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# Infinitely many nodal solutions for a class of quasilinear elliptic equations

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**Abstract.** In this paper, we study the existence of infinitely many nodal solutions for the following quasilinear elliptic equation

$$\begin{cases} -\nabla \cdot [\phi'(|\nabla u|^2)\nabla u] + |u|^{\alpha-2}u = f(u), & x \in \mathbb{R}^N, \\ u(x) \to 0, \quad \text{as } |x| \to \infty, \end{cases}$$

where $N \geq 2$, $\phi(t)$ behaves like $t^{q/2}$ for small $t$ and $t^{p/2}$ for large $t$, $1 < p < q < N$, $f \in \mathcal{C}^1(\mathbb{R}^+, \mathbb{R})$ is of subcritical, $q \leq \alpha \leq p^* q'/p'$, let $p^* = \frac{Np}{N-p}$, $p'$ and $q'$ be the conjugate exponents respectively of $p$ and $q$. For any given integer $k \geq 0$, we prove that the equation has a pair of radial nodal solution with exactly $k$ nodes.

**Keywords:** quasilinear elliptic equation, nodal solutions, multiple solutions.

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## 1 Introduction

In this paper, we consider the following quasilinear elliptic equation

$$-\nabla \cdot [\phi'(|\nabla u|^2)\nabla u] + |u|^{\alpha-2}u = f(u), \qquad x \in \mathbb{R}^N, \tag{1.1}$$

where $N \geq 2$, $\phi \in \mathcal{C}^2(\mathbb{R}_+, \mathbb{R}_+)$ has a different growth near zero and infinity. Quasilinear equation of form (1.1) can be transformed into different differential equations corresponding to various types of $\phi$. For example, when $\phi(t) = 2[(1+t)^{\frac{1}{2}} - 1]$ and $\alpha = 2$, equation (1.1) corresponds to the prescribed mean curvature equation or the capillary surface equation

$$-\nabla \cdot \Big( \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \Big) + u = f(u), \qquad x \in \mathbb{R}^N.$$

Such problem has been deeply studied since last century, under different assumptions on the nonlinearity $f$, the existence and nonexistence of solutions have been investigated by many authors, see [3,5,8,27] for example.

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Equation (1.1) also related to $(p, q)$-Laplacian equations. In fact, if $\phi(t) = \frac{2}{p}t^{\frac{p}{2}} + \frac{2}{q}t^{\frac{q}{2}}$, then equation (1.1) becomes

$$-\Delta_p u - \Delta_q u + |u|^{\alpha-2}u = f(u) \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < q < N$ and $\alpha > 2$ satisfies some conditions. Equation (1.2) originates from the following reaction diffusion system

$$\frac{\partial u}{\partial t} = \operatorname{div}\big[D(u)\nabla u\big] + c(x, u), \tag{1.3}$$

where $D(u) = (|\nabla u|^{p-2} + |\nabla u|^{q-2})$. This system has a wide range of application in physics and related sciences such as biophysics, plasma physics and chemical reaction design. In such applications, the function $u$ describes a concentration; the first term on the right hand side of (1.3) corresponds to diffusion with a diffusion coefficient $D(u)$, whereas the second one is the reaction and relates to source and loss processes. Typically, in chemical and biological applications, the reaction term $c(x, u)$ has a polynomial form with respect to the concentration $u$. For more mathematical and physical background of equations (1.2)–(1.3), we refer the reader to the papers [9, 24, 25, 31] and the references therein. In particular, when $p = q = \alpha = 2$, equation (1.2) reduced to

$$-\Delta u + u = f(u) \quad \text{in } \mathbb{R}^N. \tag{1.4}$$

There has been plenty of results on the existence, nonexistence and multiplicity of positive or sign-changing solutions for equation (1.4), see [2, 6, 7, 10, 17] and the references therein.

If $p = q = \alpha \neq 2$, then equation (1.2) becomes into the following general $p$-Laplacian equation

$$-\Delta_p u + |u|^{p-2}u = f(u) \quad \text{in } \mathbb{R}^N, \tag{1.5}$$

which was studied by many authors. Many results for equation (1.4) has been extended to equation (1.5). Deng, Guo and Wang in [12] proved the existence of nodal solutions for $p$-Laplacian equations with critical growth. Recently in [13], Deng, Li and Shuai studied the existence of solutions for a class of $p$-Laplacian equations with critical growth and potential vanishing at infinity.

Recently, Azzollini et al. [1] studied the following quasilinear elliptic equation

$$\begin{cases} -\nabla \cdot [\phi'(|\nabla u|^2)\nabla u] + |u|^{\alpha-2}u = |u|^{s-2}u, & x \in \mathbb{R}^N, \\ u(x) \to 0, \quad \text{as } |x| \to \infty, \end{cases} \tag{1.6}$$

where $N \geq 2$, $\phi(t)$ behaves like $t^{q/2}$ for small $t$ and $t^{p/2}$ for large $t$, $1 < p < q < N$, $1 < \alpha \leq p^*q'/p'$ and $\max\{q, \alpha\} < s < p^* = \frac{Np}{N-p}$, with being $p'$, $q'$ are the conjugate exponents of $p$, $q$ respectively. The authors in [1] found a sort of Orlicz–Sobolev space in which the energy functional is well defined. They also proved that the Orlicz–Sobolev space compactly embedded into certain Lebesgue spaces. Then, they obtained the existence of a sequence of nontrivial radial solutions for equation (1.6) besides a nontrivial non-negative radial solution. General quasilinear elliptic problems of (1.1) have been intensively studied, see for example, [1, 11, 15, 16, 18, 28] and the references therein.

Motivated by the above results, in this paper, we intend to find nodal solutions for the following quasilinear elliptic equation

$$\begin{cases} -\nabla \cdot \big[\phi'(|\nabla u|^2)\nabla u\big] + |u|^{\alpha-2}u = f(u), & x \in \mathbb{R}^N, \\ u(x) \to 0, \quad \text{as } |x| \to \infty, \end{cases} \tag{1.7}$$

where $N \geq 2$, $\phi(t)$ behaves like $t^{q/2}$ for small $t$ and $t^{p/2}$ for large $t$, $1 < p < q < N$, $q \leq \alpha \leq p^* q'/p'$, and the function $f$ satisfies some conditions given by $(f_1)$-$(f_3)$ in this paper. Similar as [1], we impose some restrictions on $\phi$, let $\phi \in \mathcal{C}^2(\mathbb{R}_+, \mathbb{R}_+)$ such that

($\Phi$1) $\phi(0) = 0$;

($\Phi$2) there exists a positive constant $C$ such that

$$\begin{cases} Ct^{\frac{p}{2}-1} \leq \phi'(t), & \text{if } t \geq 1, \\ Ct^{\frac{q}{2}-1} \leq \phi'(t), & \text{if } 0 \leq t \leq 1; \end{cases}$$

($\Phi$3) there exists a positive constant $C$ such that

$$\begin{cases} \phi(t) \leq Ct^{\frac{p}{2}}, & \text{if } t \geq 1, \\ \phi(t) \leq Ct^{\frac{q}{2}}, & \text{if } 0 \leq t \leq 1; \end{cases}$$

($\Phi$4) there exists $\alpha < \theta$ such that $\phi'(t)/t^{\frac{\theta-2}{2}}$ is strictly decreasing for all $t > 0$;

($\Phi$5) the map $t \mapsto \phi(t^2)$ is convex.

We also assume the nonlinearity $f$ satisfies:

$(f_1)$ $f(t) = o(t^{\alpha-1})$, as $t \to 0^+$;

$(f_2)$ $f(t) = o(t^{p^*-1})$, as $t \to +\infty$;

$(f_3)$ there exists $\alpha < \theta$ such that

$$0 < (\theta - 1)f(t) \leq f'(t)t, \quad \text{for all } t > 0.$$

Before we present our main result, we give some notions and definitions. In the following, we use $\|u\|_q$ to denote the $L^q(\mathbb{R}^N)$ norm.

**Definition 1.1** (See [1, Definition 2.1])**.** Let $1 < p < q$ and $\Omega \subset \mathbb{R}^N$. Denote $L^p(\Omega) + L^q(\Omega)$ the completion of $\mathcal{C}_c^\infty(\Omega, \mathbb{R})$ in the norm

$$\|u\|_{L^p(\Omega)+L^q(\Omega)} = \inf\left\{\|v\|_{L^p(\Omega)} + \|w\|_{L^q(\Omega)} \mid v \in L^p(\Omega),\ w \in L^q(\Omega), u = v + w\right\}.$$

Next, we denote $\|u\|_{p,q} = \|u\|_{L^p(\mathbb{R}^N)+L^q(\mathbb{R}^N)}$. Moreover, in [4], it has shown that $L^p(\Omega) + L^q(\Omega)$ can be characterized as an Orlicz spaces.

**Definition 1.2** (See [1, Definition 2.3])**.** Let $\alpha > 1$, the Orlicz–Sobolev space $\mathcal{W}(\mathbb{R}^N)$ is the completion of $\mathcal{C}_c^\infty(\mathbb{R}^N, \mathbb{R})$ in the norm

$$\|u\| = \|u\|_\alpha + \|\nabla u\|_{p,q}.$$

By Theorem 2.8 of [1], the space $\mathcal{W}(\mathbb{R}^N)$ can be precise description by

$$\mathcal{W}(\mathbb{R}^N) = \left\{u \in L^\alpha(\mathbb{R}^N) \cap L^{p^*}(\mathbb{R}^N) \mid \nabla u \in L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)\right\}.$$

In the following, we define

$$\left(\mathcal{C}_c^\infty(\mathbb{R}^N, \mathbb{R})\right)_r = \left\{u \in \mathcal{C}_c^\infty(\mathbb{R}^N, \mathbb{R}) \mid u \text{ is radially symmetric}\right\}.$$

Then $\mathcal{W}_r(\mathbb{R}^N)$ is the completion of $\left(\mathcal{C}_c^\infty(\mathbb{R}^N,\mathbb{R})\right)_r$ in the norm $\|\cdot\|$, namely

$$\mathcal{W}_r(\mathbb{R}^N) = \overline{\left(\mathcal{C}_c^\infty(\mathbb{R}^N,\mathbb{R})\right)_r}^{\|\cdot\|}.$$

Thus, $\mathcal{W}_r(\mathbb{R}^N)$ coincides with the set of radial functions of $\mathcal{W}(\mathbb{R}^N)$. Define the energy functional $I$ corresponding to equation (1.7) by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} \phi(|\nabla u|^2)\,\mathrm{d}x + \frac{1}{\alpha}\int_{\mathbb{R}^N} |u|^\alpha\,\mathrm{d}x - \int_{\mathbb{R}^N} F(u)\,\mathrm{d}x, \qquad u \in \mathcal{W}_r(\mathbb{R}^N),$$

where $F(u) = \int_0^u f(z)\,\mathrm{d}z$. The well-posedness and regularity of $I(u)$ are given by Proposition 3.1 in [1] and hypotheses $(f_1)$–$(f_2)$.

A function $u \in \mathcal{W}_r(\mathbb{R}^N)$ is called a weak solution of equation (1.7) if for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N,\mathbb{R})$, it holds

$$\int_{\mathbb{R}^N} \phi'(|\nabla u|^2)\nabla u \nabla\varphi\,\mathrm{d}x + \int_{\mathbb{R}^N} |u|^{\alpha-2}u\varphi\,\mathrm{d}x - \int_{\mathbb{R}^N} f(u)\varphi\,\mathrm{d}x = 0.$$

In particular, for $u \in \mathcal{W}_r(\mathbb{R}^N)$, we denote

$$\gamma(u) = \langle I'(u), u\rangle = \int_{\mathbb{R}^N} \phi'(|\nabla u|^2)|\nabla u|^2\,\mathrm{d}x + \int_{\mathbb{R}^N} |u|^\alpha\,\mathrm{d}x - \int_{\mathbb{R}^N} f(u)u\,\mathrm{d}x.$$

Now we state our main result. We denote $u^+ = \max\{u,0\}$ and $u^- = \min\{u,0\}$.

**Theorem 1.3.** *Suppose* $1 < p < q < \min\{N, p^*\}$, $q \le \alpha \le p^*q'/p'$, $(\Phi1)$–$(\Phi5)$ *and* $(f_1)$–$(f_3)$ *hold, then there exists a pair of radial solutions* $u_k^\pm$ *of equation* (1.7) *with the following properties:*

*(i)* $u_k^-(0) < 0 < u_k^+(0)$,

*(ii)* $u_k^\pm$ *possess exactly* $k$ *nodes* $r_i$ *with* $0 < r_1 < r_2 < \cdots < r_k < +\infty$, *and* $u_k^\pm(x)|_{|x|=r_i} = 0$, $i = 1,2,\ldots,k$.

**Remark 1.4.** The solutions $u_k$ obtained in Theorem 1.3, as we will see, is the least energy radial solution of equation (1.7) and changes sign exactly $k$ ($k \in \{0,1,2,\ldots\}$) times. We should point out that $\alpha < p^*$. The existence of $u_0$ had been proved by the Mountain Pass Theorem in [1].

**Remark 1.5.** Like [1], a specific example of the function $\phi(t)$ is

$$\phi(t) = \frac{2}{p}\left[\left(1 + t^{\frac{q}{2}}\right)^{\frac{p}{q}} - 1\right].$$

In this paper, we prove by constrained minimization method in a special space in which each function changes sign $k (k \in \{0,1,2,\ldots\})$ times. We first prove the existence of minimizer and then verify that the minimizer is indeed a solution to equation (1.7) by analyzing the least energy related to the minimizer. Here, we have to point out that it is hard to obtain radial solutions with a prescribed number of nodes by gluing method as in Bartsch–Willem [6] and Cao–Zhu [10]. Because, we obtain that all weak solutions of (1.7) by Lemma 2.7 are only of class $\mathcal{C}^{1,\gamma}_{loc}(\mathbb{R}^N)$, and it is not enough to glue the functions in each annuli by matching the normal derivative at each junction point. We will follow the approach explored by Z. Liu and Z.-Q. Wang [21,22], see Section 3 for more details. Moreover, we introduce some new analysis techniques and establish better inequalities.

This paper is organized as follows. In Section 2, we give some preliminary results, which are crucial to prove our main results. In Section 3, we will prove our main theorem.

Throughout this paper, we denote " $\to$ " and " $\rightharpoonup$ " as the strong convergence and the weak convergence, respectively. We use $\langle\cdot,\cdot\rangle$ to denote the duality pairing between $\mathcal{W}_r(\mathbb{R}^N)$ and $\mathcal{W}_r'(\mathbb{R}^N)$. We employ $C$ or $C_j$, $j = 1,2,\ldots$ to denote the generic constant which may vary from line to line.

## 2 Some preliminary lemmas

In this section, let us first recall some known facts about (1.7). From [1], we introduce the embedding result on $\mathcal{W}_r(\mathbb{R}^N)$ and a uniform decaying estimate on the functions of $\mathcal{W}_r(\mathbb{R}^N)$. The proof of lemmas can be found in the corresponding references.

**Lemma 2.1** (see [1, Remark 2.7]). *If* $1 < p < \min\{q, N\}$ *and* $1 < p^* \frac{q'}{p'}$, *then for every* $\alpha \in \big(1, p^* \frac{q'}{p'}\big]$, $\mathcal{W}_r(\mathbb{R}^N)$ *is continuously embedded into* $L^\tau(\mathbb{R}^N)$ *with* $\alpha \le \tau \le p^*$.

**Lemma 2.2** (see [1, Theorem 2.11]). *If* $1 < p < q < N$ *and* $1 < p^* \frac{q'}{p'}$, *then for every* $\alpha \in \big(1, p^* \frac{q'}{p'}\big]$, $\mathcal{W}_r(\mathbb{R}^N)$ *is compactly embedded into* $L^\tau(\mathbb{R}^N)$ *with* $\alpha < \tau < p^*$.

**Lemma 2.3** (see [1, Lemma 2.13]). *If* $1 < p < q < N$, *there exists* $C > 0$ *such that for every* $u \in \mathcal{W}_r(\mathbb{R}^N)$

$$|u(x)| \le \frac{C}{|x|^{\frac{N-q}{q}}} \|\nabla u\|_{p,q}, \quad \textit{for } |x| \ge 1.$$

Let $\Omega$ be one of the following domains:

$$\{x \in \mathbb{R}^N : |x| < R_1\}, \qquad \{x \in \mathbb{R}^N : 0 < R_2 \le |x| < R_3 < \infty\}, \qquad \{x \in \mathbb{R}^N : |x| \ge R_4 > 0\}.$$

Thus, we first consider the existence of positive least energy solution for

$$\begin{cases} -\nabla \cdot [\phi'(|\nabla u|^2)\nabla u] + |u|^{\alpha-2}u = f(u), \quad x \in \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{2.1}$$

Define

$$I_\Omega(u) = \frac{1}{2}\int_\Omega \phi(|\nabla u|^2)\,\mathrm{d}x + \frac{1}{\alpha}\int_\Omega |u|^\alpha\,\mathrm{d}x - \int_\Omega F(u)\,\mathrm{d}x,$$

$$\gamma_\Omega(u) = \langle I'_\Omega(u), u\rangle = \int_\Omega \phi'(|\nabla u|^2)|\nabla u|^2\,\mathrm{d}x + \int_\Omega |u|^\alpha\,\mathrm{d}x - \int_\Omega f(u)u\,\mathrm{d}x$$

and

$$\mathrm{M}(\Omega) = \big\{u \in \mathcal{W}_r(\Omega) : u \not\equiv 0, u|_{\partial\Omega} = 0,\ \gamma_\Omega(u) = 0\big\}.$$

Then we have the following lemmas.

**Lemma 2.4.** *Suppose* $1 < p < q < \min\{N, p^*\}$, $q \le \alpha \le p^*q'/p'$, $(\Phi1)$–$(\Phi5)$ *and* $(f_1)$–$(f_3)$ *hold and* $u \in \mathcal{W}_r(\Omega)$. *Then there exists a unique* $t > 0$ *such that* $tu \in \mathrm{M}(\Omega)$.

*Proof.* For fixed $u \in \mathcal{W}_r(\Omega)$ with $u \not\equiv 0$, $tu$ is contained in $\mathrm{M}(\Omega)$ if and only if

$$\gamma_\Omega(tu) = \int_\Omega \phi'(|t\nabla u|^2)|t\nabla u|^2\,\mathrm{d}x + \int_\Omega |tu|^\alpha\,\mathrm{d}x - \int_\Omega f(tu)tu\,\mathrm{d}x = 0. \tag{2.2}$$

Hence, the problem is reduced to verify that there is only one solution of equation (2.2) with $t > 0$. Since $1 < p < q \le \alpha$ and

$$\phi(t^2) \simeq \begin{cases} t^p, & \text{if } |t| \gg 1, \\ t^q, & \text{if } |t| \ll 1. \end{cases}$$

By $(f_1)$–$(f_2)$, for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ and $\alpha < s < p^*$ such that

$$f(u)u \le \varepsilon|u|^\alpha + C_\varepsilon|u|^s. \tag{2.3}$$

It is easy to see that $I_\Omega(tu) \to 0$ as $t \to 0$ and $I_\Omega(tu) \to -\infty$ as $t \to +\infty$. We have that $I_\Omega$ possesses a global maximum point $t \in (0, +\infty)$, i.e., $tu \in \mathrm{M}(\Omega)$.

It remains to show the uniqueness of $t$. We shall divide our proof into two cases.

**Case 1.** $u \in \mathrm{M}(\Omega)$. First of all, we note that it follows from $\gamma_\Omega(u) = 0$ that

$$\int_\Omega \phi'(|\nabla u|^2)|\nabla u|^2 \,\mathrm{d}x + \int_\Omega |u|^\alpha \,\mathrm{d}x - \int_\Omega f(u)u \,\mathrm{d}x = 0. \tag{2.4}$$

We now prove that $t = 1$ is the unique number such that $tu \in \mathrm{M}(\Omega)$. In fact, if $t > 0$ such that $\gamma_\Omega(tu) = 0$, then we have

$$\int_\Omega \phi'(|t\nabla u|^2)|t\nabla u|^2 \,\mathrm{d}x + \int_\Omega |tu|^\alpha \,\mathrm{d}x - \int_\Omega f(tu)tu \,\mathrm{d}x = 0. \tag{2.5}$$

Furthermore, combining equation (2.4) and (2.5), we have

$$\begin{aligned} \int_\Omega \left[\phi'(t^2|\nabla u|^2)t^2|\nabla u|^2 - t^\theta \phi'(|\nabla u|^2)|\nabla u|^2\right] \mathrm{d}x \\ + \int_\Omega \left[(t^\alpha - t^\theta)|u|^\alpha + \left(t^\theta f(u) - f(tu)tu\right)\right] \mathrm{d}x = 0. \end{aligned} \tag{2.6}$$

On one hand, by $(f_3)$, we can get that

$$\frac{f(t)}{t^{\theta-1}}$$

is increasing for all $t > 0$. On the other hand, by $(\Phi 4)$, we can deduce that

$$\frac{\phi'(t^2)}{t^{\theta-2}}$$

is strictly decreasing for all $t > 0$. Assume $t > 1$ for a while, then we get

$$\frac{f(u)}{u^{\theta-1}} \le \frac{f(tu)}{|tu|^{\theta-1}}, \qquad \frac{\phi'(t^2|\nabla u|^2)}{t^{\theta-2}|\nabla u|^{\theta-2}} < \frac{\phi'(|\nabla u|^2)}{|\nabla u|^{\theta-2}},$$

that is

$$t^\theta f(u) - f(tu)tu \le 0 \tag{2.7}$$

and

$$\phi'(t^2|\nabla u|^2)t^2|\nabla u|^2 - t^\theta \phi'(|\nabla u|^2)|\nabla u|^2 < 0. \tag{2.8}$$

Since $\alpha < \theta$, the left side of equation (2.6) is negative, which gives a contradiction. With a similar argument, the case $t < 1$ is also contradictory. Thus we deduce that $t = 1$.

**Case 2.** $u \notin \mathrm{M}(\Omega)$. If there exist $t_1, t_2 > 0$ such that $t_1 u, t_2 u \in \mathrm{M}(\Omega)$, we have

$$\frac{t_2}{t_1}(t_1 u) = t_2 u \in \mathrm{M}(\Omega).$$

Noticing $t_1 u \in \mathrm{M}(\Omega)$, by Case 1, we obtain $t_1 = t_2$. This completes the proof of Lemma 2.4. □

**Lemma 2.5.** *Suppose* $1 < p < q < \min\{N, p^*\}$, $q \le \alpha \le p^* q'/p'$, $(\Phi 1)$–$(\Phi 5)$ *and* $(f_1)$–$(f_3)$ *hold and* $u \in \mathrm{M}(\Omega)$, $t \in (0, \infty)$ *and* $t \neq 1$, *then* $I_\Omega(tu) < I_\Omega(u)$.

*Proof.* Define a function in $(0,\infty)$ by $g(t) = I_\Omega(tu)$

$$g(t) = I_\Omega(tu) = \frac{1}{2}\int_\Omega \phi(t^2|\nabla u|^2)\,dx + \frac{t^\alpha}{\alpha}\int_\Omega |u|^\alpha\,dx - \int_\Omega F(tu)\,dx.$$

Then

$$g'(t) = \int_\Omega t\phi'(t^2|\nabla u|^2)|\nabla u|^2\,dx + t^{\alpha-1}\int_\Omega |u|^\alpha\,dx - \int_\Omega f(tu)u\,dx.$$

By the fact $u \in \mathrm{M}(\Omega)$, i.e.,

$$\int_\Omega \phi'(|\nabla u|^2)|\nabla u|^2\,dx + \int_\Omega |u|^\alpha\,dx - \int_\Omega f(u)u\,dx = 0,$$

using a similar argument to Lemma 2.4, we obtain $g'(t) > 0$ for $0 < t < 1$ and $g'(t) < 0$ for $t > 1$. Hence $g(t) < g(1)$, that is $I_\Omega(tu) < I_\Omega(u)$ for $t \in (0,\infty)$ and $t \neq 1$. □

Next we consider the following minimization problem

$$\tilde{c} = \inf_{\mathrm{M}(\Omega)} I_\Omega(u).$$

$\mathrm{M}(\Omega)$ is nonempty in $\mathcal{W}_r(\Omega)$ by Lemma 2.4. Here we denote

$$\|u\|_\Omega = \|u\|_{L^\alpha(\Omega)} + \|\nabla u\|_{L^p(\Omega)+L^q(\Omega)},$$

and

$$\Lambda_u = \{x \in \Omega : |u| > 1\}, \qquad \Lambda_u^c = \{x \in \Omega : |u| \leq 1\}.$$

**Lemma 2.6.** *Suppose* $1 < p < q < \min\{N, p^*\}$, $q \leq \alpha \leq p^*q'/p'$, $(\Phi 1)$–$(\Phi 5)$ *and* $(f_1)$–$(f_3)$ *hold, then* $\tilde{c}$ *can be achieved by some positive function* $\tilde{u}$ *which is a solution of equation* (2.1).

*Proof.* We use the minimization method. The proof can be divided into two steps.

**Step 1.** $\tilde{c}$ is attained. By the definition of $\tilde{c}$, there exists a sequence $\{\tilde{u}_n\} \subset \mathrm{M}(\Omega)$ such that

$$I_\Omega(\tilde{u}_n) = \tilde{c} + o(1), \qquad \gamma_\Omega(\tilde{u}_n) = 0,$$

i.e.,

$$I_\Omega(\tilde{u}_n) = \frac{1}{2}\int_\Omega \phi(|\nabla \tilde{u}_n|^2)\,dx + \frac{1}{\alpha}\int_\Omega |\tilde{u}_n|^\alpha\,dx - \int_\Omega F(\tilde{u}_n)\,dx = \tilde{c} + o(1),$$

$$\int_\Omega \phi'(|\nabla \tilde{u}_n|^2)|\nabla \tilde{u}_n|^2\,dx + \int_\Omega |\tilde{u}_n|^\alpha\,dx - \int_\Omega f(\tilde{u}_n)\tilde{u}_n\,dx = 0.$$

By the Proposition 2.2 of [1], we have

$$\|\tilde{u}_n\|_{L^p(\Omega)+L^q(\Omega)} \leq \max\left\{\|\tilde{u}_n\|_{L^p(\Lambda_{\tilde{u}_n})}, \|\tilde{u}_n\|_{L^q(\Lambda_{\tilde{u}_n}^c)}\right\}.$$

It follows from $(\Phi 4)$ that $\phi''(t)t < \frac{\theta-2}{2}\phi'(t)$ for all $t > 0$. Moreover, $\phi(0) = 0$, we see that $\phi'(t)t < \frac{\theta}{2}\phi(t)$. There exists $0 < \mu < 1$ such that

$$\phi'(t)t \leq \frac{\theta\mu}{2}\phi(t), \quad \text{for all } t \geq 0.$$

Thus, by $(\Phi 2)$ and the fact that $\tilde{u}_n \in L^p(\Lambda_{\tilde{u}_n}) \cap L^q(\Lambda^c_{\tilde{u}_n})$ (see Proposition 2.2 (iv) in [1]), we get

$$\begin{aligned}
\tilde{c} + o(1) &= I_\Omega(\tilde{u}_n) - \frac{1}{\theta}\langle I'_\Omega(\tilde{u}_n), \tilde{u}_n\rangle \\
&\geq \int_\Omega \left[\frac{1}{2}\phi(|\nabla \tilde{u}_n|^2) - \frac{1}{\theta}\phi'(|\nabla \tilde{u}_n|^2)|\nabla \tilde{u}_n|^2\right] \mathrm{d}x + \left(\frac{1}{\alpha} - \frac{1}{\theta}\right)\int_\Omega |\tilde{u}_n|^\alpha \,\mathrm{d}x \\
&\geq \frac{1-\mu}{2}\int_\Omega \phi(|\nabla \tilde{u}_n|^2)\,\mathrm{d}x + \left(\frac{1}{\alpha} - \frac{1}{\theta}\right)\int_\Omega |\tilde{u}_n|^\alpha \,\mathrm{d}x \\
&\geq C_1 \int_{\Lambda^c_{\nabla \tilde{u}_n}} |\nabla \tilde{u}_n|^q \,\mathrm{d}x + C_2 \int_{\Lambda_{\nabla \tilde{u}_n}} |\nabla \tilde{u}_n|^p \,\mathrm{d}x + \left(\frac{1}{\alpha} - \frac{1}{\theta}\right)\int_\Omega |\tilde{u}_n|^\alpha \,\mathrm{d}x \\
&\geq C\Big[\min\left\{\|\nabla \tilde{u}_n\|^q_{L^p(\Omega)+L^q(\Omega)}, \|\nabla \tilde{u}_n\|^p_{L^p(\Omega)+L^q(\Omega)}\right\} + \|\tilde{u}_n\|^\alpha_{L^\alpha(\Omega)}\Big] \\
&\geq C\|\tilde{u}_n\|^\alpha_\Omega.
\end{aligned} \tag{2.9}$$

Since $C > 0$, it is easy to verify $\{\tilde{u}_n\}$ is bounded in $\mathrm{M}(\Omega)$. Then by Proposition 2.5 of [1] and Lemma 2.1, there exists $\tilde{u} \in \mathcal{W}_r(\Omega)$ such that

$$\begin{aligned}
\tilde{u}_n &\rightharpoonup \tilde{u}, \quad \text{weakly in } \mathcal{W}_r(\Omega), \\
\tilde{u}_n &\to \tilde{u}, \quad \text{in } L^s(\Omega), \\
\tilde{u}_n &\to \tilde{u}, \quad \text{a.e. in } \Omega,
\end{aligned}$$

where $\alpha < s < p^*$. By Theorem A.2 in [34], we can deduce that

$$f(\tilde{u}_n)\tilde{u}_n \to f(\tilde{u})\tilde{u} \quad \text{in } L^1(\Omega).$$

Since $\gamma_\Omega(\tilde{u}_n) = 0$, we first prove $\tilde{u} \not\equiv 0$. In fact, by equation (2.3) , Lemma 2.1 and inequality (2.9), we have

$$C_\varepsilon\|\tilde{u}_n\|^s_\Omega + \varepsilon\|\tilde{u}_n\|^\alpha_\Omega \geq \int_\Omega f(\tilde{u}_n)\tilde{u}_n \,\mathrm{d}x = \int_\Omega \phi'(|\nabla \tilde{u}_n|^2)|\nabla \tilde{u}_n|^2 \,\mathrm{d}x + \int_\Omega |\tilde{u}_n|^\alpha \,\mathrm{d}x \geq C\|\tilde{u}_n\|^\alpha_\Omega. \tag{2.10}$$

Since $s > \alpha$, we have $\|\tilde{u}_n\|_\Omega \geq C_3 > 0$. Hence

$$\begin{aligned}
C_\varepsilon\|\tilde{u}\|^s_\Omega + \varepsilon\|\tilde{u}\|^\alpha_\Omega + o(1) \geq o(1) + \int_\Omega f(\tilde{u})\tilde{u}\,\mathrm{d}x &= \int_\Omega \phi'(|\nabla \tilde{u}_n|^2)|\nabla \tilde{u}_n|^2 \,\mathrm{d}x + \int_\Omega |\tilde{u}_n|^\alpha \,\mathrm{d}x \\
&\geq C\|\tilde{u}_n\|^\alpha_\Omega \geq C_3,
\end{aligned}$$

we get $\tilde{u} \not\equiv 0$.

According to Lemma 2.4, there exists a unique $\bar{t} > 0$ which satisfies $\gamma_\Omega(\bar{t}\tilde{u}) = 0$. Using the condition $(\Phi 5)$, then

$$\frac{1}{2}\int_\Omega \phi(\bar{t}^2|\nabla \tilde{u}|^2)\,\mathrm{d}x \leq \liminf_{n\to\infty} \frac{1}{2}\int_\Omega \phi(\bar{t}^2|\nabla \tilde{u}_n|^2)\,\mathrm{d}x.$$

By the definition of $\tilde{c}$ and equation (2), we have

$$\begin{aligned}
\tilde{c} \leq I_\Omega(\bar{t}\tilde{u}) &= \frac{1}{2}\int_\Omega \phi(\bar{t}^2|\nabla \tilde{u}|^2)\,\mathrm{d}x + \frac{\bar{t}^\alpha}{\alpha}\int_\Omega |\tilde{u}|^\alpha \,\mathrm{d}x - \int_\Omega F(\bar{t}\tilde{u})\,\mathrm{d}x \\
&\leq \liminf_{n\to\infty} \int_\Omega \left[\frac{1}{2}\phi(\bar{t}^2|\nabla \tilde{u}_n|^2) + \frac{\bar{t}^\alpha}{\alpha}|\tilde{u}_n|^\alpha - F(\bar{t}\tilde{u}_n)\right] \mathrm{d}x \\
&\leq \liminf_{n\to\infty} I_\Omega(\bar{t}\tilde{u}_n) \leq \liminf_{n\to\infty} I_\Omega(\tilde{u}_n) = \tilde{c}.
\end{aligned}$$

Thus we get

$$I_\Omega(\bar{t}\tilde{u}) = \tilde{c},$$

and $\tilde{c}$ is attained by $\bar{t}\tilde{u}$.

**Step 2.** In the following, we prove that $\bar{t}\tilde{u}$ is a radial solution of equation (2.1), which is similar to the Lemma 2.7 of [14]. For simplicity, we denote $\tilde{u}$ to $\bar{t}\tilde{u}$. Suppose $\tilde{u} \in \mathrm{M}(\Omega)$, $I_\Omega(\tilde{u}) = \tilde{c}$, but the conclusion of the lemma is not true. Then we can find a function $\varphi \in \mathcal{W}_r'(\mathbb{R}^N)$ such that

$$\langle I_\Omega'(\tilde{u}), \varphi\rangle = \int_\Omega \phi'(|\nabla \tilde{u}|^2)\nabla \tilde{u} \nabla \varphi \,\mathrm{d}x + \int_\Omega |\tilde{u}|^{\alpha-2}\tilde{u}\varphi \,\mathrm{d}x - \int_\Omega f(\tilde{u})\varphi \,\mathrm{d}x \le -1. \tag{2.11}$$

Choosing $\varepsilon > 0$ small enough such that

$$\left\langle I_\Omega'(t\tilde{u} + \sigma\varphi), \varphi\right\rangle \le -\frac{1}{2}, \qquad \forall\ |t-1| + |\sigma| \le \varepsilon.$$

Let $\eta$ be a cut-off function such that

$$\eta(t) = \begin{cases} 1, & |t-1| \le \frac{1}{2}\varepsilon, \\ 0, & |t-1| \ge \varepsilon. \end{cases}$$

We estimate

$$\sup_t I_\Omega(t\tilde{u} + \varepsilon\eta(t)\varphi).$$

If $|t-1| \le \varepsilon$, then

$$\begin{aligned} I_\Omega\left(t\tilde{u} + \varepsilon\eta(t)\varphi\right) &= I_\Omega(t\tilde{u}) + \int_0^1 \langle I_\Omega'(t\tilde{u} + \sigma\varepsilon\eta(t)\varphi), \varepsilon\eta(t)\varphi\rangle \,\mathrm{d}\sigma \\ &\le I_\Omega(t\tilde{u}) - \frac{1}{2}\varepsilon\eta(t). \end{aligned} \tag{2.12}$$

For $|t-1| \ge \varepsilon$, $\eta(t) = 0$, and the above estimate is trivial. Now, since $\tilde{u} \in \mathrm{M}(\Omega)$, for $t \ne 1$, we get $I_\Omega(t\tilde{u}) < I_\Omega(\tilde{u})$ by Lemma 2.5. Hence it follows from equation (2.12) that

$$I_\Omega\left(t\tilde{u} + \varepsilon\eta(t)\varphi\right) \le \begin{cases} I_\Omega(t\tilde{u}) < I_\Omega(\tilde{u}), & t \ne 1, \\ I_\Omega(\tilde{u}) - \frac{1}{2}\varepsilon\eta(1) = I_\Omega(\tilde{u}) - \frac{1}{2}\varepsilon, & t = 1. \end{cases}$$

In any case, we have $I_\Omega(t\tilde{u} + \varepsilon\eta(t)\varphi) < I_\Omega(\tilde{u}) = \tilde{c}$. In particular,

$$\sup_{0\le t\le 2} I_\Omega\left(t\tilde{u} + \varepsilon\eta(t)\varphi\right) < \tilde{c}.$$

Since $\tilde{u} \in \mathrm{M}(\Omega)$, we have

$$\int_\Omega \phi'(|\nabla \tilde{u}|^2)|\nabla \tilde{u}|^2 \,\mathrm{d}x + \int_\Omega |\tilde{u}|^\alpha \,\mathrm{d}x - \int_\Omega f(\tilde{u})\tilde{u} \,\mathrm{d}x = 0. \tag{2.13}$$

Let

$$\begin{aligned} h(t) = \int_\Omega \big[&\phi'(|\nabla(t\tilde{u} + \varepsilon\eta(t)\varphi)|^2)|\nabla(t\tilde{u} + \varepsilon\eta(t)\varphi)|^2 + |t\tilde{u} + \varepsilon\eta(t)\varphi|^\alpha \\ &- f(t\tilde{u} + \varepsilon\eta(t)\varphi)(t\tilde{u} + \varepsilon\eta(t)\varphi)\big] \,\mathrm{d}x. \end{aligned}$$

Without loss of generality, we assume $\varepsilon < \frac{1}{4}$. For $t = 2$, we have $\eta(2) = 0$, thus from (2.7)-(2.8) and (2.13)

$$\begin{aligned} h(2) &= \int_\Omega \left[4\phi'(4|\nabla\tilde{u}|^2)|\nabla\tilde{u}|^2 + 2^\alpha|\tilde{u}|^\alpha - f(2\tilde{u})2\tilde{u}\right] \mathrm{d}x \\ &= \int_\Omega \left[4\phi'(4|\nabla\tilde{u}|^2)|\nabla\tilde{u}|^2 - 2^\theta\phi'(|\nabla\tilde{u}|^2)|\nabla\tilde{u}|^2\right] \mathrm{d}x + \int_\Omega (2^\alpha - 2^\theta)|\tilde{u}|^\alpha \,\mathrm{d}x \\ &\qquad + \int_\Omega \left[2^\theta f(\tilde{u})\tilde{u} - f(2\tilde{u})2\tilde{u}\right] \mathrm{d}x \\ &< 0. \end{aligned}$$

For $t = \frac{1}{2}$, we have

$$\begin{aligned} h\left(\frac{1}{2}\right) &= \int_\Omega \left[\frac{1}{4}\phi'\left(\frac{1}{4}|\nabla\tilde{u}|^2\right)|\nabla\tilde{u}|^2 + \frac{1}{2^\alpha}|\tilde{u}|^\alpha - f\left(\frac{1}{2}\tilde{u}\right)\frac{1}{2}\tilde{u}\right] \mathrm{d}x \\ &= \int_\Omega \left[\frac{1}{4}\phi'\left(\frac{1}{4}|\nabla\tilde{u}|^2\right)|\nabla\tilde{u}|^2 - \frac{1}{2^\theta}\phi'(|\nabla\tilde{u}|^2)|\nabla\tilde{u}|^2\right] \mathrm{d}x + \int_\Omega \left(\frac{1}{2^\alpha} - \frac{1}{2^\theta}\right)|\tilde{u}|^\alpha \,\mathrm{d}x \\ &\quad + \int_\Omega \left[\frac{1}{2^\theta} f(\tilde{u})\tilde{u} - f\left(\frac{1}{2}\tilde{u}\right)\frac{1}{2}\tilde{u}\right] \mathrm{d}x \\ &> 0. \end{aligned}$$

Consequently, we can find $\tilde{t} \in (\frac{1}{2}, 2)$ such that $h(\tilde{t}) = 0$. It implies $\tilde{t}\tilde{u} + \varepsilon\eta(\tilde{t})\varphi \in \mathrm{M}(\Omega)$, which contradicts with (2.11). From this, $\tilde{u}$ is a solution for equation (2.1).

If $\alpha \geq q$, we infer that the solution $\tilde{u}$ is positive by Theorem 1 of [30]. Thus, we complete the proof. □

We shall show any $\mathcal{W}_r(\mathbb{R}^N)$-solution of the equation (1.7) is $C^{1,\gamma}_{loc}(\mathbb{R}^N)$-solution of the equation (1.7).

**Lemma 2.7.** *Assume $u$ be a weak solution of (1.7), $1 < p < q < min\{N, p^*\}$, $q \leq \alpha \leq p^*q'/p'$, $u \in \mathcal{W}_r(\mathbb{R}^N)$, $(\Phi1)$–$(\Phi5)$ and $(f_1)$–$(f_3)$ hold, then $u \in C^{1,\gamma}_{\mathrm{loc}}(\mathbb{R}^N)$ for some $0 < \gamma < 1$.*

*Proof.* We first prove by the Moser's iteration that $u \in L^\infty(\mathbb{R}^N)$, then belongs to $C^{1,\gamma}_{\mathrm{loc}}(\mathbb{R}^N)$. Since $u \in \mathcal{W}_r(\mathbb{R}^N)$, $u \in L^{p^*}(\mathbb{R}^N)$. For $r > 0$ to be determined later, taking $\varphi = |u^T|^{pr}u$ as a test function with

$$u^T = \begin{cases} T, & u > T, \\ u, & |u| \leq T, \\ -T, & u < -T. \end{cases}$$

Moreover, without any loss of generality, we shall assume that $T > 1$. Then $\nabla u \nabla\varphi = pr|u^T|^{pr}|\nabla u^T|^2 + |u^T|^{pr}|\nabla u|^2$, $u$ is a weak solution of equation (1.7), i.e.,

$$\int_{\mathbb{R}^N} \phi'(|\nabla u|^2)\nabla u \nabla\varphi \,\mathrm{d}x + \int_{\mathbb{R}^N} |u|^{\alpha-2}u\varphi \,\mathrm{d}x = \int_{\mathbb{R}^N} f(u)\varphi \,\mathrm{d}x.$$

We have

$$\begin{aligned} (pr+1)&\int_{|u|\leq T} \phi'(|\nabla u|^2)|\nabla u|^2|u^T|^{pr} \,\mathrm{d}x \\ &+ \int_{|u|>T} \phi'(|\nabla u|^2)|\nabla u|^2|u^T|^{pr} \,\mathrm{d}x + \int_{\mathbb{R}^N} |u|^\alpha|u^T|^{pr} \,\mathrm{d}x = \int_{\mathbb{R}^N} f(u)u|u^T|^{pr} \,\mathrm{d}x. \end{aligned}$$

Define $A = \{x \in \mathbb{R}^N : |u| \le T\} \cap \Lambda^c_{\nabla u}$ and $B = \{x \in \mathbb{R}^N : |u| \le T\} \cap \Lambda_{\nabla u}$, then

$$\begin{aligned}
\int_{\mathbb{R}^N} f(u)u|u^T|^{pr}\,\mathrm{d}x &\ge (pr+1)\int_{|u|\le T}\phi'(|\nabla u|^2)|\nabla u|^2|u^T|^{pr}\,\mathrm{d}x + \int_{|u|\le T}|u|^\alpha|u^T|^{pr}\,\mathrm{d}x\\
&\ge C(1+r)^{1-p}\min\Big\{\int_A|\nabla|u|^{1+r}|^p\,\mathrm{d}x, \int_B|\nabla|u|^{1+r}|^q\,\mathrm{d}x\Big\}\\
&\quad+\frac{1}{T^{(\alpha-p)r}}\int_{|u|\le T}||u|^{1+r}|^\alpha\,\mathrm{d}x\\
&\ge C(1+r)^{1-p}\Big[\|\nabla|u|^{1+r}\|^q_{L^p(|u|\le T)+L^q(|u|\le T)} + \||u|^{1+r}\|^\alpha_{L^\alpha(|u|\le T)}\Big]\\
&\ge C(1+r)^{1-p}\||u|^{1+r}\|^p_{L^{p^*}(|u|\le T)}\\
&\ge C(1+r)^{1-p}\Big(\int_{|u|\le T}|u|^{(1+r)p^*}\,\mathrm{d}x\Big)^{\frac{p}{p^*}}.
\end{aligned}$$

Set $d = 1+r = \frac{Np-(N-p)(s-p)}{(N-p)p} > 1$, $s \in (\alpha, p^*)$. Let $T \to +\infty$, by equation (2.3) and Hölder inequality, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} f(u)u|u^T|^{pr}\,\mathrm{d}x &\le C_\varepsilon\int_{\mathbb{R}^N}|u|^{s-p}|u|^{pr+p}\,\mathrm{d}x + \varepsilon\int_{\mathbb{R}^N}|u|^{\alpha-p}|u|^{pr+p}\,\mathrm{d}x\\
&\le C_\varepsilon\Big(\int_{\mathbb{R}^N}|u|^{p^*}\,\mathrm{d}x\Big)^{\frac{p^*-pd}{p^*}}\Big(\int_{\mathbb{R}^N}|u|^{p^*}\,\mathrm{d}x\Big)^{\frac{pd}{p^*}}\\
&\quad+\varepsilon\Big(\int_{\mathbb{R}^N}|u|^{\bar\alpha}\,\mathrm{d}x\Big)^{\frac{p^*-pd}{p^*}}\Big(\int_{\mathbb{R}^N}|u|^{p^*}\,\mathrm{d}x\Big)^{\frac{pd}{p^*}}\\
&\le C\Big(\int_{\mathbb{R}^N}|u|^{p^*}\,\mathrm{d}x\Big)^{\frac{pd}{p^*}},
\end{aligned}$$

where $\alpha < \bar\alpha = \frac{(\alpha-p)(Np)}{(N-p)(s-p)} < p^*$. Then we get

$$\Big(\int_{\mathbb{R}^N}|u|^{p^*d}\,\mathrm{d}x\Big)^{\frac{p}{p^*}} \le C(1+r)^{p-1}\int_{\mathbb{R}^N} f(u)u|u^T|^{pr}\,\mathrm{d}x \le C(1+r)^{p-1}\Big(\int_{\mathbb{R}^N}|u|^{p^*}\,\mathrm{d}x\Big)^{\frac{pd}{p^*}}.$$

Hence

$$\Big(\int_{\mathbb{R}^N}|u|^{p^*d}\,\mathrm{d}x\Big)^{\frac{1}{p^*d}} \le C(1+r)^{\frac{p-1}{pd}}\Big(\int_{\mathbb{R}^N}|u|^{p^*}\,\mathrm{d}x\Big)^{\frac{1}{p^*}}.$$

Therefore

$$\Big(\int_{\mathbb{R}^N}|u|^{p^*d^k}\,\mathrm{d}x\Big)^{\frac{1}{p^*d^k}} \le (\Pi_{i=1}^k Cd^i)^{\frac{1}{d^i}}\Big(\int_{\mathbb{R}^N}|u|^{p^*}\,\mathrm{d}x\Big)^{\frac{1}{p^*}}.$$

Since $\Pi_{i=1}^\infty(Cd^i)^{\frac{1}{d^i}} \le C^*$ for some constant $C^* > 0$, we then deduce that $u \in L^\infty(\mathbb{R}^N)$. Suppose $u$ is a weak solution of the equation (1.7) and $u \in \mathcal{W}_r(\mathbb{R}^N)$, we have that $u \in C^{1,\gamma}_{loc}(\mathbb{R}^N)$ for some $\gamma > 0$ by Chapter 4 of [19] or [33]. □

## 3 Existence of sign-changing solutions

In this section, we construct infinitely many nodal solutions for equation (1.7). For any given $k$ numbers $r_j$ $(j = 1, \dots, k)$ such that $0 < r_1 < r_2 < \cdots < r_k < +\infty$, we denote $r_0 = 0, r_{k+1} = \infty$,

$$\Omega^1 = \{x \in \mathbb{R}^N : |x| < r_1\} \quad\text{and}\quad \Omega^j = \{x \in \mathbb{R}^N : r_{j-1} < |x| < r_j\}.$$

We will always extend $u_j \in \mathcal{W}_r(\Omega^j)$ to $\mathcal{W}_r(\mathbb{R}^N)$ by setting $u_j \equiv 0$ for $x \in \mathbb{R}^N \backslash \Omega^j$ for every $u_j$, $j = 1, 2, \ldots, k+1$. For convenience, we use $I(u_j)$ to replace $I_{\Omega^j}(u_j)$ and $\gamma(u_j)$ to replace $\gamma_{\Omega^j}(u_j)$. Define

$$\begin{aligned} Y_k^{\pm}(r_1, r_2, \ldots, r_{k+1}) = \Big\{ u \in \mathcal{W}_r(\mathbb{R}^N) \mid u = \pm \sum_{j=1}^{k+1} (-1)^{j-1} u_j,\ u_j \geq 0, \\ u_j \not\equiv 0,\ u_j \in \mathcal{W}_r(\Omega^j),\ j = 1, 2, \ldots, k+1 \Big\}, \end{aligned}$$

$$\begin{aligned} \mathrm{M}_k^{\pm} = \big\{ u \in \mathcal{W}_r(\mathbb{R}^N) \mid \exists\, 0 = r_0 < r_1 < r_2 < \cdots < r_k < r_{k+1} = +\infty, \\ \text{such that } u \in Y_k^{\pm}(r_1, r_2, \ldots, r_{k+1}) \text{ and } u_j \in \mathrm{M}(\Omega^j),\ j = 1, 2, \ldots, k+1 \big\}. \end{aligned}$$

Note that $\mathrm{M}_k^{\pm} \neq \varnothing$, $k = 1, 2, \ldots$ In order to prove the existence of non-negative critical points of energy functional $I$, similar to [6] or [10], we only need to extend $f(u)$ as follows

$$f^+(u) := \begin{cases} f(u), & \text{if } u \geq 0, \\ -f(-u), & \text{if } u < 0, \end{cases}$$

thus the oddness assumption on nonlinear term is actually unnecessary. The function $I^+(u)$ is defined on $\mathcal{W}_r(\mathbb{R}^N)$ by

$$I^+(u) = \frac{1}{2} \int_{\mathbb{R}^N} \phi(|\nabla u|^2)\, \mathrm{d}x + \frac{1}{\alpha} \int_{\mathbb{R}^N} |u|^{\alpha}\, \mathrm{d}x - \int_{\mathbb{R}^N} F^+(u)\, \mathrm{d}x,$$

$c_k^+ = \inf_{u \in \mathrm{M}_k^+} I^+(u)$ in the same way as those in [10]. For $\mathrm{M}_k^-$, we can complete the proof in the same way. By the arguments of the Section 2, it is not difficult to verify that

$$\forall\, u = \sum_{j=1}^{k+1} (-1)^{j-1} u_j \in \mathrm{M}_k^+ \Leftrightarrow I(u) = \max_{\substack{\alpha_j > 0 \\ 1 \leq j \leq k+1}} I\Big( \sum_{j=1}^{k+1} \alpha_j \breve{u}_j \Big),$$

where $\breve{u}_j = (-1)^{j-1} u_j$.

Set

$$c_k = \inf_{u \in \mathrm{M}_k^+} I(u), \qquad k = 1, 2, \ldots$$

**Lemma 3.1.** *$c_k$ is attained provided that $1 < p < q < \min\{N, p^*\}$, $q \leq \alpha \leq p^* q'/p'$, $(\Phi 1)$–$(\Phi 5)$ and $(f_1)$–$(f_3)$ hold.*

*Proof.* We prove by induction that for each $k$ there exists $\bar{u}_k \in \mathrm{M}_k^+$ such that

$$I(\bar{u}_k) = c_k.$$

For $k = 0$ or $\Omega = \mathbb{R}^N$, we can directly derive from Lemma 2.6. We discuss the case $k \geq 1$ in the following.

First, we prove $I$ is bounded from below on $\mathrm{M}_k^+$ by a positive constant. Let $\bar{u} \in \mathrm{M}_k^+$, then $\bar{u} = \sum_{j=1}^{k+1} (-1)^{j-1} \bar{u}_j$ and $\bar{u}_j \in \mathrm{M}(\Omega^j)$, $j = 1, 2, \ldots, k+1$. By the similar arguments of inequality (2.10), we have $\|\bar{u}_j\|_{\Omega^j} \geq C_j$. It follows from the same computations in (2.9) that

$$I(\bar{u}) = I\Big( \sum_{j=1}^{k+1} (-1)^{j-1} \bar{u}_j \Big) = \sum_{j=1}^{k+1} I(\bar{u}_j) \geq C \sum_{j=1}^{k+1} \|\bar{u}_j\|_{\Omega_j}^{\alpha} \geq C \sum_{j=1}^{k+1} C_j^{\alpha} = \bar{C}. \tag{3.1}$$

There exists a positive constant $\bar{C} > 0$ such that $I(\bar{u}) \geq \bar{C}$, for all $\bar{u} \in \mathrm{M}_k^+$.

Second, we suppose the conclusion is true for $k-1$ and let $\{\bar{u}_m\}_{m\geq 1}$ be a minimizing sequence of $c_k$ in $\mathrm{M}_k^+$, that is

$$\lim_{m\to\infty} I(\bar{u}_m) = c_k, \qquad \bar{u}_m \in \mathrm{M}_k^+, \qquad m = 1,2,\ldots$$

$\bar{u}_m$ corresponding to $k$ nodes, $r_m^1, r_m^2, \ldots, r_m^k$, with $0 < r_m^1 < r_m^2 < \cdots < r_m^k < \infty$, set

$$\Omega_m^i = \{x \in \mathbb{R}^N : r_m^{i-1} < |x| < r_m^i\},$$

and

$$\bar{u}_m^i = \begin{cases} \bar{u}_m, & \text{if } x \in \Omega_m^i, \\ 0, & \text{if } x \notin \Omega_m^i. \end{cases}$$

We can select a subsequence $\{r_m^i\}$ such that $\lim_{m\to\infty} r_m^i = r_i$, and $0 \leq r_1 \leq r_2 \leq \cdots \leq r_k \leq +\infty$.

Now we give the following claims.

**Claim 1:** Under the assumptions of Lemma 3.1, $r_i \neq r_{i-1}$, $i = 1,2,\ldots,k$. Here we denote $r_0 = 0$.

If $r_i = r_{i-1}$ for some $i \in \{1,\ldots,k\}$. Suppose there exists $i_0 \in \{1,\ldots,k\}$ such that $r_{i_0} = r_{i_0-1}$, then $\lim_{m\to\infty} r_m^{i_0} = \lim_{m\to\infty} r_m^{i_0-1}$. We denote the measure of $\Omega_m^{i_0}$ by $|\Omega_m^{i_0}|$, so that $|\Omega_m^{i_0}| \to 0$ as $m \to \infty$. Since $\bar{u}_m^{i_0} \in \mathrm{M}(\Omega_m^{i_0})$, by Proposition 2.2 of [1] and Lemma 2.1, we have

$$\begin{aligned}
\|\bar{u}_m^{i_0}\|_{\Omega_m^{i_0}}^\alpha &\leq C\left\{ \|\nabla \bar{u}_m^{i_0}\|^q_{L^p(\Omega_m^{i_0})+L^q(\Omega_m^{i_0})} + \|\bar{u}_m^{i_0}\|^\alpha_{\Omega_m^{i_0}} \right\} \\
&\leq C\left\{ \max\left\{ \int_{\{x\in\Omega_m^{i_0}:|\nabla\bar{u}_m^{i_0}|\leq 1\}} |\nabla \bar{u}_m^{i_0}|^q \,\mathrm{d}x, \int_{\{x\in\Omega_m^{i_0}:|\nabla\bar{u}_m^{i_0}|> 1\}} |\nabla \bar{u}_m^{i_0}|^p \,\mathrm{d}x \right\} + \int_{\Omega_m^{i_0}} |\bar{u}_m^{i_0}|^\alpha \,\mathrm{d}x \right\} \\
&\leq C\left\{ \int_{\Omega_m^{i_0}} \phi'(|\nabla \bar{u}_m^{i_0}|^2)|\nabla \bar{u}_m^{i_0}|^2 \,\mathrm{d}x + \int_{\Omega_m^{i_0}} |\bar{u}_m^{i_0}|^\alpha \,\mathrm{d}x \right\} \\
&\leq C \int_{\Omega_m^{i_0}} f(\bar{u}_m^{i_0})\bar{u}_m^{i_0} \,\mathrm{d}x \\
&\leq C_\varepsilon \int_{\Omega_m^{i_0}} |\bar{u}_m^{i_0}|^s \,\mathrm{d}x + \varepsilon \int_{\Omega_m^{i_0}} |\bar{u}_m^{i_0}|^\alpha \,\mathrm{d}x \\
&\leq C_\varepsilon \int_{\Omega_m^{i_0}} |\bar{u}_m^{i_0}|^s \,\mathrm{d}x + \varepsilon \|\bar{u}_m^{i_0}\|^\alpha.
\end{aligned}$$

Let $\varepsilon = \frac{1}{2}$, then

$$\|\bar{u}_m^{i_0}\|^\alpha_{\Omega_m^{i_0}} \leq C\left(\int_{\Omega_m^{i_0}} |\bar{u}_m^{i_0}|^{p^*} dx\right)^{\frac{s}{p^*}} |\Omega_m^{i_0}|^{1-\frac{s}{p^*}} \leq C\|\bar{u}_m^{i_0}\|^s_{\Omega_m^{i_0}} |\Omega_m^{i_0}|^{1-\frac{s}{p^*}}.$$

Since $C$ is positive constants and $\alpha < s < p^*$, we deduce that

$$\|\bar{u}_m^{i_0}\|_{\Omega_m^{i_0}} \to \infty, \quad \text{as } m \to \infty.$$

By inequality (3.1),

$$I(\bar{u}_m^{i_0}) \to \infty, \quad \text{as } m \to \infty. \tag{3.2}$$

From the inductive assumption and equation (3.2), for $\varepsilon > 0$ fixed we can choose $L > 0$ such that

$$I(\bar{u}_m^{i_0}) > c_k - c_{k-1} + \varepsilon, \qquad |I(\bar{u}_m) - c_k| < \varepsilon, \quad \text{as } m \geq L.$$

Then we define $\bar{v}(x) \in \mathrm{M}_{k-1}^{+}$ by

$$\bar{v}(x) = \begin{cases} \bar{u}_m^l(x), & \text{if } x \in \Omega_m^l \text{ as } l < i, \\ 0, & \text{if } x \in \Omega_m^{i_0}, \\ \bar{u}_m^l(x), & \text{if } x \in \Omega_m^l \text{ as } l > i. \end{cases}$$

Hence

$$I(\bar{v}(x)) = I(\bar{u}_m) - I(\bar{u}_m^{i_0}) < c_k + \varepsilon - (c_k - c_{k-1} + \varepsilon) = c_{k-1}, \quad \text{as } m \geq L,$$

which contradicts with $c_{k-1} = \inf_{u \in \mathrm{M}_{k-1}^{+}} I(u)$. Thus $r_i \neq r_{i-1}$, $i = 1, 2, \ldots, k$. Then the proof of Claim 1 is completed.

**Claim 2:** Under the assumptions of Lemma 3.1, $r^k < \infty$.

If $r^k \to \infty$, then $r_m^k \to \infty$. It follows from Claim 1 and $\bar{u}_m^k \in \mathrm{M}(\Omega_m^k)$ that

$$\begin{aligned} \|\bar{u}_m^k\|_{\Omega_m^k}^{\alpha} &\leq C\bigg\{ \max\bigg\{ \int_{\{x \in \Omega_m^k : |\nabla \bar{u}_m^k| \leq 1\}} |\nabla \bar{u}_m^k|^q \,\mathrm{d}x, \int_{\{x \in \Omega_m^k : |\nabla \bar{u}_m^k| > 1\}} |\nabla \bar{u}_m^k|^p \,\mathrm{d}x \bigg\} + \int_{\Omega_m^k} |\bar{u}_m^k|^{\alpha} \,\mathrm{d}x \bigg\} \\ &\leq C\bigg\{ \int_{\Omega_m^k} \phi'(|\nabla \bar{u}_m^k|^2) |\nabla \bar{u}_m^k|^2 \,\mathrm{d}x + \int_{\Omega_m^k} |\bar{u}_m^k|^{\alpha} \,\mathrm{d}x \bigg\} \\ &\leq C_{\varepsilon} \int_{\Omega_m^k} |\bar{u}_m^k|^s \,\mathrm{d}x + \varepsilon \int_{\Omega_m^k} |\bar{u}_m^k|^{\alpha} \,\mathrm{d}x. \end{aligned}$$

Using Lemma 2.3, we deduce that

$$\|\bar{u}_m^k\|_{\Omega_m^k}^{\alpha} \leq C \|\bar{u}_m^k\|_{\Omega_m^k}^{s-\alpha} \int_{\Omega_m^k} |\bar{u}_m^k|^{\alpha} |x|^{\frac{(q-N)(s-\alpha)}{q}} \,\mathrm{d}x \leq C \|\bar{u}_m^k\|_{\Omega_m^k}^{s} |r_m^k|^{\frac{(q-N)(s-\alpha)}{q}},$$

so $\|\bar{u}_m^k\|_{\Omega_m^k} \geq C |r_m^k|^{\frac{N-q}{q}}$. By inequality (3.1) we find

$$I(\bar{u}_m^k) \to \infty, \quad \text{as } m \to \infty. \tag{3.3}$$

Similar to the proof of Claim 1, we can obtain $r^k < \infty$. Claim 2 is therefore proved.

From the above two claims, by selecting a subsequence, we may assume that $\lim_{m \to \infty} r_m^i = r^i$, and clearly $0 < r^1 < r^2 < \cdots < r^k < \infty$. Define $\Omega^i = \{x \in \mathbb{R}^N \mid r^{i-1} < |x| < r^i\}$, for all $i = 1, 2, \ldots, k+1$, $r^0 = 0$, $r^{k+1} = +\infty$. Lemma 2.6 implies that $\bar{c} = \inf_{u \in \mathrm{M}(\Omega^i)} I(u)$ is attained by some positive function $\hat{u}^i$ which satisfies the following boundary value problem

$$\begin{cases} -\nabla \cdot [\phi'(|\nabla u|^2) \nabla u] + |u|^{\alpha-2} u = f(u), & x \in \Omega^i, \\ u|_{\partial \Omega^i} = 0. \end{cases}$$

Define $\bar{u}_k = \sum_{i=1}^{k+1} (-1)^{i-1} \hat{u}^i(x)$, $(\hat{u}^i = 0, x \notin \Omega^i)$. Thus $\bar{u}_k \in \mathrm{M}_k^{+}$.

We define functions $v_m^i : [r^{i-1}, r^i] \to \mathbb{R}$ such that

$$\begin{cases} v_m^i := a_m^i \bar{u}_m^i \left( \frac{r_m^i - r_m^{i-1}}{r^i - r^{i-1}} (t - r^{i-1}) + r_m^{i-1} \right), & \text{for } i = 1, \ldots, k, \\ v_m^{k+1} := a_m^{k+1} \bar{u}_m^{k+1} \left( \frac{r_m^k}{r^k} t \right), \end{cases}$$

where $r_m^0 = 0$, $r_m^{k+1} = \infty$ and $a_m^i$ is a unique positive real number such that $v_m^i \in \mathrm{M}(\Omega^i)$, for all $i = 1, 2, \ldots, k+1$. For $m$ large enough, we can compute that

$$\int_{\Omega^i} \phi'(|\nabla v_m^i|^2) |\nabla v_m^i|^2 \,\mathrm{d}t = \int_{\Omega_m^i} \phi'(|a_m^i|^2 |\nabla \bar{u}_m^i|^2) |a_m^i|^2 |\nabla \bar{u}_m^i|^2 \,\mathrm{d}x + o(1),$$

$$\int_{\Omega^i} |v_m^i|^\alpha \,\mathrm{d}t = |a_m^i|^\alpha \int_{\Omega_m^i} |\bar{u}_m^i|^\alpha \,\mathrm{d}x + o(1),$$

$$\int_{\Omega^i} f(v_m^i) v_m^i \,\mathrm{d}t = \int_{\Omega_m^i} f(a_m^i \bar{u}_m^i) a_m^i \bar{u}_m^i \,\mathrm{d}x + o(1).$$

Since $v_m^i \in \mathrm{M}(\Omega^i)$, it follows

$$\int_{\Omega_m^i} \phi'(|a_m^i|^2 |\nabla \bar{u}_m^i|^2) |a_m^i|^2 |\nabla \bar{u}_m^i|^2 \,\mathrm{d}x + |a_m^i|^\alpha \int_{\Omega_m^i} |\bar{u}_m^i|^\alpha \,\mathrm{d}x - \int_{\Omega_m^i} f(a_m^i \bar{u}_m^i) a_m^i \bar{u}_m^i \,\mathrm{d}x = o(1), \quad (3.4)$$

for all $i = 1, 2, \dots, k+1$. Note that it also holds

$$\int_{\Omega_m^i} \phi'(|\nabla \bar{u}_m^i|^2) |\nabla \bar{u}_m^i|^2 \,\mathrm{d}x + \int_{\Omega_m^i} |\bar{u}_m^i|^\alpha \,\mathrm{d}x - \int_{\Omega_m^i} f(\bar{u}_m^i) \bar{u}_m^i \,\mathrm{d}x = 0, \quad (3.5)$$

for each $i$. Using an argument similar to that in the proof of Lemma 2.4, by (3.4) and (3.5), we can obtain that $\lim_{m\to\infty} a_m^i = 1$ for all $i$. Therefore we deduce that

$$\lim_{m\to\infty} I\left(a_m^i \bar{u}_m^i(x)\right) = \lim_{m\to\infty} I\left(\bar{u}_m^i(x)\right).$$

On the other hand, since $I(\hat{u}^i) = \inf_{u \in \mathrm{M}(\Omega^i)} I(u)$ and $a_m^i \bar{u}_m^i(x) \in \mathrm{M}(\Omega^i)$, we have

$$I(\hat{u}^i) \le I\left(a_m^i \bar{u}_m^i(x)\right).$$

Thus

$$\lim_{m\to\infty} I\left(\bar{u}_m^i(x)\right) \ge I(\hat{u}^i),$$

and

$$c_k = \lim_{m\to\infty} I(\bar{u}_m(x)) = \lim_{m\to\infty} \sum_{i=1}^{k+1} I(\bar{u}_m^i(x)) \ge \sum_{i=1}^{k+1} I(\hat{u}^i) = I(\bar{u}_k).$$

Since $\bar{u}_k \in \mathrm{M}_k^+$, which means that $c_k$ is attained. □

Now, we begin to prove Theorem 1.3. Because the weak solutions of (1.7) are of class $C_{loc}^{1,\gamma}(\mathbb{R}^N)$, as stated in Lemma 2.7. We apply some ideas of in [21,22,35] to prove the minimizer of $c_k$ is the weak solution of (1.7) instead of glue the function in each annuli by matching the normal derivative at each junction point.

*Proof of Theorem 1.1.* By Lemma 3.1, there exists $\bar{u}_k \in \mathrm{M}_k^+$ which attains $c_k$. Thus we get $k$ nodes:

$$r_1, r_2, \cdots, r_k, \qquad 0 < r_1 < r_2 < \cdots < r_k < +\infty, \qquad \Omega^i = \{x \in \mathbb{R}^N : r_{i-1} < |x| < r_i\}$$

and

$$(\bar{u}_k)^i = \begin{cases} \bar{u}_k(x), & x \in \Omega^i, \\ 0, & x \notin \Omega^i. \end{cases}$$

For convenience, $u := \bar{u}_k$, and $u$ satisfies equation (1.7) in $\{x \in \mathbb{R}^N : |x| \ne r_i, i = 1, 2, \dots, k\}$.

In order to show that $u$ is a critical point of $I$. We assume by contradiction that there exists $\psi \in \mathcal{W}_r'(\mathbb{R}^N)$ such that

$$\langle I'(u), \psi \rangle = -2.$$

Similarly to the proof of Step 2 in Lemma 2.6 we choose $\delta \in (0,1)$ such that if $s = (s_1, s_2, \ldots, s_{k+1}) \in D$ and $0 \le \epsilon \le \delta$, then

$$\left\langle I'\left(\sum_{i=1}^{k+1} s_i u^i + \epsilon\psi\right), \psi\right\rangle < -1,$$

where

$$D = \{(s_1, \ldots, s_{k+1}) \in \mathbb{R}^{k+1} \; : \; |s_i - 1| \le \delta, \text{ for all } i \in \{1, \ldots, k+1\}\}.$$

There is a sufficiently small $\epsilon$ such that $\sum_{i=1}^{k+1} s_i u^i + \epsilon\psi$ changes sign exactly $k$ times with $k$ nodes $0 < r_1(s,\epsilon) < \cdots < r_k(s,\epsilon) < \infty$. Here $r_j(s,\epsilon)$ denotes that $r_j$ depends on $s, \epsilon$ for all $j = 1, \cdots, k$. Let $\eta \in C_0^\infty(\mathbb{R}^N)$ be a cut-off function which satisfies $\eta(s) = 0$ in a neighborhood of $\partial D$, $\eta(1, \ldots, 1) = 1$ and $0 \le \eta(s) \le 1$ for all $s \in D$. If $\delta$ is small enough, we see that $\sum_{i=1}^{k+1} s_i u^i + \delta\eta(s)\psi$ also has exactly $k$ nodes $0 < r_1(s) < \cdots < r_k(s) < \infty$ for all $s \in D$, $r_j(s)$ is continuous about $s$ for every $j = 1, \ldots, k$, and

$$\left\langle I'\left(\sum_{i=1}^{k+1} s_i u^i + \delta\eta(s)\psi\right), \psi\right\rangle < -1. \tag{3.6}$$

We claim that there exists $s \in D$ such that $\sum_{i=1}^{k+1} s_i u^i + \delta\eta(s)\psi \in \mathrm{M}_k^+$. Let

$$H_i(s) = \int_{\mathbb{R}^N} \left[\phi'(|\nabla g_i(s)|^2)|\nabla g_i(s)|^2 + |g_i(s)|^\alpha - f(g_i(s))g_i(s)\right] \mathrm{d}x, \qquad \forall\, 1 \le i \le k+1,$$

and

$$g_i(s) = \left(\sum_{i=1}^{k+1} s_i u^i + \delta\eta(s)\psi\right)\Big|_{\Omega_s^i},$$

where $\Omega_s^i = \{x \in \mathbb{R}^N : r_{i-1}(s) < |x| < r_i(s)\}$ for all $1 \le i \le k+1$, $r_0(s) = 0$ and $r_{k+1}(s) = \infty$. Suppose that $s \in \partial D$, then $\eta(s) = 0$, $g_i(s) = s_i u^i$. For $s_i = 1 + \delta$, by (2.7)–(2.8), we have

$$\begin{aligned}
H_i(1+\delta) &= \int_{\Omega^i} \left[(1+\delta)^2\phi'((1+\delta)^2|\nabla u^i|^2)|\nabla u^i|^2 + (1+\delta)^\alpha|u^i|^\alpha - f((1+\delta)u^i)(1+\delta)u^i\right] \mathrm{d}x \\
&= \int_{\Omega^i} \left[(1+\delta)^2\phi'((1+\delta)^2|\nabla u^i|^2)|\nabla u^i|^2 - (1+\delta)^\theta\phi'(|\nabla u^i|^2)|\nabla u^i|^2\right] \mathrm{d}x \\
&\quad + \int_{\Omega^i}((1+\delta)^\alpha - (1+\delta)^\theta)|u^i|^\alpha \,\mathrm{d}x + \int_{\Omega^i} \left[(1+\delta)^\theta f(u^i)u^i - f((1+\delta)u^i)(1+\delta)u^i\right] \mathrm{d}x \\
&< 0.
\end{aligned}$$

For $s_i = 1 - \delta$, we get

$$\begin{aligned}
H_i(1-\delta) &= \int_{\Omega^i} \left[(1-\delta)^2\phi'((1-\delta)^2|\nabla u^i|^2)|\nabla u^i|^2 + (1-\delta)^\alpha|u^i|^\alpha - f((1-\delta)u^i)(1-\delta)u^i\right] \mathrm{d}x \\
&= \int_{\Omega^i} \left[(1-\delta)^2\phi'((1-\delta)^2|\nabla u^i|^2)|\nabla u^i|^2 - (1-\delta)^\theta\phi'(|\nabla u^i|^2)|\nabla u^i|^2\right] \mathrm{d}x \\
&\quad + \int_{\Omega^i}((1-\delta)^\alpha - (1-\delta)^\theta)|u^i|^\alpha \,\mathrm{d}x + \int_{\Omega^i} \left[(1-\delta)^\theta f(u^i)u^i - f((1-\delta)u^i)(1-\delta)u^i\right] \mathrm{d}x \\
&> 0.
\end{aligned}$$

By the homotopy invariance of the topological degree (or Miranda's Theorem [23]), we see that there exists $s \in D$ such that $H_i(s) = 0$ for all $1 \le i \le k+1$. That is $\sum_{i=1}^{k+1} s_i u^i + \delta\eta(s)\psi \in \mathrm{M}_k^+$.

From the claim, we get $I\big(\sum_{i=1}^{k+1} s_i u^i + \delta\eta(s)\psi\big) \geq c_k$. On the other hand, by (3.6), there holds that

$$I\bigg(\sum_{i=1}^{k+1} s_i u^i + \delta\eta(s)\psi\bigg) = I\bigg(\sum_{i=1}^{k+1} s_i u^i\bigg) + \int_0^1 \bigg\langle I'\bigg(\sum_{i=1}^{k+1} s_i u^i + \sigma\delta\eta(s)\psi\bigg), \delta\eta(s)\psi\bigg\rangle \,\mathrm{d}\sigma$$
$$\leq I\bigg(\sum_{i=1}^{k+1} s_i u^i\bigg) - \delta\eta(s).$$

If $s_i = 1$ for all $1 \leq i \leq k+1$, then we have

$$c_k \leq I\bigg(\sum_{i=1}^{k+1} u^i\bigg) - \delta\eta(1,\dots,1) = c_k - \delta,$$

which is impossible. If $s_i \neq 1$ for some $1 \leq i \leq k+1$, then we obtain

$$c_k \leq I\bigg(\sum_{i=1}^{k+1} s_i u^i\bigg) = \sum_{i=1}^{k+1} \int_{\Omega_i} \bigg[\phi'\Big(s_i^2|\nabla u^i|^2\Big) s_i^2 |\nabla u^i|^2 + s_i^\alpha |u^i|^\alpha - f(s_i u^i)(s_i u^i)\bigg] \,\mathrm{d}x$$
$$= \sum_{i=1}^{k+1} I_{\Omega_i}(s_i u^i) < \sum_{i=1}^{k+1} I_{\Omega_i}(u^i) = I\bigg(\sum_{i=1}^{k+1} u^i\bigg) = c_k,$$

which is also a contradiction.

Therefore, the function $u$ is indeed a radial solution of (1.7), which changes sign exactly $k$ times. We complete the proof. □

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