_g)$. Invoking statement e) of Theorem 2.4 completes the proof of statement b).

Reasoning analogous to that used in the proof of statement b) proves statement c), and so we omit the details. □

Remark 2.7.

(i) If $\left(g^{-1}\right)'$ is strictly convex, then using the first Hermite–Hadamard inequality in (1.4) gives

$$\sigma_g(u) = \frac{-1}{u}\int_{-u}^{u}(g^{-1})'(s)ds < -2(g^{-1})'(0) = \frac{-2}{g'(y_g)} = \sigma_g(0) \quad \forall\, u\in(0,b_g),$$

and, consequently, σ_g attains a global maximum at 0. Hence, statement b) in Proposition 2.6 may be proven by statement e) of Theorem 2.4 directly together with the first inequality in the Hermite–Hadamard inequality, instead of the second inequality as was done above. A similar comment is valid for statement c) of Proposition 2.6.

(ii) Assume that $b_g = +\infty$. Since

$$\left(g^{-1}\right)'(u) = \frac{1}{g'(g^{-1}(u))} < 0 \quad \forall\, u\in(-\infty,+\infty),$$

as g is strictly decreasing, it follows that $-\left(g^{-1}\right)'(u) > 0$. In particular, if $\left(g^{-1}\right)'$ is convex in $\mathbb{R}$, then $-\left(g^{-1}\right)'$ is concave and positive in $\mathbb{R}$, and hence must be constant. Therefore, $\left(g^{-1}\right)'$ cannot be strictly convex. This implies that (2.3) cannot be negative.

In light of the above, when $b_g = +\infty$, statement b) of Proposition 2.6 cannot be applied.

We finish the section by showing how Theorem 2.4 and Proposition 2.6 can be used to study a particular type of positive difference equation via topological conjugacy. We define

$$\mathfrak{C} := \bigcup_{-\infty\leq a<b\leq\infty} \mathcal{C}(a,b) \qquad \text{and} \qquad \mathfrak{C}_+ := \bigcup_{0\leq a<b\leq\infty} \mathcal{C}_+(a,b),$$

with $\mathcal{C}_+(J) = \{d \in \mathcal{C}(J) : d > 0\}$, and define $\mathfrak{T}\colon \mathfrak{C}_+ \to \mathfrak{C}$ by $\mathfrak{T}(d) = \ln \circ d \circ \exp$. Clearly, $\mathfrak{T}$ is bijective, with inverse $\mathfrak{T}^{-1}\colon \mathfrak{C} \to \mathfrak{C}_+$ given by $\mathfrak{T}^{-1}(g) = \exp \circ g \circ \ln$. Define

$$\mathfrak{D} := \mathfrak{T}^{-1}(\mathfrak{G}) = \bigcup_{0 \le a < b \le \infty} \mathfrak{D}(a,b)\,,$$

with

$$\mathfrak{D}(a,b) := \left\{ d \in \mathcal{C}^1(a,b) : a < \mathrm{id} \cdot d < b,\ d' < 0,\ 1 \in d\big((a,b)\big) \right\},$$

and consider the difference equation

$$x_{n+1} = x_n d(x_n), \quad x_0 \in \operatorname{dom} d\,, \tag{2.4}$$

where $d \in \mathfrak{D}$. Note that for each $d \in \mathfrak{D}$ there exists a unique $x_d \in \operatorname{dom} d$ such that $d(x_d) = 1$, and consequently x_d is an equilibrium of (2.4).

A routine calculation shows that $x = (x_n)$ is a solution of (2.1) if, and only if, $z = e^x$ is a solution of (2.4), where g and d are related by $d = \mathfrak{T}^{-1}(g)$. Therefore, stability properties of (2.4) may be studied by applying Theorem 2.4 and Proposition 2.6 to the transformed version (2.4).

3 Sharpest interval containing the attractor

We will make use of the following result (see [7, Theorems 2.2 and 2.3]).

Lemma 3.1. *If there exists an interval $I_0 \subset I$ such that*

$$\inf I_0 \le \liminf_{n\to+\infty} f^{(n)}(x) \le \limsup_{n\to+\infty} f^{(n)}(x) \le \sup I_0 \quad \forall\, x \in I\,,$$

then the solutions of (1.1) satisfy

$$\inf I_0 \le \liminf_{t\to+\infty} x(t,\xi) \le \limsup_{t\to+\infty} x(t,\xi) \le \sup I_0 \quad \forall\, \tau > 0,\ \forall\, \xi \in \mathcal{C}([-\tau,0], I)\,.$$

In particular, if K is G.A.S. for the difference equation (1.3), then

$$\lim_{t\to+\infty} x(t;\xi) = K \quad \forall\, \tau > 0,\ \forall\, \xi \in \mathcal{C}([-\tau,0], I)\,.$$

The following theorem is the main result of this paper. It provides a partial answer to the problem of finding the sharpest attracting interval for the delay-differential equation (1.1) under condition **(L)** by establishing a dichotomy, in the flavour of that of Theorem 1.1.

Theorem 3.2. *Assume that **(L)** holds, that f is three times differentiable and satisfies*

$$3(f'')^2 - (f'-1)f''' > 0\,, \tag{3.1}$$

on the interval (α,β). Then exactly one of the following holds:

1. *$f'(K) \ge -1$ and the global attractor of (1.1) for all values of the delay τ is $\{K\}$.*

2. *$f'(K) < -1$ and the sharpest invariant and attracting interval containing the global attractor of (1.1) for all values of the delay τ is $[\bar\alpha, \bar\beta]$, where $\{\bar\alpha, \bar\beta\}$ is the unique nontrivial 2-cycle of the map f in $[\alpha,\beta]$.*

Proof. Using condition **(L)**, it is not hard, but tedious since several cases need to be considered, to see that for any $x_0 \in I$ there exists $n \in \mathbb{N}$ such that $f^{(n)}(x_0) \in (\alpha, \beta)$, and $f([\alpha, \beta]) \subset [\alpha, \beta)$.

Define $g := f - \mathrm{id}$. We claim that g belongs to $\mathfrak{G}(\alpha, \beta)$. To see this, note that g is strictly decreasing in $[\alpha, \beta]$ since f is. Also, note that

$$g(x) + x = f(x) \in (\alpha, \beta) \quad \forall\, x \in (\alpha, \beta)$$

and, since $f([\alpha, \beta]) \subset [\alpha, \beta)$,

$$f(\beta) - \beta < 0 < f(\alpha) - \alpha,$$

so $0 \in g\big((\alpha, \beta)\big)$, and we have that $g \in \mathfrak{G}(\alpha, \beta)$.

Assume first that $f'(K) \geq -1$. Using that for any $x_0 \in I$ there exists $n \in \mathbb{N}$ such that $f^{(n)}(x_0) \in (\alpha, \beta)$ and invoking the second part of Lemma 3.1, it is enough to show that K is G.A.S. for the difference equation (2.1). And this follows from the second part of Proposition 2.6 after noting that $g'(K) \geq -2$, because $f'(K) \geq -1$, and that the function in (2.3) is positive, because (3.1) holds.

Assume now that $f'(K) < -1$. Since $f([\alpha, \beta]) \subset [\alpha, \beta)$, by a celebrated result of Coppel [3], f has at least one nontrivial 2-cycle $\{\bar{\alpha}, \bar{\beta}\}$ with $[\bar{\alpha}, \bar{\beta}] \subsetneq [\alpha, \beta]$. Moreover, by Proposition 2.6, it is the unique nontrivial 2-cycle contained in $[\alpha, \beta]$.

Next, invoking [11, Lemma 2], $[\bar{\alpha}, \bar{\beta}]$ is an attracting and forward invariant interval for the map f. Therefore, by Lemma 3.1, the interval $[\bar{\alpha}, \bar{\beta}]$ contains the global attractor of (1.1). Finally, using [11, Proposition 5] we see that any closed subinterval of $[\bar{\alpha}, \bar{\beta}]$ does not contain the global attractor of (1.1) for all $\tau > 0$ because we can find slowly oscillating periodic solutions of (1.1) taking values as close as desired to $\bar{\alpha}$ and $\bar{\beta}$. □

It is interesting to note that the previous result is based on rewriting the difference equation (1.3) in the form (2.1). In Theorem 3.2, we have used the natural choice $g = f - \mathrm{id}$. However, this transformation is not the unique and any topologically conjugate difference equation of (1.1) belonging to model (2.1) will give a different condition on f for the validity of the dichotomy. In particular, if f is positive and $x \mapsto f(x)/x$ is decreasing, then we obtain the following result from the topological conjugacy described at the end of Section 2.

Proposition 3.3. *Assume that **(L)** holds, that $d(x) := f(x)/x$ is three times differentiable with $d' < 0$, and that*

$$3(g'')^2 - g'g''' > 0\,,$$

on the interval $(\ln \alpha, \ln \beta)$, where $g := \ln \circ d \circ \exp$. Then the conclusions of Theorem 3.2 hold.

4 Examples

This section provides several examples demonstrating the applicability of Theorem 3.2 and Proposition 3.3. The first example shows that Theorem 3.2 can be applied in situations where Theorem 1.1 can not.

Example 4.1. Consider equation (1.1) with $f\colon (0,1) \to (0,1)$ given by

$$f(x) = \frac{19}{20} x(1-x)(5 - 4x + 2x^3). \tag{4.1}$$

The graph of f is plotted in Panel A in Figure 4.1. Using Sturm's Theorem, it is easy to see that neither f nor $f - 1$ have any real roots in the open interval $(0, 1)$. Moreover, $f(1/2) =$

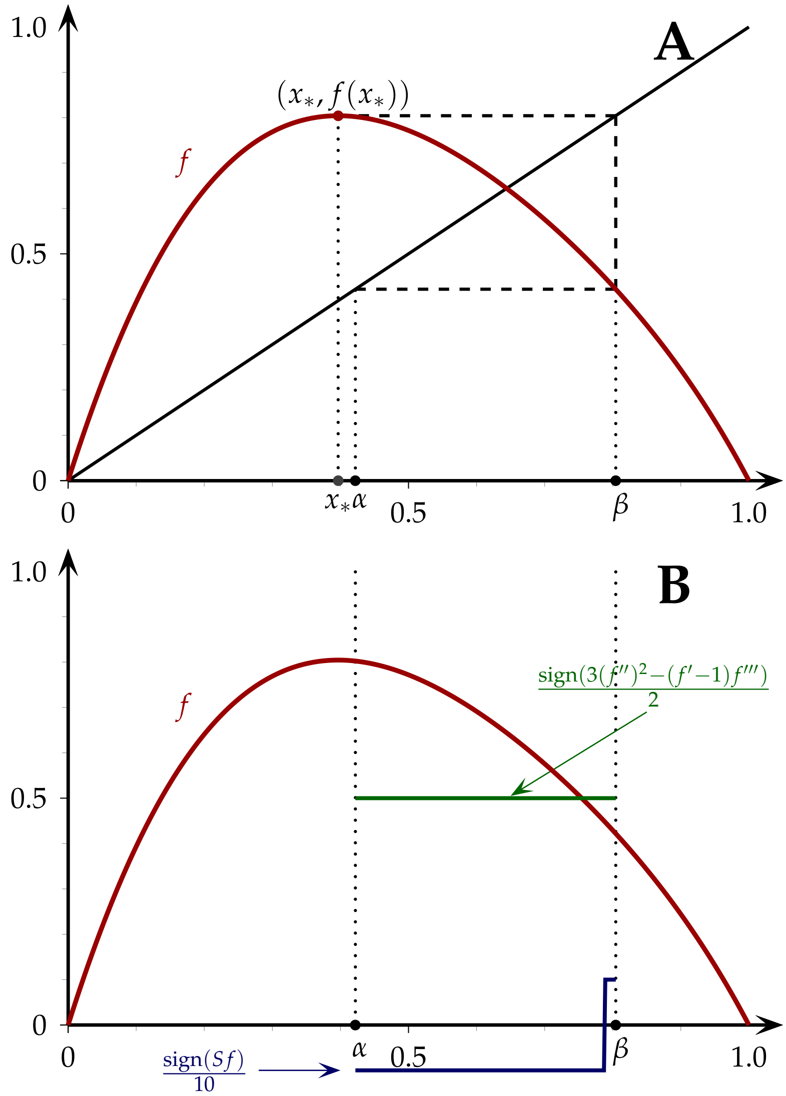

Figure 4.1: Panel A shows the graph of $f(x) = \frac{19}{20}x(1-x)(5-4x+2x^3)$. Observe that condition **(U)** holds. Also note that $f(f(x_*)) > x_*$ and condition **(L)** holds. Panel B shows, in the interval $[\alpha, \beta]$, the graphs of scaled versions of the sign function composed with, respectively, the Schwarzian derivative of f and $3(f'')^2 - (f'-1)f'''$. Observe that the sign of $3(f'')^2 - (f'-1)f'''$ remains positive, meanwhile Sf changes sign in the interval $[\alpha, \beta]$.

$247/320 \in (0,1)$. Hence, f is well-defined. On the other hand, $f'(x) = -\frac{19}{20}(10x^4 - 8x^3 - 12x^2 + 18x - 5)$ and so $f'(0) = \frac{19}{4} > 1$. Moreover, invoking again Sturm's Theorem, f' has exactly one real root x_* (which one can calculate explicitly since f' is a polynomial of degree 4) in the interval $(0,1)$. At $x_* \approx 0.3966$ the function f attains a local maximum because $f(0^+) = f(1^-) = 0$. Solving the equation $f(x) = x$, we find that f has a unique solution $K \in (0,1)$, which again can be explicitly calculated, with $K \approx 0.6441$; and so $x_* < K$. Thus, f satisfies the unimodal condition **(U)** with $a = 0$ and $b = 1$. Observe in Panel A in Figure 4.1 that condition **(L)** holds for f because $x_* < \alpha = f(f(x_*))$.

Panel B in Figure 4.1 illustrates that Theorem 1.1 cannot be used to study the behaviour of equation (1.1) with f given by (4.1). Indeed, we observe that the condition **(S)** is violated, i.e.,

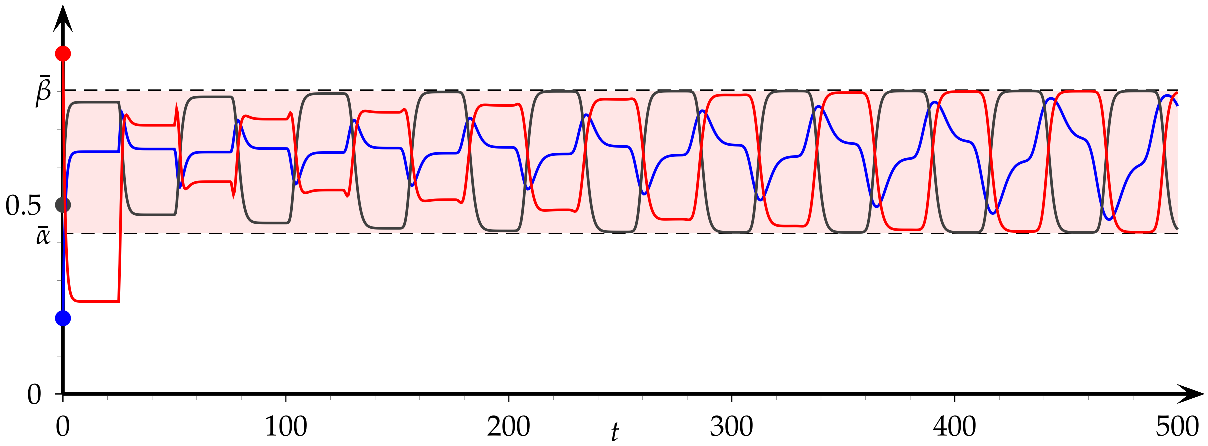

Figure 4.2: The figure shows three different solutions of equation 1.1 with $\mu = 1$, $\tau = 25$ and f as in (4.1). The initial condition is a constant function $\xi \in \mathcal{C}([-\tau, 0], \mathbb{R})$, namely, in the blue curve $\xi = 0.2$, the red curve $\xi = 0.9$, and the black curve $\xi = 0.5$. The pink region is determined by the 2-cycle $\bar{\alpha}, \bar{\beta}$. Observe how the three solutions are asymptotically trapped in this region.

the Schwarzian derivative, Sf, is not negative in the interval $[\alpha, \beta]$. In contrast, the function $3(f'')^2 - (f' - 1)f'''$ has positive sign (again this is easily verified using Sturm's Theorem in the interval $[0, 1]$, which contains the interval $[\alpha, \beta]$). Thus, f satisfies the assumptions of Theorem 3.2.

Since $f'(K) \approx -1.1390$, invoking Theorem 3.2 we conclude that the sharpest invariant and attracting interval containing the attractor of equation (1.1) for all values of the delay τ is determined by the unique nontrivial 2-cycle $\{\bar{\alpha}, \bar{\beta}\}$ of f in the interval $[\alpha, \beta]$. Numerically, we find that $\bar{\alpha} \approx 0.4269$ and $\bar{\beta} \approx 0.8013$.

In Figure 4.2, we plot three solutions of equation (1.1) with f as in (4.1), $\mu = 1$, $\tau = 25$ and different constant initial conditions. Observe that all the solutions asymptotically take values in the interval determined by the 2-cycle $\{\bar{\alpha}, \bar{\beta}\}$ as the result predicts. Moreover, observe that as $t \to \infty$ the solutions oscillate in a range that it is close to the length of the interval $[\bar{\alpha}, \bar{\beta}]$. $\diamondsuit$

The next example shows that Proposition 3.3 can be applied in situations where the assumptions in Theorem 1.1, and in Theorem 3.2, do not hold.

Example 4.2. Consider equation (1.1) with $f\colon (0,1) \to (0,1)$ given by

$$f(x) = \frac{3}{10} x \left(1 - \frac{1}{10} \ln(x)\right)^{15}. \tag{4.2}$$

Differentiating, we have

$$f'(x) = \frac{3}{10}\left(1 - \frac{\ln(x)}{10}\right)^{15} - \frac{9}{20}\left(1 - \frac{\ln(x)}{10}\right)^{14} = -\frac{3(\ln(x) - 10)^{14}(\ln(x) + 5)}{10^{16}}.$$

Therefore, f has a critical point at $x_* = \mathrm{e}^{-5} \in (0, 1)$. Moreover,

$$f''(x) = -\frac{9(\ln(x) - 10)^{13}(\ln(x) + 4)}{2 \cdot 10^{15} x},$$

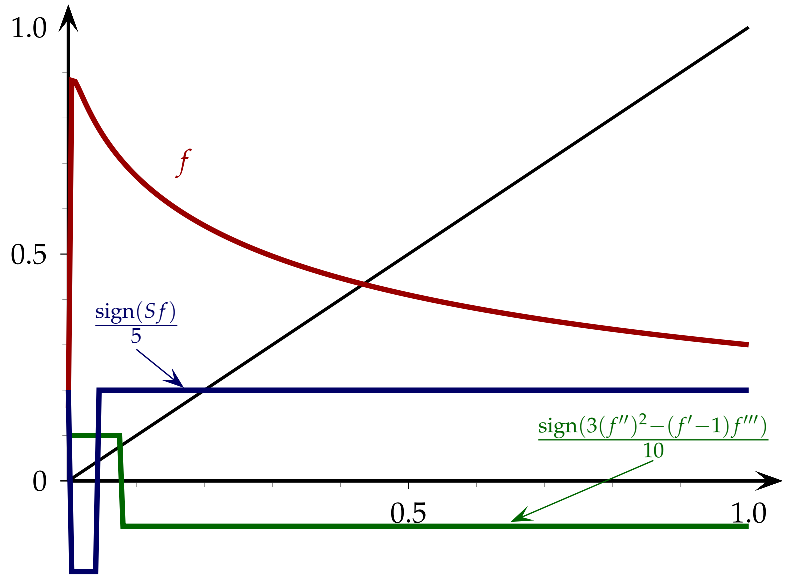

Figure 4.3: Graphs of the function $f(x) = \frac{3}{10}x(1 - \frac{1}{10}\ln(x))^{15}$ (red curve), and the graphs of scaled versions of the sign function composed with, respectively, the Schwarzian derivative of f (blue curve) and $3(f'')^2 - (f' - 1)f'''$ (green curve). Observe that at the fixed point K, Sf is positive and $3(f'')^2 - (f' - 1)f'''$ is negative. Since $x_* \in [\alpha, \beta]$, the assumptions of Theorem 1.1 and Theorem 3.2 are not satisfied.

and so $f''(x_*) < 0$. Noting that $f(0^+) = 0$ and $f(1^-) = 3/10$, we conclude that f is well-defined and unimodal in the interval $(0, 1)$. Now, note that f is convex in the interval $(0, \mathrm{e}^{-4})$ and $\lim_{x\to 0^+} f(x)/x = +\infty$. Consequently, f has a unique fixed point K in the interval $(0, 1)$ and it satisfies $x_* < K$. This shows that **(U)** hold for (4.2)

Next, we obtain that

$$\beta = f(x_*) = \frac{43046721\mathrm{e}^{-5}}{327680} \approx 0.8852\,,$$

and

$$\alpha = f(f(x_*)) = \frac{129140163\mathrm{e}^{-5}}{3276800}\left(1 - \frac{1}{10}\ln\left(\frac{43046721\mathrm{e}^{-5}}{327680}\right)\right)^{15} \approx 0.3185\,.$$

Recalling that $x_* = \mathrm{e}^{-5}$, we have that condition **(L)** holds. In this case, neither Theorem 1.1 nor Theorem 3.2 can be used because the Schwarzian derivative and the function $3(f'')^2 - (f' - 1)f'''$ do not satisfy the sign restrictions in the interval $[\alpha, \beta]$, cf. Figure 4.3.

Nevertheless, Proposition 3.3 holds. We need to verify that

$$d(x) = \frac{f(x)}{x} = \frac{3}{10}\left(1 - \frac{1}{10}\ln(x)\right)^{15}$$

is decreasing, which is trivial, and $g(x) = \ln \circ d \circ \exp$ satisfies $3(g'')^2 - g'g''' > 0$ in the interval $[\ln\alpha, \ln\beta]$. Deriving, we obtain

$$3(g''(x))^2 - g'(x)g'''(x) = \frac{225}{(x - 10)^4}, \tag{4.3}$$

and $3(g''(x))^2 - g'(x)g'''(x)$ is positive in the interval $[\ln\alpha, \ln\beta]$.

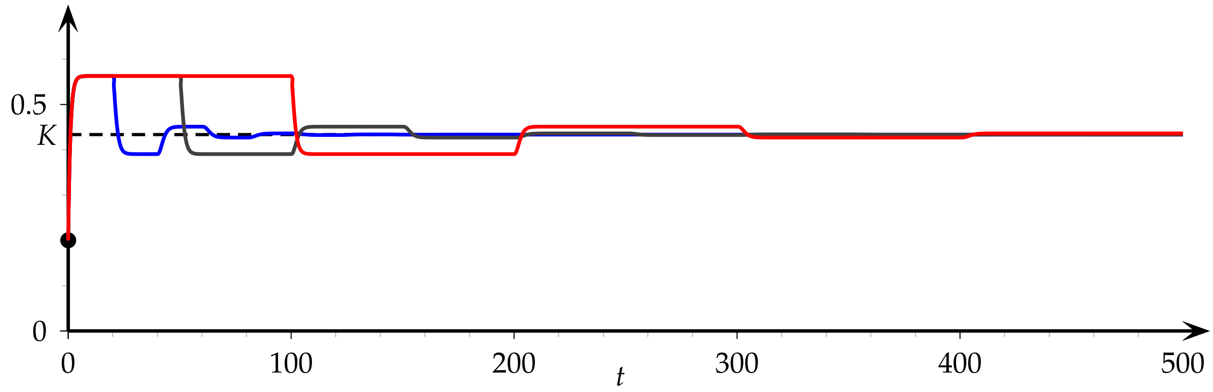

Figure 4.4: The figure shows three solutions of equation 1.1 with $\mu = 1$ and f as in (4.2). The initial condition is a constant function $\xi \in \mathcal{C}([-\tau, 0], \mathbb{R})$, namely, $\xi = 0.2$, but the delay τ is different. For the blue curve we fixed $\tau = 20$, for the black curve $\tau = 50$, and for the red curve $\tau = 100$. Observe how independently of τ the three solutions tend to K.

Computing the derivative of f at its fixed point K, we obtain that this derivative is greater than -1 (approx. -0.3843). By Proposition 3.3 for any initial condition $\xi \in \mathcal{C}([-\tau, 0], (0, 1))$ the solutions of (1.1) tend to K as t tends to $+\infty$, with independence of the size of the delay $\tau > 0$ and the value of $\mu > 0$ as Figure 4.4 illustrates. $\diamondsuit$

Probably, the most famous representatives of equation (1.1) are the Nicholson's blowflies equation and the Mackey–Glass equation. In the Nicholson's blowflies equation f is given by

$$f(x) = \frac{1}{\mu} x e^{-x}, \tag{4.4}$$

whereas in the Mackey–Glass equation f is given by

$$f(x) = \frac{1}{\mu} \frac{ax}{1 + x^b}, \quad a > 0,\ b \geq 1. \tag{4.5}$$

Both (4.4) and (4.5) have negative Schwarzian derivative, and therefore Theorem 1.1 can be used to study them. This was illustrated in [11, Section 3] with a couple of examples. We notice that Proposition 3.3 can be used to obtain the same conclusions as in those examples. Indeed, $d(x) = f(x)/x$ is decreasing both for (4.4) and (4.5). Therefore, to invoke Proposition 3.3 we need to check that $g(x) = \ln \circ d \circ \exp$ satisfies $3(g'')^2 - g'g''' > 0$ in the interval $(\ln \alpha, \ln \beta)$. The following examples show that the inequality holds not only in the interval $(\ln \alpha, \ln \beta)$ but in the whole $\mathbb{R}$.

Example 4.3. Nicholson's blowflies equation. In this case, $g(x) = \ln(1/\mu) - e^x$ and trivially

$$3(g'')^2 - g'g''' = 2e^{2x} > 0\,. \qquad \diamondsuit$$

Example 4.4. The Mackey–Glass equation. In this case, $g(x) = \ln(a/\mu) - \ln(1 + e^{bx})$ and after some straightforward calculations we obtain that

$$3(g'')^2(x) - g'(x)g'''(x) = \frac{b^4 e^{2bx}(2 + e^{bx})}{(1 + e^{bx})^4} > 0\,. \qquad \diamondsuit$$

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