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Stability results for the functional differential equations associated to water hammer in hydraulics

Dedicated to László Hatvani, outstanding scholar and long life friend

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Abstract. We consider a system of two sets of partial differential equations describing the water hammer in a hydroelectric power plant containing the dynamics of the tunnel, turbine penstock, surge tank and hydraulic turbine. Under standard simplifying assumptions (negligible Darcy–Weisbach losses and dynamic head variations), a system of functional differential equations of neutral type, with two delays, can be associated to the aforementioned partial differential equations and existence, uniqueness and continuous data dependence can be established. Stability is then discussed using a Lyapunov functional deduced from the energy identity. The Lyapunov functional is "weak" i.e. its derivative function is only non-positive definite. Therefore only Lyapunov stability is obtained while for asymptotic stability application of the Barbashin–Krasovskii–LaSalle invariance principle is required. A necessary condition for its validity is the asymptotic stability of the difference operator associated to the neutral system. However, its properties in the given case make the asymptotic stability *non-robust* (fragile) in function of some arithmetic properties of the delay ratio.
Keywords: differential equations, neutral functional differential equations, energy Lyapunov functional, asymptotic stability, water hammer.

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1 The engineering model and problem statement

The transient processes of the hydraulic power plants are quite important since, if uncontrolled, they can produce technical and/or environment catastrophes. Consequently their theoretical analysis is of major concern: we can cite but a few references on this subject [3, 21, 24, 27].

We shall consider here, as in other papers of ours, the (relatively) standard structure of a hydroelectric power plant consisting of a water reservoir ("lake"), tunnel, penstock and hydraulic turbine (Fig. 1.1)

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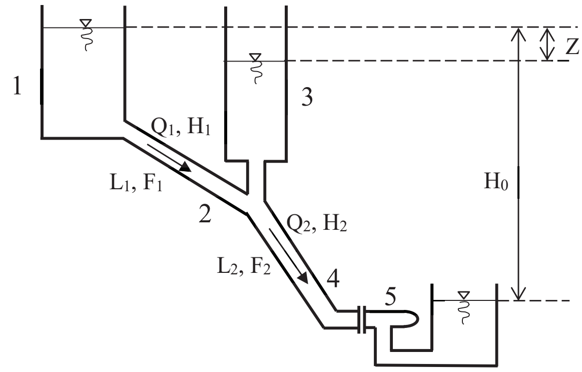

Figure 1.1: Hydroelectric plant structure. 1. Lake. 2. Tunnel. 3. Surge tank. 4. Penstock. 5. Hydraulic turbine.

This structure is common for the hydroelectric power plants throughout the world: such examples as *"Bicaz"* and *"Someș Mărișelu"* in Romania [27] or *"Tanzmühle"* in Germany [25,26] illustrate this assertion.

If distributed parameters are considered along the two water conduits, the adapted Saint Venant partial differential equations have to be used and the following mathematical model is obtained – we reproduce it after [10]

$$
\begin{aligned}
&\partial_{x_i}\left(H_i+\frac{V_i^2}{2g}\right)+\frac{1}{g}\partial_t V_i+\frac{\lambda_i}{2D_i}\frac{1}{g}V_i|V_i|=0, \quad \partial_t H_i+\frac{a_i^2}{g}\partial_{x_i}V_i=0, \quad i=1,2,\\
&H_1(0,t)=H_0; \quad H_1(L_1,t)+\frac{V_1^2(L_1,t)}{2g}=Z(t)+R_s\frac{\mathrm{d}Z}{\mathrm{d}t}=H_2(0,t)+\frac{V_2^2(0,t)}{2g},\\
&Q_i=F_iV_i, \quad i=1,2; \quad \bar{Q}=\alpha_q F_{\theta\,\max}\sqrt{H_0}; \quad F_s\frac{\mathrm{d}Z}{\mathrm{d}t}=Q_1(L_1,t)-Q_2(0,t),\\
&Q_2(L_2,t)=(1-k)\alpha_q F_\theta(t)\sqrt{H_2(L_2,t)}+k\bar{Q}\Omega(t)/\Omega_c,\\
&J\Omega_c\frac{\mathrm{d}\Omega}{\mathrm{d}t}=\eta_\theta\frac{\gamma}{2g}Q_2(L_2,t)H_2(L_2,t)-N_g,
\end{aligned}
\tag{1.1}
$$

where the notations are the usual ones in the field and are enumerated in the Appendix (also reproduced after the Appendix of [10]). In (1.1) the flow crossing the wicket gates of the turbine, namely $Q_2(L_2,t)$ (the subscript 2 accounts for the penstock state variables and parameters) is expressed according to an improved formula of [3], thus being dependent of the turbine rotating speed. The terms depending on V_i^2 account for the dynamic heads and those in $V_i|V_i|$ for the Darcy–Weisbach losses. It is worth mentioning that all conduits are assumed to be described by distributed parameters. The throttling of the surge tank is represented by its parameter R_s; letting $R_s=0$ means assuming a surge tank without throttling. Also the flow $\bar{Q}$ as defined in (1.1) represents the maximally available flow at the wicket gates and serves to flow rating (at this point we do not yet discuss the state variables ratings). It is worth mentioning however that this model reproduces the usual models of the hydraulic plants incorporating the dynamic (velocity) heads $(Q_i/F_i)^2$ and the distributed Darcy–Weisbach losses $(\lambda_i/(2D_ig))(Q_i/F_i))(|Q_i|/F_i)$. The boundary condition for the water flow $Q_2(L_2,t)$ is borrowed from [3] and incorporates the turbine rotating speed effect on the flow: it is stated that $0<k<0.3$ but in general there is taken $k=0$; k is thus a numeric coefficient having a corrective character from the engineering point of view. As it will appear in the following, k is irrelevant in water hammer analysis. As it can be seen in the Appendix containing the notation list, F_θ is the cross section area of the hydraulic turbine wicket gates. In the equations of the model (1.1) it acts as an input (forcing) signal, being defined by the speed controller of the turbine; its amplitude is limited physically: $0 \leq F_\theta \leq F_{\theta\max}$. During water hammer the hydraulic turbine is decoupled from the hydraulic system upstream and the forcing signal is blocked at some constant value, being thus irrelevant for the water hammer dynamics.

In hydraulic engineering two are the types of transients which are discussed: the normal and the abnormal ones. The normal exploitation regimes of the hydraulic power plants are concerned firstly with the so called *frequency/megawatt control* of the Electric Grid. The frequency/megawatt control is achieved by the control of the turbine rotating speed through water flow admission – controlled by the cross section area $F_\theta(t)$ of the wicket gates. The turbine controllers can be mechanical, hydraulic or electro-hydraulic (as technical implementation); the most recent control approach is based on predictive control [24]. The turbine controller is not included in (1.1) since normal regimes are outside the aim of this paper.

The abnormal regimes are concerned with sudden large power changes including turbine shut down. Especially in the last case, the turbine with the rotating speed controller are "cut" from the upstream dynamics; the only stabilizing device for the upstream dynamics remains the surge tank.

The present paper is concerned with the second case – the dynamics of the abnormal regimes. Again, two will be the problems analyzed. The first one will be the inherent stability of the surge tank as stabilizing device. The problem occurs from the engineering conviction that a stabilizing device incorporated in a feedback structure must be stable itself. Moreover, the surge tank is not a miniaturized electronic device but a construction which cannot be rebuilt in case of a design error. The second problem, already mentioned, is the stability of the upstream dynamics of the turbine (tunnel, surge tank, penstock) under water hammer.

Several simplifying assumptions are introduced, considered as covering from the engineering point of view (this aspect will be explained in what follows). The newly obtained model will allow a rigorous mathematical study by associating certain functional differential equations of neutral type.

2 Rated variables and parameters – the basic working model

A specific feature of the analysis of the real world mathematical models is the use of the rated (scaled) variables: the real physical variables are rated to certain reference values, the aim being at least twofold: to use relative i.e. comparable values and to reduce numerical ill conditioning. In our case the flows will be rated to the maximally available water flow at the wicket gates of the hydraulic turbine $\bar{Q} = \alpha_q F_{\theta\max}\sqrt{H_0}$; the piezometric heads are rated to the maximal head H_0 of the reservoir; the rotating speed of the turbine is rated to the synchronous speed Ω_c. These are the scalings of the state variables. The next scalings are those of the conduit coordinates $x_i (i = 1,2)$ to the conduit lengths L_i namely $\xi_i = x_i/L_i$.

We introduce further the following time constants of the conduits

- the starting time constant $T_{wi} = (L_i\bar{Q})(F_iH_0g)^{-1}$ $(i = 1,2)$;
- the fill up time constant $T_i = (L_iF_i)/\bar{Q}$ $(i = 1,2)$;
- the wave propagation time $T_{pi} = L_i/a_i$ $(i = 1,2)$,

and also

– the fill up time constant of the surge tank $T_s = F_s H_0/\bar{Q}$;

– the starting time constant of the turbine

$$T_a = \frac{J\Omega_c^2}{\eta_\theta \frac{\gamma}{2g}\bar{Q}H_0}.$$

The time constants notations are also listed in the Appendix.

After some simple and straightforward manipulation the following equations are obtained

$$\begin{aligned}
&\partial_{\xi_i}\left(h_i + \frac{1}{2}\frac{T_{wi}}{T_i}q_i^2\right) + T_{wi}\partial_t q_i + \frac{\lambda_i L_i}{D_i}\frac{1}{2}\frac{T_{wi}}{T_i}q_i|q_i| = 0, \quad \frac{T_{pi}^2}{T_{wi}}\partial_t h_i + \partial_{\xi_i} q_i = 0,\\
&h_1(0,t) \equiv 1; \quad h_1(1,t) = 1 + z(t) + R_s\frac{\mathrm{d}z}{\mathrm{d}t} = h_2(0,t),\\
&T_s\frac{\mathrm{d}z}{\mathrm{d}t} = q_1(1,t) - q_2(0,t); \quad q_2(1,t) = (1-k)f_\theta(t)\sqrt{h_2(1,t)} + k\varphi(t),\\
&T_a\frac{\mathrm{d}\varphi}{\mathrm{d}t} = q_2(1,t)h_2(1,t) - \nu_g.
\end{aligned} \tag{2.1}$$

By lower case letters q_i, h_i, z we denoted the rated state variables – flows, piezometric heads and water level in the surge tank respectively. We introduced also the rated rotating speed $\varphi = \Omega/\Omega_c$, the rated load mechanical power

$$\nu_g = \frac{N_g}{\eta_\theta \frac{\gamma}{2g}\bar{Q}H_0} \tag{2.2}$$

and the rated cross section area of the wicket gates $f_\theta = F_\theta/F_{\theta\max}$. The aforementioned rated variables are also listed in the Appendix. Observe also in (2.1) that the local dynamic heads $V_1^2(L_1,t)/(2g)$ and $V_2^2(0,t)/(2g)$ have been neglected, as it is customary in hydropower engineering. Another remark concerns the water level in the surge tank: again, as it is customary in hydraulic engineering, the rated level is "counted" from the maximal head H_0 of the lake i.e. $z := (Z - H_0)/H_0$ and this explains the presence of 1 in the boundary conditions at $\xi_1 = 1$ and $\xi_2 = 0$ where the surge tank is located – equations (2.1).

The next model transformation is connected with the rating of the time to the largest time constant T_1 (this assertion – T_1 being the largest – holds for most hydroelectric power plants). We shall have $\tau = t/T_1$; only the equations containing time derivatives will be modified since the corresponding time constant will be now rated to T_1 – from the chain rule differentiation.

Before writing down the modified model, an explanation for this time rating appears as necessary. Model (1.1) is considered in hydraulics as *fundamental* in the sense that various particular models for various analysis are deduced from it according to corresponding assumptions (as it will appear throughout this paper also). Among other features of the model – which correspond to a physical reality – is the property of *several time scales*. This property follows by comparison of the time constants introduced previously: if we refer again to the aforementioned hydroelectric power plants of Romania (for which numerical data are available) we can see e.g. that $T_1 = 1005$ sec., $T_s = 502.25$ sec., $T_{w1} = 14.71$ sec., $T_{p1} = 3.81$ sec., $T_2 = 44.33$ sec., $T_{w2} = 0.38$ sec., $T_a \approx 8$ sec. etc. Several time scales are usually tackled within the framework of the *singular perturbations*. Therefore it is useful for a basic model to have the "small parameters" as ratios of time constants ensuring their dimensionless.

Denoting by $\theta_{wi} = T_{wi}/T_1$, $\theta_i = T_i/T_1$ ($\theta_1 = 1$) etc. the rated to T_1 time constants, the following work model is obtained

$$\begin{aligned}
&\partial_{\xi_i}\left(h_i + \frac{1}{2}\frac{\theta_{wi}}{\theta_i}q_i^2\right) + \theta_{wi}\partial_\tau q_i + \frac{\lambda_i L_i}{D_i}\frac{1}{2}\frac{\theta_{wi}}{\theta_i}q_i|q_i| = 0, \quad \frac{\theta_{pi}^2}{\theta_{wi}}\partial_\tau h_i + \partial_{\xi_i} q_i = 0,\\
&h_1(0,\tau) \equiv 1; \quad h_1(1,\tau) = 1 + z(\tau) + \lambda_s\frac{\mathrm{d}z}{\mathrm{d}\tau} = h_2(0,\tau),\\
&\theta_s\frac{\mathrm{d}z}{\mathrm{d}\tau} = q_1(1,\tau) - q_2(0,\tau); \quad q_2(1,\tau) = (1-k)f_\theta(\tau)\sqrt{h_2(1,\tau)} + k\varphi(\tau),\\
&\theta_a\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = q_2(1,\tau)h_2(1,\tau) - \nu_g
\end{aligned} \tag{2.3}$$

with $\lambda_s := R_s/T_1$; the term $\lambda_s \mathrm{d}z/\mathrm{d}\tau$ in the boundary conditions at the surge tank accounts for a surge tank with throttling [15,27,39].

The rated time constants are also listed in the Appendix. The fact that $\theta_1 = 1$ appears in the equations is due to the similarity of the Saint Venant partial differential equations for the two conduits, suggesting the more compact writing of the equations.

Model (2.3) is, generally speaking, completed with certain equations of the speed controller for the hydraulic turbine. We already mentioned at Section 1 that this controller is decoupled (even blocked) during the abnormal regime of the water hammer hence that its dynamics will be irrelevant throughout this paper. Its equations are nevertheless given in order to make clearer the passage from normal to abnormal exploitation.

The speed controller has various engineering implementations (mechanical, mechanic-hydraulic, electro-hydraulic). For the aforementioned purpose we can write down a general form

$$\begin{aligned}
\dot{x}_c &= A_c x_c + b_c(\varphi_0 - \varphi),\\
f_\theta &= f_c^T x_c + \gamma_c(\varphi_0 - \varphi),
\end{aligned} \tag{2.4}$$

where $x_c \in \mathbb{R}^n$ is the state vector of the controller dynamics and $\varphi_0 = 1 = \Omega/\Omega_c$ – the rated synchronous speed of the hydraulic turbine, imposed by the Power Grid. The controller's coefficients A_c, b_c, f_c, γ_c have appropriate dimensions.

Obviously the speed controller acts by modifying the cross section area f_θ of the turbine wicket gates. Its role is firstly to ensure a stable steady state for system (2.3)–(2.4). Let us compute it, by letting the "time" derivatives from (2.3)–(2.4) go to zero. We obtain firstly for the steady state flows $\bar{q}_i(\xi_i)$ that they are constant and equal i.e. $\bar{q}_1(\xi_1) \equiv \bar{q}_2(\xi_2) \equiv \bar{q}$. Let $\bar{h}_i(\xi_i)$ be the steady state values for the piezometric heads. The steady state boundary condition at $\xi_2 = 1$ that is

$$\bar{q} = (1-k)\bar{f}_\theta\sqrt{\bar{h}_2(1)} + k\bar{\varphi}$$

shows that $\bar{h}_2(1) > 0$. Therefore the steady state load condition $\bar{q}\bar{h}_2(1) = \nu_g > 0$ shows that $\bar{q} > 0$ what is only natural since no normal exploitation would require an upstream flow. Therefore we deduce the differential steady state equations for the piezometric heads

$$\frac{\mathrm{d}\bar{h}_i}{\mathrm{d}\xi_i} + \frac{1}{2}\frac{\lambda_i L_i}{D_i}\frac{\theta_{wi}}{\theta_i}\bar{q}^2 = 0.$$

From here it follows

$$\begin{aligned}
\bar{h}_1(\xi_1) &= 1 - \frac{1}{2}\frac{\lambda_1 L_1}{D_1}\frac{\theta_{w1}}{\theta_1}\bar{q}^2\xi_1, \quad 0 \le \xi_1 \le 1,\\
\bar{h}_2(\xi_2) &= 1 - \frac{1}{2}\frac{\lambda_1 L_1}{D_1}\frac{\theta_{w1}}{\theta_1}\bar{q}^2 - \frac{1}{2}\frac{\lambda_2 L_2}{D_2}\frac{\theta_{w2}}{\theta_2}\bar{q}^2\xi_2, \quad 0 \le \xi_2 \le 1
\end{aligned}$$

and, therefore

$$\bar{h}_2(1) = 1 - \frac{1}{2}\left(\frac{\theta_{w1}}{\theta_1}\frac{\lambda_1 L_1}{D_1} + \frac{\theta_{w2}}{\theta_2}\frac{\lambda_2 L_2}{D_2}\right)\bar{q}^2.$$

The load condition $\bar{q}\bar{h}_2(1) = \nu_g$ will then send to the following equation of third degree which allows determination of the flow as function of the mechanical load ν_g

$$\frac{1}{2}\left(\frac{\theta_{w1}}{\theta_1}\frac{\lambda_1 L_1}{D_1} + \frac{\theta_{w2}}{\theta_2}\frac{\lambda_2 L_2}{D_2}\right)\bar{q}^3 - \bar{q} + \nu_g = 0. \tag{2.5}$$

It is worth mentioning that the design is such that the coefficients of (2.5) allow existence of a solution $\bar{q} > 0$. Observe that this solution results from the steady state equations of the water supply of the turbine (upstream it), being imposed by the steady state mechanical load ν_g of the turbine. Afterwards the piezometric heads, which are linearly decreasing, will follow, also $\bar{z} = \bar{h}_1(1) = \bar{h}_2(0)$. Now it becomes possible to obtain the steady state of the hydraulic turbine and of its controller by solving the equations

$$A_c\bar{x}_c + b_c(\varphi_0 - \bar{\varphi}) = 0; \quad \bar{f}_\theta = f_c^T\bar{x}_c + \gamma_c(\varphi_0 - \bar{\varphi}),$$
$$(1-k)\bar{f}_\theta\sqrt{\nu_g/\bar{q}} + k\bar{\varphi} = \bar{q}$$

allowing to find $\bar{\varphi}$, $\bar{f}_\theta$, $\bar{x}_c$, the reference φ_0 being given.

However, the steady state of the normal exploitation, just computed, is not of interest in this paper. We just mention that stability of this normal exploitation steady state is ensured by both the surge tank – which regulates the upstream water flow $q_2(\xi_2,\tau)$ *via* the water level oscillations $z(\tau)$ of the tank – and the speed controller – which regulates the water flow admitted in the turbine to realize the frequency/megawatt (φ *versus* ν_g) control of the Power Grid.

As already mentioned in Section 1, during the abnormal regimes generating water hammer – the sudden turbine load discharge – the turbine with its speed controller are "cut" from the upstream dynamics and f_θ – the controlled cross-section area of the wicket gates – is assumed "blocked" at a constant value.

3 Inherent stability of the surge tank

It has been just shown that the surge tank has regulatory role for the water flow upstream the turbine. This regulatory role is more obvious during water hammer, when the turbine and its speed controller are "cut" from the upstream. A standard engineering philosophy states that a stabilizing device should display inherent stability itself. The stability analysis for the surge tank is done under some unanimously accepted assumptions going back to the early period of hydraulic power engineering [3,20,24,27].

3.1 The inferred engineering model

According the the aforementioned literature (and not only), the stability model for the surge tank relies on three equations: the dynamics equation, the continuity equation and the load control equation.

The dynamics equation is the so called *inelastic water column, upstream the surge tank, equation* - in fact the water column in the tunnel; the term inelastic defines the lumped flow parameters.

It is adopted as such and it reads

$$\frac{L_1}{g}\frac{\mathrm{d}V_1}{\mathrm{d}t} + (Z - H_0) + P_1|V_1|V_1 + R_s|V_s|V_s = 0 \tag{3.1}$$

where $P_1|V_1|V_1$ accounts for the hydraulic losses at the input of the surge tank and $R_s|V_s|V_s$ are the losses through the tank throttling: we have $V_s = \mathrm{d}Z/\mathrm{d}t$ and in some cases this term is linearized. The introduction of the modulus in the losses terms is done for the case of the reverse (upstream) flow which might appear during transients.

The continuity equation is nothing more but the mass balance equation for the surge tank

$$F_s\frac{\mathrm{d}Z}{\mathrm{d}t} = F_1V_1 - Q_T \tag{3.2}$$

where Q_T is the "load" water flow reaching the hydraulic turbine. This flow, which should ensure delivery of the required mechanical power, is defined by the *load control equation*, which is static – of the form

$$Q_T = f_T(N_g, Z). \tag{3.3}$$

The static load control function is an inference, deduced from several facts: firstly, the hydroelectric plants had relatively small powers and, as a consequence, the penstocks were short and the turbines located near the surge tanks. At its turn this fact allowed neglecting the dynamics of the penstock and of the turbine, also of the hydraulic losses. Following the load instantaneously induces also a more difficult dynamic condition for the surge tank and may therefore be considered as covering ("worst case") from the engineering point of view.

Starting from the hydraulic power definition, namely $N_g = \eta_\theta(\gamma/(2g))H_TQ_T$, H_T being the piezometric head at the wicket gates of the turbine, taking into account that head losses between the surge tank and the hydraulic turbine are negligible and neglected, it follows that $H_T = Z$ and

$$f_T(N_g, Z) = \frac{N_g}{\eta_\theta \frac{\gamma}{2g} Z}. \tag{3.4}$$

The model is thus given by (3.1), (3.2), (3.4) and, as already specified, represents an inference – at the engineering level of rigor – from certain equations of the Hydraulic engineering. From this moment, however, no additional physical or engineering assumptions can be introduced and the analysis will deal with the differential equations

$$\begin{aligned} &\frac{L_1}{g}\frac{\mathrm{d}V_1}{\mathrm{d}t} + (Z - H_0) + P_1|V_1|V_1 + R_s\left|\frac{\mathrm{d}Z}{\mathrm{d}t}\right|\frac{\mathrm{d}Z}{\mathrm{d}t} = 0, \\ &F_s\frac{\mathrm{d}Z}{\mathrm{d}t} = F_1V_1 - \frac{N_g}{\eta_\theta \frac{\gamma}{2g} Z}. \end{aligned} \tag{3.5}$$

In order to use a unitary framework, we rate the flows at $\bar{Q}$ (as in Section 2), the piezometric heads to H_0 and denote

$$q_1 := F_1V_1/\bar{Q}\,, \qquad z := (Z - H_0)/H_0$$

Therefore equations (3.5) become

$$\begin{aligned} &T_{w1}\frac{\mathrm{d}q_1}{\mathrm{d}t} + z + \frac{P_1\bar{Q}^2}{F_1^2H_0}|q_1|q_1 + \frac{R_s}{H_0^2}\left|\frac{\mathrm{d}z}{\mathrm{d}t}\right|\frac{\mathrm{d}z}{\mathrm{d}t} = 0 \\ &T_s\frac{\mathrm{d}z}{\mathrm{d}t} = q_1 - \frac{\nu_g}{1+z} \end{aligned} \tag{3.6}$$

with T_{w1} and T_s as defined in Section 2. While the fill up time constant T_1 does not appear in the present inference, it is however possible to introduce the rated time $\tau = t/T_1$ to transform (3.6) as below

$$\begin{aligned} &\theta_{w1}\frac{\mathrm{d}q_1}{\mathrm{d}\tau} + z + P_1'|q_1|q_1 + \lambda_s\left|\frac{\mathrm{d}z}{\mathrm{d}\tau}\right|\frac{\mathrm{d}z}{\mathrm{d}\tau} = 0, \\ &\theta_s\frac{\mathrm{d}z}{\mathrm{d}\tau} = q_1 - \frac{\nu_g}{1+z}, \end{aligned} \tag{3.7}$$

where the rated coefficients of the losses and of the throttling are as follows

$$P_1' := \frac{P_1\bar{Q}^2}{F_1^2 H_0}\,, \qquad \lambda_s := \frac{R_s}{H_0^2 T_1^2} = \frac{R_s\bar{Q}^2}{F_1^2 L_1^2 H_0}. \tag{3.8}$$

We proceed now to analyze stability of the surge tank based on (3.7), following [14, 16]. The steady state (equilibrium) imposed by following the load ν_g is given by

$$\bar{z} + P_1'|\bar{q}_1|\bar{q}_1 = 0\,, \qquad \bar{q}_1 = \frac{\nu_g}{1+\bar{z}}$$

The physically significant steady states correspond to positive flows (flowing downstream), what implies $1+\bar{z} > 0$ i.e. the water level in the surge tank, usually lower than the lake water level ($\bar{z} < 0$ since $\nu_g < 1$), cannot be under the basic reference level. Therefore $\bar{z}$ is a real solution of the third degree equation

$$\bar{z}(1+\bar{z})^2 + P_1'\nu_g^2 = 0. \tag{3.9}$$

If $P_1'\nu_g^2 > 4/27$, equation (3.9) has a single real root which is lower than -1 hence this case is not acceptable from the engineering point of view. In practice the parameters are chosen to have the reverse i.e. $P_1'\nu_g^2 < 4/27$ – when (3.9) has three real roots: $\bar{z}_1 \in (-1/3, 0)$, $\bar{z}_2 \in (-1, -1/3)$, $\bar{z}_3 \in (-\infty, -1)$. The third has no engineering significance, as already mentioned, while $\bar{z}_1$ is the acceptable one. We shall discuss the stability of the equilibrium defined by it. We introduce firstly the deviations

$$\zeta := z - \bar{z}_1\,, \qquad v := \frac{\mathrm{d}\zeta}{\mathrm{d}\tau} \tag{3.10}$$

which are subject to the following differential equations

$$\begin{aligned} \frac{\mathrm{d}\zeta}{\mathrm{d}\tau} &= v, \\ \theta_s\frac{\mathrm{d}v}{\mathrm{d}\tau} &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\theta_s\frac{\mathrm{d}\zeta}{\mathrm{d}\tau}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(q_1 - \frac{\nu_g}{1+\bar{z}_1+\zeta}\right) \\ &= \frac{\nu_g}{(1+\bar{z}_1+\zeta)^2}v - \frac{1}{\theta_{w1}}[\bar{z}_1 + \zeta + P_1'|q_1|q_1 + \lambda_s|v|v] \\ &= \frac{\nu_g}{(1+\bar{z}_1+\zeta)^2}v \\ &\quad - \frac{1}{\theta_{w1}}\left[\bar{z}_1 + \zeta + P_1'\left|\theta_s v + \frac{\nu_g}{1+\bar{z}_1+\zeta}\right|\left(\theta_s v + \frac{\nu_g}{1+\bar{z}_1+\zeta}\right) + \lambda_s|v|v\right]. \end{aligned} \tag{3.11}$$

To system (3.11) it is attached the following Lyapunov function

$$\mathcal{V}(\zeta, v) = \frac{1}{2}\theta_s v^2 + \frac{1}{2\theta_{w1}}\left[1 - 2P_1'\frac{\nu_g^2}{(1+\bar{z}_1)^2(1+\bar{z}_1+\zeta)}\right]\zeta^2. \tag{3.12}$$

This function is positive definite in the following domain of the phase plane (ζ, v)

$$1 - 2P_1' \frac{\nu_g^2}{(1+\bar{z}_1)^2(1+\bar{z}_1+\zeta)} = 1 + \frac{2\bar{z}_1}{1+\bar{z}_1+\zeta} = \frac{1+3\bar{z}_1+\zeta}{1+\bar{z}_1+\zeta} > 0. \tag{3.13}$$

The acceptable condition is the strip $\zeta > -(1+3\bar{z}_1)$ which contains $(0,0)$ – the equilibrium of (3.11) corresponding to the equilibrium of (3.9) $(\bar{z}_1, \nu_g(1+\bar{z}_1)^{-1})$.

The next condition is given by positiveness of the water flow q_1 (flowing downstream – what happens in most time even during water hammer transients). The condition

$$q_1 > 0 \;\Leftrightarrow\; \theta_s v + \frac{\nu_g}{1+\bar{z}_1+\zeta} > 0 \;\Leftrightarrow\; v(1+\bar{z}_1+\zeta) > -\frac{\nu_g}{\theta_s} \tag{3.14}$$

defines a domain also containing the origin $(0,0)$.

Consider firstly the simpler case of following a zero load – the completely discharged turbine. Therefore $\nu_g = 0$, $\bar{z}_1 = 0$. System (3.11) becomes

$$\frac{d\zeta}{d\tau} = v \;;\; \theta_s \frac{dv}{d\tau} - \frac{1}{\theta_{w1}}[\zeta + (P_1'\theta_s^2 + \lambda_s)|v|v] \tag{3.15}$$

with the Lyapunov function

$$\mathcal{V}(\zeta, v) = \frac{1}{2}(\theta_s v^2 + \frac{1}{\theta_{w1}}\zeta^2) > 0. \tag{3.16}$$

The derivative function will be

$$\mathcal{W}(\zeta, v) = -\frac{1}{\theta_{w1}}(P_1'\theta_s^2 + \lambda_s)|v|v^2 \le 0.$$

The asymptotic stability follows immediately by applying the Barbashin–Krasovskii–LaSalle invariance principle. The same result is however straightforward from (3.7), where $\nu_g = 0$ will show $(0,0)$ to be the unique steady state. Further, system (3.7) can be written as a single second order differential equation

$$\frac{\theta_s^2}{\theta_{w1}} \frac{d^2 z}{d\tau^2} + (P_1'\theta_s^2 + \lambda_s)\left|\frac{dz}{d\tau}\right| \frac{dz}{d\tau} + z = 0 \tag{3.17}$$

which describes an oscillator with nonlinear damping. The Lyapunov function (3.16) is just oscillator's total energy.

Let now $\nu_g > 0$ i.e. the load discharge is not full. The stability domain is delimited by (3.13) and (3.14). The derivative function of (3.12) will be now, under conditions (3.13) and (3.14)

$$\begin{aligned}
\mathcal{W}(\zeta, v) = & \left\{ \frac{\nu_g}{(1+\bar{z}_1+\zeta)^2} v - \frac{1}{\theta_{w1}} \left[\bar{z}_1 + \zeta + P_1' \left(\theta_s v + \frac{\nu_g}{1+\bar{z}_1+\zeta} \right)^2 + \lambda_s |v| v \right] \right\} v \\
& + \frac{1}{\theta_{w1}} \left\{ \left[1 - 2P_1' \frac{\nu_g^2}{(1+\bar{z}_1)^2(1+\bar{z}_1+\zeta)} \right] \zeta + P_1' \frac{\nu_g^2}{(1+\bar{z}_1)^2(1+\bar{z}_1+\zeta)^2} \zeta^2 \right\} v \\
= & \frac{\nu_g}{(1+\bar{z}_1+\zeta)^2} v^2 - \frac{1}{\theta_{w1}} \left\{ -\frac{P_1'\nu_g^2}{(1+\bar{z}_1)^2} + P_1' \left(\theta_s v + \frac{\nu_g}{1+\bar{z}_1+\zeta} \right)^2 + \right. \\
& \left. + 2P_1' \frac{\nu_g^2}{(1+\bar{z}_1)^2(1+\bar{z}_1+\zeta)} \zeta - P_1' \frac{\nu_g^2}{(1+\bar{z}_1)^2(1+\bar{z}_1+\zeta)^2} \zeta^2 + \lambda_s |v| v \right\} v
\end{aligned}$$

We compute

$$-\frac{P_1'\nu_g^2}{(1+\bar{z}_1)^2}\left[1-\frac{2\zeta}{1+\bar{z}_1+\zeta}+\frac{\zeta^2}{(1+\bar{z}_1+\zeta)^2}\right]=-\frac{P_1'\nu_g^2}{(1+\bar{z}_1+\zeta)^2}$$

to obtain further

$$\begin{aligned}\mathcal{W}(\zeta,v)&=\frac{\nu_g}{(1+\bar{z}_1+\zeta)^2}v^2-\frac{1}{\theta_{w1}}\left[P_1'\left(\theta_s v+\frac{\nu_g}{1+\bar{z}_1+\zeta}\right)^2-\frac{P_1'\nu_g^2}{(1+\bar{z}_1+\zeta)^2}+\lambda_s|v|v\right]v\\&=-\left[\frac{\theta_s}{\theta_{w1}}P_1'\left(\theta_s v+\frac{2\nu_g}{1+\bar{z}_1+\zeta}\right)-\frac{\nu_g}{(1+\bar{z}_1+\zeta)^2}+\frac{\lambda_s}{\theta_{w1}}|v|\right]v^2.\end{aligned} \tag{3.18}$$

We seek conditions for $\mathcal{W}(\zeta,v)\leq 0$ under (3.13) and (3.14). A *necessary* (not sufficient) condition would be fulfilment of $\mathcal{W}(\zeta,v)\leq 0$ in a *small neighborhood* of the origin $(0,0)$. This condition reads

$$2\frac{\theta_s}{\theta_{w1}}P_1'\frac{\nu_g}{1+\bar{z}_1}-\frac{\nu_g}{(1+\bar{z}_1)^2}>0 \;\Leftrightarrow\; 2\frac{\theta_s}{\theta_{w1}}P_1'(1+\bar{z}_1)>1 \tag{3.19}$$

which imposes a lower limit for the time constant θ_s of the surge tank, in fact for the cross-section area of the surge tank. Taking into account the definitions of θ_s, θ_{w1}, P_1' and $\bar{z}_1$ we shall have

$$\begin{aligned}2\frac{\theta_s}{\theta_{w1}}P_1'(1+\bar{z}_1)&=2\frac{T_s}{T_{w1}}P_1'(1+\bar{z}_1)=2\frac{F_sH_0}{\bar{Q}}\frac{F_1H_0g}{L_1\bar{Q}}\frac{P_1\bar{Q}^2}{F_1^2H_0}\frac{H_0+\bar{Z}_1}{H_0}\\&=2\frac{F_s}{F_1}\frac{g}{L_1}P_1(H_0+\bar{Z}_1)>1\end{aligned}$$

or

$$F_s>\frac{1}{2}\frac{L_1}{g}\frac{1}{P_1(H_0+\bar{Z}_1)}F_1=F_{Th} \tag{3.20}$$

where F_{Th} is the so called *Thoma cross-section area* introduced by D. Thoma in his doctoral thesis [38]. Since, as mentioned, (3.19) is necessary, not sufficient for $\mathcal{W}(\zeta,v)\leq 0$, we turn again to (3.18) and re-write it as follows

$$\mathcal{W}(\zeta,v)=\left[\frac{\theta_s^2}{\theta_{w1}}P_1'(\alpha|v|+v)+2\frac{\theta_s}{\theta_{w1}}P_1'\frac{\nu_g}{1+\bar{z}_1+\zeta}-\frac{\nu_g}{(1+\bar{z}_1+\zeta)^2}\right]v^2$$

where $\alpha=\lambda_s(P_1'\theta_s^2)^{-1}$. Suppose $\alpha>1$, then $\mathcal{W}(\zeta,v)\leq 0$ provided

$$2\frac{\theta_s}{\theta_{w1}}P_1'(1+\bar{z}_1+\zeta)>1. \tag{3.21}$$

It can be seen that (3.21) implies (3.19) which thus appears genuinely as necessary but not sufficient. Even if $\alpha<1$, then (3.21) becomes a necessary condition for $\mathcal{W}(\zeta,v)\leq 0$ with (3.19) as necessary for the fulfilment of (3.21).

Consider now the expressions of the aforementioned parameters. We find

$$\alpha=\frac{R_sH_0}{T_1^2}\frac{F_1^2H_0}{P_1\bar{Q}^2}\frac{T_1^2}{T_s^2}=\frac{R_s}{P_1}\left(\frac{F_1}{F_s}\right)^2$$

and, for (3.21)

$$2\frac{F_s}{F_1}\frac{g}{L_1}P_1(\bar{Z}_1+Y)>1\,,\qquad Y:=Z-\bar{Z}_1. \tag{3.22}$$

The last inequality is easily re-written as

$$\bar{Z}_1 + Y > \bar{Z}_1 \frac{F_{Th}}{F_s} \tag{3.23}$$

and it implies (3.20) which again appears as a necessary condition.

Unfortunately condition $\alpha > 1$ turns to be completely *unrealistic* since normally $R_s < P_1$ and $F_1 \ll F_s$. We have thus to consider the case $\alpha < 1$. Following [16], we consider the following function

$$\Phi(\zeta, v) = v - \frac{\theta_{w1}}{\theta_s^2 P_1'} \frac{\nu_g}{1 + \bar{z}_1 + \zeta} \left[\frac{1}{1 + \bar{z}_1 + \zeta} - \frac{2}{\theta_s} \right] \tag{3.24}$$

and if $\Phi(\zeta, v) > 0$, it follows that $\mathcal{W}(\zeta, v) \leq 0$. Together with the Barbashin–Krasovskii–LaSalle invariance principle, this will give asymptotic stability.

Concerning the estimate of the attraction domain of the equilibrium $(0,0)$, we have to consider the interior of the domain defined by (3.13), (3.14) and $\Phi(\zeta, v) > 0$, together with the family of curves $\Psi_c(\zeta, v) = \{(\zeta, v) \mid \mathcal{V}(\zeta, v) = c > 0\}$ which are closed for $c > 0$ small enough. An estimate of the attraction domain is the domain inside Ψ_c completely included in the domain defined by (3.13), (3.14) and $\Phi(\zeta, v) > 0$, $c > 0$ being maximal from this point of view. Summarizing, the mathematical result is as follows.

Theorem 3.1. *Consider the system* (3.11)*, the associated Lyapunov function* (3.12) *and its derivative function along* (3.11) *– its Lie derivative* $\mathcal{W}(\zeta, v)$ *– given by* (3.18)*. The equilibrium* $(0,0)$ *of (3.11) is asymptotically stable with the attraction domain contained in the set of the phase plane* (ζ, v) *defined by* (3.13)*,* (3.14) *and* $\Phi(\zeta, v) > 0$*. A standard estimate of this domain is given by inequalities of the form* $\mathcal{V}(\zeta, v) < c$ *with* $c > 0$ *maximally possible in order to have* $\mathcal{V}(\zeta, v) = c$ *closed curves and the domain inside fully contained in the aforementioned set defined by* (3.13)*,* (3.14) *and* $\{(\zeta, v) | \Phi(\zeta, v) > 0\}$*.*

Finally, let us remark that, since the attraction domain does not encompass the entire phase plane, the analysis should be completed with additional studies dealing with limit cycles and hidden attractors [2,22,32,33]. This extended analysis is outside the aims of this paper.

3.2 Modeling the surge tank in the context of several time scales

In this subsection we shall consider modeling of the surge tank stability dynamics as resulting from the model (2.3)–(2.4). This model displays distributed parameters, being defined by partial differential equations: it is valid for larger hydroelectric power plants unlike those for which the model considered in the previous subsection was inferred. Nevertheless, in the contemporary water hammer analysis, a difference is made between fast water mass oscillations – where partial differential equations are used in modeling – and slow water mass oscillations. This last case is more suitable for surge tank stability analysis. The explanation is that model (2.3)–(2.4) has several time scales, as follows e.g. from the size analysis of the occurring time constants. Taking as examples the two hydroelectric plants of Romania, mentioned at the beginning of Section 1, we can see that

a) for the “Bicaz” plant: $\theta_{w1} = 14.71 \times 10^{-3}$, $\theta_{p1} = 3.81 \times 10^{-3}$, $\theta_{w2} = 0.38 \times 10^{-3}$, $\theta_{p2} = 0.14 \times 10^{-3}$, $\theta_s = 0.502$, $\theta_a = 5.1 \times 10^{-3}$;

b) for the “Someș-Mărișelu” plant: $\theta_{w1} = 2.34 \times 10^{-3}$, $\theta_{p1} = 2.7 \times 10^{-3}$, $\theta_{w2} = 0.36 \times 10^{-3}$, $\theta_{p2} = 0.27 \times 10^{-3}$, $\theta_s = 0.108$, $\theta_a = 1.3 \times 10^{-3}$.

Consequently, the surge tank stability has to be studied at its time scale – given by the time constant θ_s. We take therefore the approach of (formal) singular perturbations, following also the standard engineering assumptions enumerated in the previous subsection.

By taking $\theta_{p1}^2/\theta_{w1} = \delta_1^2\theta_{w1} \approx 0$ ($\delta_1 = \theta_{p1}/\theta_{w1} = 0.26$), it follows that $\partial_{\xi_1} q_1 = 0$ hence $q_1(\xi_1, \tau) \equiv q_1(\tau)$; also $\partial_{\xi_1} q_1^2 = 0$. The equation for $h_1(\xi_1, \tau)$ becomes

$$\partial_{\xi_1} h_1 + \theta_{w1}\frac{dq_1}{d\tau} + \frac{1}{2}\frac{\theta_{w1}}{\theta_1}\frac{\lambda_1 L_1}{D_1}|q_1|q_1 = 0$$

and can be integrated with respect to ξ_1 from 0 to 1 to obtain

$$h_1(1,\tau) - h_1(0,\tau) + \theta_{w1}\frac{dq_1}{d\tau} + \frac{1}{2}\frac{\theta_{w1}}{\theta_1}\frac{\lambda_1 L_1}{D_1}|q_1|q_1 = 0,$$
$$h_1(1,\tau) = 1 + z(\tau) + \lambda_s\frac{dz}{d\tau}\,; \qquad h_1(0,\tau) \equiv 1.$$

Therefore

$$\theta_{w1}\frac{dq_1}{d\tau} + \frac{1}{2}\frac{\theta_{w1}}{\theta_1}\frac{\lambda_1 L_1}{D_1}|q_1|q_1 + z(\tau) + \lambda_s\frac{dz}{d\tau} = 0. \tag{3.25}$$

A comparison with the first equation of (3.7) is useful. This first equation of (3.7) and (3.25) are almost identical – the difference appears in the model of the surge tank throttling. In the model (2.3)–(2.4) it was considered linear [15,27], thus migrating in (3.25) while in other studies is taken quadratic [19,27,41]. In fact the modeling of the local hydraulic losses is made of engineering inferences: starting from the general laws of the Fluid Mechanics, a formula is inferred and then verified experimentally. Many constructive elements in engineering are modeled in this way, based on steady state behavior and measurements, then put together in a comprehensive dynamical model thus extending the steady state properties to dynamics. This explains the necessary validation of the mathematical model [28].

The second equation of the surge tank model is the continuity one i.e.

$$\theta_s\frac{dz}{d\tau} = q_1(\tau) - q_2(0,\tau) \tag{3.26}$$

(we already took $q_1(\tau)$ from the equation (3.25)). The remaining modeling problem is to represent the load flow. The engineering requirement is that the load flow should follow a *static* external mechanical load. Since in statics (but rated variables) we have the formula $\nu_g = q_2h_2$ with all terms – constant, it follows that $q_2 = \nu_g/h_2 = \nu_g(1+z)^{-1}$. This is an engineering inference which *has not been deduced* from (2.3). The model

$$\begin{aligned} &\theta_{w1}\frac{dq_1}{d\tau} + \frac{1}{2}\frac{\theta_{w1}}{\theta_1}\frac{\lambda_1 L_1}{D_1}|q_1|q_1 + z(\tau) + \lambda_s\frac{dz}{d\tau} = 0,\\ &\theta_s\frac{dz}{d\tau} = q_1 - \frac{\nu_g}{1+z} \end{aligned} \tag{3.27}$$

is very much alike to (3.7) but it is obtained (partly) from (2.3)–(2.4). We can try however to point out possible assumptions leading to the second equation of (3.27) and/or (3.7).

Following a static load means firstly letting to zero all time constants multiplying the time derivatives downstream the surge tank: $\theta_{p2}^2/\theta_{w2} = \delta_2^2\theta_{w2} = 0$, $\theta_{w2} = 0$, $\theta_a = 0$. It follows that $q_2(\xi_2, \tau) \equiv q_2(\tau)$ and

$$\frac{dh_2}{d\xi_2} + \frac{1}{2}\frac{\theta_{w2}}{\theta_2}|q_2|q_2 = 0.$$

Since the penstock is much shorter than the tunnel, the engineering assumption is that the losses along the penstock are negligible; this inference is consistent with $\theta_{w2} \approx 0$ (according to the numerical data $\theta_{w2}/\theta_2 < \theta_{w2}$). Therefore $h_2(\xi_2, \tau) \equiv h_2(\tau)$. This implies

$$h_2(\tau) = 1 + z(\tau) + \lambda_s\frac{dz}{d\tau}\,; \qquad q_2 = \nu_g\left(1 + z(\tau) + \lambda_s\frac{dz}{d\tau}\right)^{-1}.$$

The resulting model will be

$$\begin{aligned} \theta_{w1}\frac{\mathrm{d}q_1}{\mathrm{d}\tau} + \frac{1}{2}\frac{\theta_{w1}}{\theta_1}\frac{\lambda_1 L_1}{D_1}|q_1|q_1 + z(\tau) + \lambda_s\frac{\mathrm{d}z}{\mathrm{d}\tau} = 0, \\ \theta_s\frac{\mathrm{d}z}{\mathrm{d}\tau} = q_1 - \nu_g\left(1 + z(\tau) + \lambda_s\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^{-1} \end{aligned} \tag{3.28}$$

being different from (3.27) by the term $\lambda_s(\mathrm{d}z/\mathrm{d}\tau)$ in the second equation. Its significance is that now *the surge tank load flow must follow a dynamic load*.

Some additional comments are necessary. Model (3.28) *is not equivalent* to (2.3)–(2.4), but *it is obtained from it* by letting some small time constants to zero and neglecting the losses along the penstock (the water conduit corresponding to $i = 2$). The connection of the two models can be viewed at the level of their solutions by: a) neglecting the losses along the penstock in (2.3) also; b) comparing the solutions of the two mathematical models for the "small" time constants sufficiently small.

Other details are also to be specified. They are related to the hydraulic turbine and its speed controller and we have to consider again the engineering assumptions and inferences.

Surge tank stability is related to water hammer – an abnormal transient occurring as a result of a sudden, rather large load discharge. This load discharge initiates a safety maneuver of decoupling the controller (2.4), stopping the turbine ($\varphi = 0$) and blocking the wicket gates crossing area f_θ at a constant value $\bar{f}_\theta$. The boundary condition

$$q_2 = (1-k)f_\theta\sqrt{h_2(\tau)} + k\varphi$$

combined with $q_2(\tau)h_2(\tau) = \nu_g$ and $\varphi = 0$ will give

$$\nu_g = (1-k)f_\theta(h_2(\tau))^{3/2} \;\Rightarrow\; f_\theta = \frac{\nu_g}{1-k}(h_2(\tau))^{3/2} = \frac{\nu_g}{1-k}\left(1 + z(\tau) + \lambda_s\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^{-1}$$

and

$$\bar{f}_\theta = \lim_{\tau\to\infty}\frac{\nu_g}{1-k}(h_2(\tau))^{3/2} = (1+\bar{z}_1)^{3/2}\frac{\nu_g}{1-k}.$$

It follows that the blocking value $\bar{f}_\theta$ is reached after a transient process due to the surge tank which, again, must be stable.

To end these considerations, let us mention that they contain several engineering inferences resulting from practice, some of them being assumed here for the sake of completeness, because civil hydraulic engineers, hydroelectric power engineers and automatic control engineers have rather few interactions: each of them is following the prescriptions and the experience of the corresponding domain of expertise.

3.3 Asymptotic stability and total stability

We shall consider here stability for the models (3.27) and (3.28). Both models have the same steady state given by

$$\frac{1}{2}\frac{\theta_{w1}}{\theta_1}\frac{\lambda_1 L_1}{D_1}\bar{q}_1^2 + \bar{z} = 0\,, \qquad \bar{q}_1 = \frac{\nu_g}{1+\bar{z}} \tag{3.29}$$

which reduces to the third degree equation

$$\bar{z}(1+\bar{z})^2 + A_q\nu_g^2 = 0\,; \qquad A_q := \frac{1}{2}\frac{\theta_{w1}}{\theta_1}\frac{\lambda_1 L_1}{D_1} \tag{3.30}$$

like in the case discussed in Subsection 3.1 In fact (3.7) and (3.27) have the same structure. For $A_q \nu_g^2 < 4/27$ equation (3.30) has three real roots, among which $\bar{z}_1 \in (-1/3, 0)$ is the acceptable one in applications. It is interesting to give computed data for the aforementioned inequality. Using the same data for the two already mentioned hydroelectric power plants of Romania, we shall have $A_q \approx 0.073$ for "Bicaz" and $A_q \approx 0.032$ for "Someș-Mărișelu". Since $0 < \nu_g < 1$ and $4/27 \approx 0.148$, the fulfilment of $A_q \nu_g^2 < 4/27$ is obvious.

Model (3.27) having the same structure as (3.7), we can introduce again the deviations (3.10) which are subject to the system in deviations – much alike to (3.11)

$$\begin{aligned} \frac{\mathrm{d}\zeta}{\mathrm{d}\tau} &= v, \\ \theta_s \frac{\mathrm{d}v}{\mathrm{d}\tau} &= \frac{\nu_g}{(1+\bar{z}_1+\zeta)^2} v \\ &\quad - \frac{1}{\theta_{w1}} \left[\bar{z}_1 + \zeta + A_q \left| \theta_s v + \frac{\nu_g}{1+\bar{z}_1+\zeta} \right| \left(\theta_s v + \frac{\nu_g}{1+\bar{z}_1+\zeta} \right) + \lambda_s v \right]. \end{aligned} \tag{3.31}$$

The only differences in comparison to (3.11) are to have A_q instead of P_1' and $\lambda_s v$ instead of $\lambda_s |v| v$; the last difference is introduced as a result of having a linear throttling model. Associate the same (in fact) Lyapunov function

$$\mathcal{V}(\zeta, v) = \frac{1}{2}\theta_s v^2 + \frac{1}{2\theta_{w1}} \left[1 - 2A_q \frac{\nu_g^2}{(1+\bar{z}_1)^2(1+\bar{z}_1+\zeta)} \right] \zeta^2 \tag{3.32}$$

which is strictly positive definite in the strip $\zeta > -(1+3\bar{z}_1)$ containing the equilibrium $(0,0)$. Computing the derivative function

$$\mathcal{W}(\zeta, v) = -\left[\frac{\theta_s}{\theta_{w1}} A_q \left(\theta_s v + \frac{2\nu_g}{1+\bar{z}_1+\zeta} \right) - \frac{\nu_g}{(1+\bar{z}_1+\zeta)^2} + \frac{\lambda_s}{\theta_{w1}} \right] v^2$$

valid in the phase plane domain $q_1 > 0$ i.e. $\theta_s v + \nu_g (1+\bar{z}_1+\zeta)^{-1} > 0$, we remark that the "helpful" term $\lambda_s/\theta_{w1} > 0$ is now constant everywhere in the phase plane. The necessary condition of (3.19) type is now

$$2\frac{\theta_s}{\theta_{w1}} A_q \frac{\nu_g}{1+\bar{z}_1} - \frac{\nu_g}{(1+\bar{z}_1)^2} + \frac{\lambda_s}{\theta_{w1}} > 0 \tag{3.33}$$

and it is relaxed in comparison to (3.19), because of the term $\lambda_s/\theta_{w1} > 0$. Condition (3.24) also can be relaxed since now we shall have

$$\Phi(\zeta, v) = v - \frac{\theta_{w1}}{\theta_s^2 A_q} \frac{\nu_g}{1+\bar{z}_1+\zeta} \left[\frac{1}{1+\bar{z}_1+\zeta} - \frac{2}{\theta_s} \right] + \frac{\lambda_s}{\theta_s^2 A_q} > 0 \tag{3.34}$$

the term $\lambda_s (\theta_s^2 A_q)^{-1}$ being again helpful.

Consider now the model (3.28). While the model (3.27) relies on the load curve $q_2 = \nu_g (1 + z)^{-1}$ which is inferred while accepted by the hydraulic engineering community, model (3.28) is obtained partly from (2.3)–(2.4) and this imposes a dynamic load curve defined by $q_2 = \nu_g (1 + z + \lambda_s \mathrm{d}z/\mathrm{d}\tau)^{-1}$. This model is not *homologated* within the hydraulic engineering community, possibly because for $\lambda_s = 0$ (surge tank without throttling) the two models coincide, also because for most surge tanks the throttling effect is neglected ($\lambda_s \approx 0$ in real data).

Constructing a Lyapunov function for (3.28) is a new and distinct problem which is outside the mainstream of the present paper, dealing with a model which has to be adopted as such.

As a preliminary analysis, we can however consider (3.28) written as

$$\begin{aligned}
&\theta_{w1}\frac{\mathrm{d}q_1}{\mathrm{d}\tau}+\frac{1}{2}\frac{\theta_{w1}}{\theta_1}\frac{\lambda_1 L_1}{D_1}|q_1|q_1+z(\tau)+\lambda_s\frac{\mathrm{d}z}{\mathrm{d}\tau}=0,\\
&\theta_s\frac{\mathrm{d}z}{\mathrm{d}\tau}=q_1-\frac{\nu_g}{1+z}+\frac{\nu_g\lambda_s}{(1+z)(1+z+\lambda_s\mathrm{d}z/\mathrm{d}\tau)}\frac{\mathrm{d}z}{\mathrm{d}\tau}.
\end{aligned}\tag{3.35}$$

The last term in the second equation of (3.35) is a *persistent perturbation*. It is thus possible to search for the total stability [30,40] i.e. stability with respect to persistent perturbations [13,23]. The basic result on stability under persistent perturbations (total stability) is from 1944 and is due to Malkin – see [23], pages 301 and next, also [30] (Theorems II.4.4 and II.4.5), [40], pages 118 and next.

The assumptions of these basic results of Malkin are fulfilled by (3.35) and its associated Lyapunov function (3.32) – re-written in the state variables of (3.27), (3.28), (3.35), provided they are considered on bounded domains of the state space (e.g. of the form $\mathcal{V}(\zeta,v)\le c$). Obviously the attraction domain of the origin under persistent perturbations is rather small. Improvements can be sought using more refined results on perturbed dynamical systems [4–7,34–37]

4 Water hammer stability analysis

For this analysis we shall start from the mathematical model (2.3) under the assumptions of [15,27]. The basic one is to neglect the Darcy–Weisbach losses along the two water conduits (the tunnel and the penstock). From the engineering point of view, with an argument at the physical level of rigor, this assumption is covering: water hammer oscillation quenching is connected to energy dissipation and the analysis is done precisely without certain energy dissipation terms. Next, neglecting the dynamic (velocity) heads variation $\partial_{\xi_i}q_i^2$ is standard in hydraulic engineering – see all the cited "hydraulic" literature – and we shall not elaborate on this assumptions. Therefore the starting model will be now the following

$$\begin{aligned}
&\theta_{wi}\partial_\tau q_i+\partial_{\xi_i}h_i=0,\quad \frac{\theta_{pi}^2}{\theta_{wi}}\partial_\tau h_i+\partial_{\xi_i}q_i=0;\quad i=1,2,\\
&h_1(0,\tau)\equiv 1,\quad h_1(1,\tau)=1+z(\tau)+\lambda_s\frac{\mathrm{d}z}{\mathrm{d}\tau}=h_2(0,\tau),\\
&q_2(1,\tau)=(1-k)f_\theta(\tau)\sqrt{h_2(1,\tau)}+k\varphi(\tau),\\
&\theta_s\frac{\mathrm{d}z}{\mathrm{d}\tau}=q_1(1,\tau)-q_2(0,\tau),\quad \theta_a\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}=q_2(1,\tau)h_2(1,\tau)-\nu_g.
\end{aligned}\tag{4.1}$$

This form of the equations will turn to be helpful for the basic theory for (4.1).

The ignition of the water hammer takes place as follows: system (4.1) starts from a normal steady state defined by

$$\begin{aligned}
&\bar{h}_i(\xi_i)\equiv\text{const},\quad \bar{q}_i(\xi_i)\equiv\text{const}\\
&\bar{h}_1(0)=1,\quad \bar{h}_1(1)=1+\bar{z}=\bar{h}_2(0)\quad\Rightarrow\quad \bar{z}=0,\quad \bar{h}_1\equiv 1,\quad \bar{h}_2\equiv 1\\
&\bar{q}_1=\bar{q}_2=\bar{q},\quad \bar{q}=(1-k)\bar{f}_\theta+k\bar{\varphi};\quad \bar{q}=\nu_g
\end{aligned}\tag{4.2}$$

The steady state value of $\bar{\varphi}$ – the rotating speed of the hydraulic turbine – is imposed by the frequency/megawatt control of the Grid (together with the power level ν_g) and is ensured by the speed controller of the turbine.

From this steady state the system is moved to an abnormal operation by turbine shutdown $f_\theta \equiv 0$. Usually the case $k = 0$ is considered; the case $k \neq 0$ [3] is somehow unusual and we do not know if k does change during turbine shutdown. If k is kept at its previous value and the turbine is not unloaded instantaneously then the tendency will be to have the steady state (4.2) but with $\bar{\varphi} = \nu_g/k$. The turbine is probably unloaded before this steady state is reached (provided it is asymptotically stable); nevertheless the stability of the steady state defined by $\bar{f}_\theta = 0$, $\bar{\varphi} = \nu_g/k$ – starting from (4.2) is interesting in itself and its study was not undertaken (prior to our knowledge).

We shall however deal with basic theory for (4.1) with $\bar{f}_\theta(\tau) \equiv 0$ in order to deal with a linear Boundary value problem of nonstandard type. We call it nonstandard because its boundary conditions (of Dirichlet type) are controlled by ordinary differential equations which at their turn are controlled by the boundary conditions (an internal feedback).

4.1 Basic theory

We shall take the approach arising from the papers of A.D. Myshkis [1] and K.L. Cooke [8] (this one summarized and completely proven in [28]). This approach consists in associating to (4.1) a system of functional differential equations with deviated argument and establishing a one to one correspondence between the solutions of the two mathematical objects. As a consequence, *any property proven for one mathematical object is thus projected back on the other.*

We shall thus turn to (4.1) with $f_\theta(\tau) \equiv 0$. Introduce first the Riemann invariants $r_i^\pm(\xi_i, \tau)$ by

$$r_i^\pm(\xi_i,\tau) = \frac{1}{2}\left[\frac{\theta_{pi}}{\theta_{wi}}h_i(\xi_i,\tau) \pm q_i(\xi_i,\tau)\right] \tag{4.3}$$

with their inverses

$$h_i(\xi_i,\tau) = \frac{\theta_{wi}}{\theta_{pi}}\left[r_i^+(\xi_i,\tau) + r_i^-(\xi_i,\tau)\right], \quad q_i(\xi_i,\tau) = r_i^+(\xi_i,\tau) - r_i^-(\xi_i,\tau). \tag{4.4}$$

Consequently the boundary value problem (4.1) – with $f_\theta(\tau) \equiv 0$ – will be written with respect to the Riemann invariants as follows

$$\begin{aligned}
&\theta_{pi}\partial_\tau r_i^\pm \pm \partial_{\xi_i} r_i^\pm = 0, \quad i = 1,2,\\
&r_1^+(0,\tau) + r_1^-(0,\tau) = \frac{\theta_{p1}}{\theta_{w1}},\\
&\frac{\theta_{w1}}{\theta_{p1}}[r_1^+(1,\tau) + r_1^-(1,\tau)] = 1 + z(\tau) + \lambda_s\frac{\mathrm{d}z}{\mathrm{d}\tau} = \frac{\theta_{w2}}{\theta_{p2}}[r_2^+(0,\tau) + r_2^-(0,\tau)],\\
&\theta_s\frac{\mathrm{d}z}{\mathrm{d}\tau} = r_1^+(1,\tau) - r_1^-(1,\tau) - r_2^+(0,\tau) + r_2^-(0,\tau),\\
&r_2^+(1,\tau) - r_2^-(1,\tau) = k\varphi(\tau),\\
&\theta_a\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = k\frac{\theta_{w2}}{\theta_{p2}}[r_2^+(1,\tau) + r_2^-(1,\tau)]\varphi - \nu_g.
\end{aligned} \tag{4.5}$$

In the strip $[0,1] \times \mathbb{R}$ we define the characteristic lines crossing $(\xi_i,\tau) \in [0,1] \times \mathbb{R}^+$, $i = 1,2$

$$\tau_i^\pm(\sigma;\xi_i,\tau) = \tau \pm \theta_{pi}(\sigma - \xi_i), \quad i = 1,2. \tag{4.6}$$

We now make use of the following properties of the characteristic lines and of the Riemann invariants along them: a) any increasing characteristic τ_i^+ can be extended "to the right" up to $\xi_i = 1$ and any decreasing characteristic τ_i^- can be extended "to the left" up to $\xi_i = 0$; b) the Riemann invariant r_i^+ (the forward wave) is constant along the increasing characteristic τ_i^+ and the Riemann invariant r_i^- (the backward wave) is constant along the decreasing characteristic τ_i^-. Consequently, the following representation formulae for the Riemann invariants, based on their boundary values, are obtained

$$r_i^+(\xi_i,\tau) = r_i^+(1,\tau+\theta_{pi}(1-\xi_i)), \quad r_i^-(\xi_i,\tau) = r_i^-(0,\tau+\theta_{pi}\xi)\ , \quad i = 1,2. \tag{4.7}$$

Consider now those characteristics which can be extended on $(0,1)$ – r_i^+ "to the left" and r_i^- "to the right" – to obtain, after denoting $y_i^+(\tau) := r_i^+(1,\tau)$, $y_i^-(\tau) := r_i^-(0,\tau)$

$$\begin{aligned} r_i^+(0,\tau) &= r_i^+(1,\tau+\theta_{pi}) = y_i^+(\tau+\theta_{pi}), \\ r_i^-(1,\tau) &= r_i^-(0,\tau+\theta_{pi}) = y_i^+(\tau+\theta_{pi}). \end{aligned} \tag{4.8}$$

The functions $y_i^{\pm}(\tau)$ are then substituted in the boundary conditions of (4.5) to obtain

$$\begin{aligned} & y_1^+(\tau+\theta_{p1}) + y_1^-(\tau) = \frac{\theta_{p1}}{\theta_{w1}}, \\ & \frac{\theta_{w1}}{\theta_{p1}}(y_1^+(\tau) + y_1^-(\tau+\theta_{p1})) = 1 + z(\tau) + \lambda_s \frac{\mathrm{d}z}{\mathrm{d}\tau} = \frac{\theta_{w2}}{\theta_{p2}}(y_2^+(\tau+\theta_{p2}) + y_2^-(\tau)), \\ & \theta_s \frac{\mathrm{d}z}{\mathrm{d}\tau} = y_1^+(\tau) - y_1^-(\tau+\theta_{p1}) - y_2^+(\tau+\theta_{p2}) + y_2^-(\tau), \\ & y_2^+(\tau) - y_2^-(\tau+\theta_{p2}) = k\varphi(\tau), \\ & \theta_a \frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = k\frac{\theta_{w2}}{\theta_{p2}}[y_2^+(\tau) + y_2^-(\tau+\theta_{p2})]\varphi - \nu_g. \end{aligned} \tag{4.9}$$

We introduce now the new functions $w_i^{\pm}(\tau) := y_i^{\pm}(\tau+\theta_{pi})$ to give (4.9) a form which is more "at hand" in the study of the systems with deviated argument

$$\begin{aligned} & w_1^+(\tau) + w_1^-(\tau-\theta_{p1}) = \frac{\theta_{p1}}{\theta_{w1}}, \\ & \frac{\theta_{w1}}{\theta_{p1}}(w_1^-(\tau) + w_1^+(\tau-\theta_{p1})) = 1 + z(\tau) + \lambda_s \frac{\mathrm{d}z}{\mathrm{d}\tau} = \frac{\theta_{w2}}{\theta_{p2}}(w_2^+(\tau) + w_2^-(\tau-\theta_{p2})), \\ & \theta_s \frac{\mathrm{d}z}{\mathrm{d}\tau} = w_1^+(\tau-\theta_{p1}) - w_1^-(\tau) - w_2^+(\tau) + w_2^-(\tau-\theta_{p2}), \\ & w_2^-(\tau) - w_2^+(\tau-\theta_{p2}) = -k\varphi(\tau), \\ & \theta_a \frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = k\frac{\theta_{w2}}{\theta_{p2}}[w_2^-(\tau) + w_2^+(\tau-\theta_{p2})]\varphi - \nu_g. \end{aligned} \tag{4.10}$$

The differential and difference system (4.10) should be expressed in a form allowing the construction by steps of its solution. Our main concern is the two boundary conditions corresponding to those of the surge tank, namely

$$\begin{aligned}\frac{\theta_{w1}}{\theta_{p1}}(w_1^-(\tau)+w_1^+(\tau-\theta_{p1})) &= z(\tau)+\frac{\lambda_s}{\theta_s}(w_1^+(\tau-\theta_{p1})-w_1^-(\tau)\\ &\quad -w_2^+(\tau)+w_2^-(\tau-\theta_{p2})),\\ \frac{\theta_{w2}}{\theta_{p2}}(w_2^+(\tau)+w_2^-(\tau-\theta_{p2})) &= z(\tau)+\frac{\lambda_s}{\theta_s}(w_1^+(\tau-\theta_{p1})-w_1^-(\tau)\\ &\quad -w_2^+(\tau)+w_2^-(\tau-\theta_{p2})).\end{aligned} \tag{4.11}$$

Denoting for the simplicity of the writing $\delta_i := \theta_{pi}/\theta_{wi}$, $\lambda_s' := \lambda_s/\theta_s$, we obtain, after a straightforward manipulation including the inversion of a 2×2 matrix, the following system of coupled delay differential and difference equations

$$\begin{aligned}\theta_s\frac{\mathrm{d}z}{\mathrm{d}\tau} &= \frac{1}{1+(\delta_1+\delta_2)\lambda_s'}[-(\delta_1+\delta_2)z(\tau)+2w_1^+(\tau-\theta_{p1})+2w_2^-(\tau-\theta_{p2})],\\ \delta_2\theta_a\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} &= k(2w_2^+(\tau-\theta_{p2})-k\varphi)\varphi-\nu_g,\\ w_1^+(\tau) &= \delta_1-w_1^-(\tau-\theta_{p1});\quad w_2^-(\tau)=w_2^+(\tau-\theta_p2)-k\varphi(\tau),\\ w_1^-(\tau) &= \frac{1}{1+(\delta_1+\delta_2)\lambda_s'}[\delta_1 z(\tau)-(1+(\delta_2-\delta_1)\lambda_s')w_1^+(\tau-\theta_{p1})\\ &\quad +2\delta_1\lambda_s' w_2^-(\tau-\theta_{p2})],\\ w_2^+(\tau) &= \frac{1}{1+(\delta_1+\delta_2)\lambda_s'}[\delta_2 z(\tau)+2\delta_2\lambda_s' w_1^+(\tau-\theta_{p1})\\ &\quad -(1+(\delta_1-\delta_2)\lambda_s')w_2^-(\tau-\theta_{p2})]\end{aligned} \tag{4.12}$$

Observe that the equation of φ is a Riccati equation and this might ignite finite time escape. Now, the solution of (4.12) can be constructed by steps provided initial conditions are given on $(-\theta_{pi},0)$ for $w_i^\pm(\tau)$, $i=1,2$. If $\varphi(0),z(0)$ are given, as well as $w_i^\pm(\tau)$ on $(-\theta_{pi},0)$, then $(\varphi(\tau),z(\tau))$ can be obtained on $(0,\theta_{pi})$. Next, using the initial data and $(\varphi(\tau),z(\tau)$ on $(0,\theta_{pi})$, $w_i^\pm(\tau)$ can be obtained on $(0,\theta_{pi})$ from the difference equations. The process is then iterated on the following interval. The resulting solution appears to be continuous and piecewise differentiable – the state variables z and φ – while $w_i^\pm$ have the smoothness of their initial conditions and, in general, have finite discontinuities ("jumps") in $\tau=m_1\theta_{p1}+m_2\theta_{p2}$, where m_i are integers. It is also quite clear that the solution can be constructed also backwards.

All this construction is conditioned by the knowledge of the initial conditions $w_{io}^\pm(\tau)$, $-\theta_{pi}\le\tau<0$, $i=1,2$. These initial conditions can be obtained starting from the initial conditions of (4.3): starting from the initial conditions of (4.1) namely $(q_i^o(\xi_i),h_i^o(\xi_i))$ given on $(0,1)$, we use (4.3) to obtain $r_{io}^\pm(\xi_i)$ on $(0,1)$.

Consider those points (ξ_i,τ) which are such that the characteristic $\tau_i^+(\sigma;\xi_i,\tau)$ cannot be extended "to the left" up to $\xi_i=0$ but only to the point where $\tau+\theta_{pi}(\sigma-\xi_i)=0$ i.e. up to $\sigma=\xi_i-\tau/\theta_{pi}$. It follows that

$$r_i^+(\xi_i-\tau/\theta_{pi},0)=r_i^+(1,\tau+\theta_{pi}(1-\xi_i))=w_i^+(\tau-\theta_{pi}\xi_i).$$

Since $0\le\xi_i-\tau/\theta_{pi}\le 1$, it follows that $w_{io}^+(\theta)=r_{io}^+(-\theta/\theta_{pi})$ with $-\theta_{pi}\le\theta\le 0$. In the same way, using those characteristic lines $\tau_i^-(\sigma;\xi_i,\tau)$ which cannot be extended to $\sigma=1$ but only to the point where $\tau-\theta_{pi}(\sigma-\xi_i)=0$, i.e. to $\sigma=\xi_i+\tau/\theta_{pi}$, the following initial condition is obtained

$$r_i^-(\xi_i+\tau/\theta_{pi},0)=r_i^-(0,\tau+\theta_{pi}\xi_i)=w_i^-(\tau+\theta_{pi}(\xi_i-1)),$$

hence $w_{io}^-(\theta) = r_{io}^-(1+\theta/\theta_{pi})$ with $-\theta_{pi} \leq \theta \leq 0$.

Consider now the converse: let $\{z(0),\varphi(0),w_{io}^{\pm}(\theta),-\theta_{pi} \leq \theta \leq 0\}$ be a set of initial conditions for (4.12). The procedure by steps allows construction of the corresponding solution for (4.12). Define

$$\begin{aligned} & r_i^+(\xi_i,\tau) = w_i^+(\tau - \theta_{pi}\xi_i) \;,\; r_i^-(\xi_i,\tau) = w_i^-(\tau + \theta_{pi}(\xi_i - 1)), \\ & h_i(\xi_i,\tau) = \frac{1}{\delta_i}[w_i^+(\tau - \theta_{pi}\xi_i) + w_i^-(\tau + \theta_{pi}(\xi_i - 1))], \\ & q_i(\xi_i,\tau) = w_i^+(\tau - \theta_{pi}\xi_i) - w_i^-(\tau + \theta_{pi}(\xi_i - 1)). \end{aligned} \tag{4.13}$$

Then, if $w_{io}^{\pm}(\theta)$ are sufficiently smooth, the set of functions $\{z(\tau),\varphi(\tau);h_i(\xi_i,\tau),q_i(\xi_i,\tau)\}$ is a (possibly discontinuous) classical solution of (4.1) with the initial conditions $\{z(0),\varphi(0); h_i(\xi_i,0),q_i(\xi_i,0)\}$. Summarizing, the following result has been obtained and proven.

Theorem 4.1. *Consider the boundary value problem defined by (4.1) with $f_\theta(\tau) \equiv 0$ and a set of initial conditions $\{z(0),\varphi(0);h_i^o(\xi_i),q_i^o(\xi_i),0 \leq \xi_i \leq 1,i = 1,2\}$ with $\{h_i^o,q_i^o\}$ sufficiently smooth to define a classical solution for (4.1). Let $r_i^{\pm}(\xi_i,\tau)$ – defined by (4.3) – be the corresponding Riemann invariants of this solution. Let $y_i^{\pm}(\tau)$ defined by (4.8) and $w_i^{\pm}(\tau) := y_i^{\pm}(\tau+\theta_{pi})$. Then $\{z(\tau),\varphi(\tau);w_i^{\pm}(\tau)\}$ is a solution of (4.12) with the initial conditions $\{z(0),\varphi(0);w_{io}^{\pm}(\tau),-\theta_{pi} \leq \tau \leq 0\}$ where $w_{io}^{\pm}(\tau)$ are obtained as defined above.*

Conversely, let $\{z(\tau),\varphi(\tau);w_i^{\pm}(\tau)\}$ be a solution of (4.12) defined by the initial conditions $\{z(0), \varphi(0);w_{io}^{\pm}(\tau),-\theta_{pi} \leq \tau \leq 0\}$ with $w_{io}^{\pm}(\tau)$ sufficiently smooth e.g. of class C^1. Then the set of functions $\{z(\tau),\varphi(\tau);h_i(\xi_i,\tau),q_i(\xi_i,\tau)\}$ with $\{h_i(\xi_i,\tau),q_i(\xi_i,\tau)\}$ defined by (4.13) is a (possibly discontinuous) solution of (4.1) with $f_\theta(\tau) \equiv 0$ and the initial conditions following by taking $\tau = 0$ in the aforementioned set of functions.

4.2 Steady state for the turbine shutdown

A. Consider $f_\theta(\tau) \equiv 0$ in (4.1) and let the time derivatives be zero: this will give the steady state at shutdown after water hammer (if the turbine is not unloaded and this steady state is stable). Its equations are given by (4.2) with $\bar{f}_\theta = 0$. Consider now (4.1) and, after taking $f_\theta(\tau) \equiv 0$, we introduce the deviations of the state variables with respect to the steady state

$$\begin{aligned} & \chi_i(\xi_i,\tau) = h_i(\xi_i,\tau) - 1, \quad \varpi_i(\xi_i,\tau) = q_i(\xi_i,\tau) - \bar{q} = q_i(\xi_i,\tau) - \nu_g; \\ & \zeta(\tau) \equiv z(\tau), \quad s(\tau) = \varphi(\tau) - \bar{\varphi} = \varphi(\tau) - \nu_g/k. \end{aligned} \tag{4.14}$$

Re-write now (4.1) with respect to the deviations

$$\begin{aligned} & \theta_{wi}\partial_\tau\varpi_i + \partial_{\xi_i}\chi_i = 0, \quad \frac{\theta_{pi}^2}{\theta_{wi}}\partial_\tau\chi_i + \partial_{\xi_i}\varpi_i = 0; \quad i = 1,2, \\ & \chi_1(0,\tau) = 0 \;,\; \chi_1(1,\tau) = z(\tau) + \lambda_s\frac{\mathrm{d}z}{\mathrm{d}\tau} = \chi_2(0,\tau), \\ & \theta_s\frac{\mathrm{d}z}{\mathrm{d}\tau} = \varpi_1(1,\tau) - \varpi_2(0,\tau); \quad \varpi_2(1,\tau) = ks(\tau), \\ & \theta_a\frac{\mathrm{d}s}{\mathrm{d}\tau} = \varpi_2(1,\tau)(1+\chi_2(1,\tau)) + \nu_g\chi_2(1,\tau). \end{aligned} \tag{4.15}$$

Except the last equation (of the hydraulic turbine), equations (4.15) are linear. If the standard dependence flow – piezometric head is considered ($k = 0$), system (4.15) becomes fully linear since the boundary condition at $\xi_2 = 1$ becomes $\varpi_2(1,\tau) = 0$ and the equation for s is decoupled being thus independent. Due to linearity, the steady state is 0 but the basic steady state with respect to which the deviations (4.14) has no importance.

B. Suppose we continue with $k \neq 0$ but take into account that the turbine is unloaded simultaneously with shutdown ($f_\theta(\tau) \equiv 0$, $\nu_g = 0$). In this case the steady state of (4.1) is defined by

$$\bar{z} = 0, \quad \bar{h}_1 = 1\ , \bar{h}_2 = 1; \quad \bar{q}_1 = \bar{q}_2 = \bar{q}; \quad \bar{q} = k\bar{\varphi}, \quad \bar{q} = \nu_g = 0 \tag{4.16}$$

(see also (4.2)). Also the deviations are given by (4.14) with $\nu_g = 0$, $\bar{\varphi} = 0$. The system in deviations will be (4.15) but with $\varpi_2(1,\tau) = 0$ and without the equation for s which is decoupled and, therefore, not involved in the stability analysis under water hammer.

In what follows we shall consider only the case $k = 0$ i.e. of the decoupled turbine under water hammer. This option is motivated by the fact that the expression of $q_2(1,\tau)$ in (2.3) has to be connected to an adequate expression for the active torque of the turbine – see [3]. In fact the expression in (2.3) i.e. $q_2(1,\tau)h_2(1,\tau)$ is used with the boundary condition with $k = 0$. Moreover, the aforementioned expression for the torque in the case $k \neq 0$ is determined in steady state and its use during transients might be questionable.

4.3 The energy identity for stability analysis

We start from the energy identity as deduced following e.g. [12]

$$\frac{1}{2} \cdot \frac{\mathrm{d}}{\mathrm{d}\tau} \int_0^1 \left(\theta_{wi}\varpi_i(\xi_i,\tau)^2 + \frac{\theta_{pi}^2}{\theta_{wi}}\chi_i(\xi_i,\tau)^2 \right) \mathrm{d}\xi_i + \varpi_i(\xi_i,\tau)\chi_i(\xi_i,\tau)|_0^1 \equiv 0. \tag{4.17}$$

The energy identity suggests the following Lyapunov functional, written along the solutions of (4.15) with $k = 0$

$$\mathcal{V}(z(\tau),\varpi_i(\cdot,\tau),\chi_i(\cdot,\tau)) = \frac{1}{2}\left\{ \theta_s z(\tau)^2 + \sum_1^2 \theta_{wi} \int_0^1 \left[\varpi_i(\xi_i,\tau)^2 + \delta_i^2 \chi_i(\xi_i,\tau)^2\right] \mathrm{d}\xi_i \right\}. \tag{4.18}$$

We differentiate (4.18) with respect to (4.15), taking into account the boundary conditions. After a straightforward manipulation we obtain

$$\mathcal{W}(\varpi_1(\cdot,\tau),\varpi_2(\cdot,\tau)) = -\frac{\lambda_s}{\theta_s}(\varpi_1(1,\tau) - \varpi_2(0,\tau))^2 = -\lambda_s' \left(\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^2 \leq 0. \tag{4.19}$$

It is clear that (4.18) and (4.19) imply Lyapunov stability of the zero solution of (4.15) in the sense of the metrics induced by the Lyapunov function (4.18):

$$\mathcal{V}(z,\varpi_i(\cdot,\tau),\chi_i(\cdot,\tau)) \leq \mathcal{V}(z,\varpi_i^o(\cdot),\chi_i^o(\cdot)). \tag{4.20}$$

It remains now to discuss asymptotic stability. Since $\mathcal{W}$ is only non-positive definite, application of the invariance principle Barbashin–Krasovskii–LaSalle is required. This principle is known to be valid for neutral functional differential equations [18]; therefore we make use of Theorem 4.1 and consider the system of functional differential equations (4.12) – but associated to the system in deviations (4.15) with $k = 0$

$$\begin{aligned}
\theta_s \frac{\mathrm{d}z}{\mathrm{d}\tau} &= \frac{1}{1 + (\delta_1 + \delta_2)\lambda_s'}[-(\delta_1 + \delta_2)z(\tau) - 2\eta_1^-(\tau - 2\theta_{p1}) + 2\eta_2^+(\tau - 2\theta_{p2})], \\
\eta_1^-(\tau) &= \frac{1}{1 + (\delta_1 + \delta_2)\lambda_s'}[\delta_1 z(\tau) + (1 + (\delta_2 - \delta_1)\lambda_s')\eta_1^-(\tau - 2\theta_{p1}) + 2\delta_1\lambda_s'\eta_2^+(\tau - 2\theta_{p2})], \\
\eta_2^+(\tau) &= \frac{1}{1 + (\delta_1 + \delta_2)\lambda_s'}[\delta_2 z(\tau) - 2\delta_2\lambda_s'\eta_1^-(\tau - 2\theta_{p1}) - (1 + (\delta_1 - \delta_2)\lambda_s')\eta_2^+(\tau - 2\theta_{p2})]
\end{aligned} \tag{4.21}$$

(for stability analysis, where large τ are concerned, one can consider $\tau > 2\max\{\theta_{p1},\theta_{p2}\}$ and the variables η_1^+, η_2^- can be substituted from their equations in the remaining ones).

Consider now the representation formulae (4.13) written for the system in deviations

$$\begin{aligned}\chi_i(\xi_i,\tau) &= \frac{1}{\delta_i}[\eta_i^+(\tau-\theta_{pi}\xi_i)+\eta_i^-(\tau+\theta_{pi}(\xi_i-1))],\\ \varpi_i(\xi_i,\tau) &= \eta_i^+(\tau-\theta_{pi}\xi_i)-\eta_i^-(\tau+\theta_{pi}(\xi_i-1))]\end{aligned}\tag{4.22}$$

to re-write $\mathcal{V}$ and $\mathcal{W}$

$$\begin{aligned}\chi_1(\xi_1,\tau) &= \frac{1}{\delta_1}[\eta_1^-(\tau+\theta_{p1}(\xi_1-1))-\eta_1^-(\tau-\theta_{p1}(\xi_1+1))],\\ \varpi_1(\xi_1,\tau) &= -\eta_1^-(\tau+\theta_{p1}(\xi_1-1))-\eta_1^-(\tau-\theta_{p1}(\xi_1+1)),\\ \chi_2(\xi_2,\tau) &= \frac{1}{\delta_2}[\eta_2^+(\tau-\theta_{p2}\xi_2)+\eta_2^+(\tau+\theta_{p2}(\xi_2-2))],\\ \varpi_2(\xi_2,\tau) &= \eta_2^+(\tau-\theta_{p2}\xi_2)-\eta_2^+(\tau+\theta_{p2}(\xi_2-2)).\end{aligned}\tag{4.23}$$

Therefore

$$\begin{aligned}&\mathcal{V}(z(\tau),\eta_1^-(\cdot,\tau),\eta_2^+(\cdot,\tau))\\ &\quad=\frac{1}{2}\left\{\theta_s z(\tau)^2+\theta_{w1}\int_{-2\theta_{p1}}^0\eta_1^-(\tau+\lambda)^2\mathrm{d}\lambda+\theta_{w2}\int_{-2\theta_{p2}}^0\eta_2^+(\tau+\lambda)^2\mathrm{d}\lambda\right\},\\ &\mathcal{W}(\eta_1^-(\cdot,\tau),\eta_2^+(\cdot,\tau)),\\ &\quad=-\lambda_s'[-\eta_1^-(\tau)-\eta_1^-(\tau-2\theta_{p1})-\eta_2^+(\tau)+\eta_2^+(\tau-2\theta_{p2})]^2=-\lambda_s'\left(\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^2\le 0.\end{aligned}\tag{4.24}$$

Since $\mathcal{W}\le 0$ we shall try to apply the invariance Barbashin–Krasovskii–LaSalle principle hence we seek first for the largest invariant set with respect to the solutions of (4.21), contained in the set where $\mathcal{W}=0$. Since $\mathrm{d}z/\mathrm{d}\tau=0$ we deduce from (4.21)

$$z(\tau)=\frac{2}{\delta_1+\delta_2}[-\eta_1^-(\tau-2\theta_{p1})+\eta_2^+(\tau-2\theta_{p2})]$$

and substitute $z(\tau)$ in the remaining difference equations; a simple manipulation will show that on the set where $\mathcal{W}=0$ the system is restricted to

$$\begin{aligned}\eta_1^-(\tau) &= \frac{\delta_2-\delta_1}{\delta_1+\delta_2}\eta_1^-(\tau-2\theta_{p1})+\frac{2\delta_1}{\delta_1+\delta_2}\eta_2^+(\tau-2\theta_{p2}),\\ \eta_2^+(\tau) &= -\frac{2\delta_2}{\delta_1+\delta_2}\eta_1^-(\tau-2\theta_{p1})-\frac{\delta_1-\delta_2}{\delta_1+\delta_2}\eta_2^+(\tau-2\theta_{p2}).\end{aligned}\tag{4.25}$$

The invariant solutions with respect to τ are the constant solutions. The non-zero determinant

$$\begin{vmatrix}1-\frac{\delta_2-\delta_1}{\delta_1+\delta_2} & -\frac{2\delta_1}{\delta_1+\delta_2}\\ \frac{2\delta_2}{\delta_1+\delta_2} & 1+\frac{\delta_1-\delta_2}{\delta_1+\delta_2}\end{vmatrix}=\frac{4\delta_1}{(\delta_1+\delta_2)}\neq 0$$

shows that the only invariant set located in the set where $\mathcal{W}=0$ is the origin $\{0;0,0\}$. Application of the Barbashin–Krasovskii–LaSalle invariance principle [18], Theorem 9.8.2 will give asymptotic stability.

There exists however a restriction to the application of the Barbashin–Krasovskii–LaSalle invariance theorem – *the stability if not even the strong stability of the difference subsystem* of (4.21). We shall consider this problem as applied to the stability of (4.21) and, *via* Theorem 4.1, to (4.15) – both with $k = 0$. We make first the following notations

$$\rho_1 = \frac{1+(\delta_2-\delta_1)\lambda_s'}{1+(\delta_1+\delta_2)\lambda_s'}\,, \qquad \rho_2 = \frac{1+(\delta_1-\delta_2)\lambda_s'}{1+(\delta_1+\delta_2)\lambda_s'} \tag{4.26}$$

to give (4.21) the following form

$$\begin{aligned} \theta_s\frac{\mathrm{d}z}{\mathrm{d}\tau} &= \frac{\rho_1+\rho_2}{2}[-(\delta_1+\delta_2)z - 2\eta_1^-(\tau-2\theta_{p1}) + 2\eta_2^+(\tau-2\theta_{p2})],\\ \eta_1^-(\tau) &= \frac{\rho_1+\rho_2}{2}\delta_1 z(\tau) + \rho_1\eta_1^-(\tau-2\theta_{p1}) + (1-\rho_1)\eta_2^+(\tau-2\theta_{p2}),\\ \eta_2^+(\tau) &= \frac{\rho_1+\rho_2}{2}\delta_2 z(\tau) - (1-\rho_2)\eta_1^-(\tau-2\theta_{p1}) - \rho_2\eta_2^+(\tau-2\theta_{p2}). \end{aligned} \tag{4.27}$$

The restriction in applying Theorem 9.8.2 of [18], page 293 thus refers to (asymptotic) stability of a difference system having the form

$$y(t) = \sum_1^p A_k y(t-r_k), \qquad t \geq 0. \tag{4.28}$$

In order to make the development which follows more clear, we shall recall in brief certain development of [18], Section 9.3, the part tackling difference equations and operators – pp. 274–276. With the notations of *op. cit.*, we consider the homogeneous and non-homogeneous difference equations

$$\mathrm{D}y_t = 0\,, \quad t \geq 0; \qquad \mathrm{D}y_t = h(t), \quad t \geq 0, \tag{4.29}$$

where $h \in \mathcal{C}([0,\infty);\mathbb{R}^n)$ and the difference operator $\mathrm{D} : \mathcal{C}(-r,0;\mathbb{R}^n) \mapsto \mathbb{R}^n$ is continuous and atomic at 0 being thus defined as

$$\mathrm{D}\phi = \phi(0) - \int_{-r}^0 \mathrm{d}[\mu(\theta)]\phi(\theta). \tag{4.30}$$

In (4.30) the kernel $\mu : \mathbb{R} \mapsto \mathbb{R}^{n\times n}$ is measurable, normalized such that $\mu(\theta) \equiv 0$ for $\theta \geq 0$ and $\mu(\theta) \equiv \mu(-r)$ for $\theta \leq -r$; the kernel is continuous from the left and of bounded variation. The following assumption is supposed to hold for the kernel μ.

Assumption 4.2 (Assumption (J) of [18], page 271). *The entries μ_{ij} of μ have an atom before they become constant i.e. there is a t_{ij} such that $\mu_{ij}(t) \equiv \mu_{ij}(t_{ij}+0)$ for $t \geq t_{ij}$ and $\mu_{ij}(t_{ij}-0) \neq \mu_{ij}(t_{ij}+0)$.*

This assumption is particularly true for (4.28) where $\mu(\theta)$ is reduced to a stepwise function. Let $\Delta_0(\lambda)$ defined below be the characteristic function of (4.30)

$$\Delta_0(\lambda) = \det\left(I - \int_{-r}^0 \mathrm{e}^{\lambda\theta}\mathrm{d}[\mu(\theta)]\right) \tag{4.31}$$

which in the case of (4.28) reads

$$\Delta_0(\lambda) = \det\left(I - \sum_1^p A_k \mathrm{e}^{-\lambda r_k}\right) \tag{4.32}$$

Let $a_D := \sup\{\Re e(\lambda) \mid \Delta_0(\lambda) = 0\}$. We state firstly

Definition 4.3 (Definition 9.3.1 of [18], page 275)**.** Suppose D is linear, continuous and atomic at 0 – see (4.30). The operator D is said to be stable if the zero solution of the homogeneous equation of (4.29) with the initial condition $\psi \in \mathcal{C}(-r,0;\mathbb{R}^n)$ subject to $\mathrm{D}\psi = 0$ is asymptotically stable

The following result [18], p. 275, concerns the aforementioned stability property.

Theorem 4.4 (Theorem 9.3.5 of [18], p. 275)**.** *The following statements are equivalent*

(i) D *is (asymptotically) stable in the sense of Definition 4.3.*

(ii) $a_D < 0$.

(iii) There exists constants $\alpha > 0$ *and* $\gamma(\alpha) > 0$ *such that for any* $h \in \mathcal{C}([0,\infty);\mathbb{R}^n)$ *any solution of the non-homogeneous equation of* (4.29) *satisfies*

$$|y(\psi,h)(t)| \leq \gamma(\alpha)[|\psi|\mathrm{e}^{-\alpha t} + \sup_{0\leq s\leq t} |h(s)|].$$

(iv) If D *is given by* (4.31) *with* $\lim_{s\to 0}\mathrm{Var}_{[-s,0]}\mu = 0$ *and* μ *is also subject to Assumption 4.2 – Assumption (J), then there exists a* $\delta > 0$ *such that all roots of the characteristic equation*

$$\Delta_0(\lambda) := \det\left(I - \int_{-r}^{0} \mathrm{e}^{\lambda\theta}\mathrm{d}[\mu(\theta)]\right) = 0$$

satisfy $\Re e(\lambda) \leq -\delta < 0$.

Observe that (iii) shows that (asymptotic) stability of D in the sense of Definition 4.3 is equivalent to exponential stability (the principle of K. P. Persidskii). Therefore stability of (4.28) means in fact *exponential stability*. Moreover, (iv) shows – along the same line – that $a_D < 0$ ensures that the roots of the characteristic equation (4.32) *have their real parts well delimited from* 0. In fact Theorem 4.1 states *equivalence of (apparently) weak properties with other, stronger ones*.

Turning to (4.29), its (asymptotic, exponential) stability is equivalent to the location of the roots of the characteristic equation $\Delta_0(\lambda) = 0$ with $\Delta_0(\lambda)$ given by (4.32) in the open left half plane $\mathbb{C}^-$. But for difference operators there exists another property called *strong stability*. This property is introduced also in [18], Section 9.6, for difference operators occurring in (4.28) i.e. defined by

$$\mathrm{D}(r,A)\phi = \phi(0) - \sum_{1}^{p} A_k\phi(-r_k). \tag{4.33}$$

Observe that the difference operator (4.33) is a special case of (4.30) with μ containing only the stepwise component with a finite number of steps.

Let $r = \mathrm{col}(r_1,\ldots,r_p)$ be the vector of the delays $r_k > 0$, $\forall k$.

Definition 4.5 (Definitions 9.6.1 and 9.6.2 of [18], p. 285)**.** The operator $\mathrm{D}(r,A)$ is said to be stable locally in the delays if there is an open neighborhood $I(r) \subset \mathbb{R}^p_+$ of r such that $\mathrm{D}(v,A)$ is stable in the sense of Definition 4.3 for each $v \in I(r)$.

The operator $\mathrm{D}(r,A)$ is said to be stable globally in the delays (strongly stable) if it is stable for each $r \in \mathbb{R}^p_+$.

For *strong stability* the following result is true.

Theorem 4.6 (Theorem 9.6.1 of [18], p. 286)**.** *The following statements are equivalent.*

(i) For some $r \in \mathbb{R}_+^p$, $r = \mathrm{col}(r_1, \dots, r_p)$ with $r_k > 0$ rationally independent, $\mathrm{D}(r, A)$ is stable in the sense of Definition 4.3.

(ii) If $\gamma(B)$ is the spectral radius of a matrix B, then $\gamma_0(A) < 1$ where

$$\gamma_0(A) := \sup\left\{\gamma\left(\sum_1^p A_k e^{\imath\theta_k}\right) \,\middle|\, \theta_k \in [0, 2\pi),\ k = 1, 2, \dots, p\right\} \tag{4.34}$$

(iii) $\mathrm{D}(r, A)$ is stable locally in the delays in the sense of Definition 4.5.

(iv) $\mathrm{D}(r, A)$ is stable globally in the delays (strongly stable) in the sense of Definition 4.5.

We are now in position to consider the stability properties of the difference subsystem of (4.27) namely the linear difference subsystem

$$\begin{aligned} \eta_1^-(\tau) &= \rho_1\eta_1^-(\tau - 2\theta_{p1}) + (1-\rho_1)\eta_2^+(\tau - 2\theta_{p2}), \\ \eta_2^+(\tau) &= -(1-\rho_2)\eta_1^-(\tau - 2\theta_{p1}) - \rho_2\eta_2^+(\tau - 2\theta_{p2}). \end{aligned} \tag{4.35}$$

Therefore we shall have

$$A_1 = \begin{pmatrix} \rho_1 & 0 \\ -(1-\rho_2) & 0 \end{pmatrix}; \qquad A_2 = \begin{pmatrix} 0 & (1-\rho_1) \\ 0 & -\rho_2 \end{pmatrix}. \tag{4.36}$$

The characteristic equation of (4.35) results in

$$\left(1 - \rho_1 e^{-2\lambda\theta_{p1}}\right)\left(1 + \rho_2 e^{-2\lambda\theta_{p2}}\right) + (1-\rho_1)(1-\rho_2)e^{-2\lambda(\theta_{p1}+\theta_{p2})} = 0 \tag{4.37}$$

the two delays being, generally speaking, rationally independent; this equation ought to have its roots in a left half plane $\Re e(\lambda) \leq -\alpha_0 < 0$. Denoting

$$e^{2\lambda\theta_{p2}} =: s\,, \qquad \nu = \theta_{p1}/\theta_{p2}, \tag{4.38}$$

the aforementioned condition reduces to the condition for the equation

$$(s^\nu - \rho_1)(s + \rho_2) + (1-\rho_1)(1-\rho_2) = 0 \tag{4.39}$$

to have its roots with $|s| < 1$ (inside the unit disk of $\mathbb{C}$).

Let firstly $s = re^{\imath\varphi}$, $r > 1$. We deduce

$$\begin{aligned} |r^\nu e^{\imath\nu\varphi} - \rho_1|^2 \cdot |re^{\imath\varphi} + \rho_2|^2 &= \left(r^{2\nu} + \rho_1^2 - 2r^\nu\rho_1\cos\nu\varphi\right)\left(r^2 + \rho_2^2 + 2\rho_2\cos\varphi\right) \\ &> (r^\nu - \rho_1)^2(r - \rho_2)^2. \end{aligned}$$

Further

$$\begin{aligned} &(r^\nu - \rho_1)(r - \rho_2) > (1-\rho_1)(1-\rho_2) \\ &(r^\nu - \rho_1)(r + \rho_2) > (1-\rho_1)(1-\rho_2) + 2\rho_2(1-\rho_1) > (1-\rho_1)(1-\rho_2). \end{aligned}$$

It follows that (4.39) cannot have roots with $|s| > 1$.

Let now $\nu = p/q \in \mathbb{Q}$ – an irreducible ratio i.e. assume the delays to be rationally dependent. Equation (4.39) becomes by taking again $s := re^{\imath\varphi}$

$$(r^{p/q}e^{\imath p\varphi/q} - \rho_1)(re^{\imath\varphi} + \rho_2) + (1-\rho_1)(1-\rho_2) = 0. \tag{4.40}$$

Denote $(re^{\imath\varphi})^{1/q} =: z$. Consequently equation (4.40) reads

$$(z^p - \rho_1)(z^q + \rho_2) + (1-\rho_1)(1-\rho_2) = 0. \tag{4.41}$$

Observe that if $z := \rho e^{\imath\theta}$ then $\varphi = q\theta$ and since $\varphi \in [0, 2\pi)$ it follows that $\theta \in [0, 2\pi/q)$. Also if $r \leq 1$ then $\rho \leq 1$.

Let $\rho = 1$. Equation (4.41) becomes

$$(e^{\imath p\theta} - \rho_1)(e^{\imath q\theta} + \rho_2) + (1-\rho_1)(1-\rho_2) = 0; \qquad \theta \in [0, 2\pi/q). \tag{4.42}$$

This equation is equivalent to

$$\begin{aligned} &(1 - 2\rho_1 \cos p\theta + \rho_1^2)(1 + 2\rho_2 \cos q\theta + \rho_2^2) = (1-\rho_1)^2(1-\rho_2)^2, \\ &\frac{\sin p\theta}{\cos p\theta - \rho_1} + \frac{\sin q\theta}{\cos q\theta + \rho_2} = 0. \end{aligned} \tag{4.43}$$

(The modulus and phase equations.) The first (modulus) equation is re-written as

$$\left[1 + \frac{2\rho_1}{(1-\rho_1)^2}(1 - \cos p\theta)\right] \cdot \left[1 + \frac{2\rho_2}{(1-\rho_2)^2}(1 + \cos q\theta)\right] = 1 \tag{4.44}$$

and holds for the unique combination

$$\cos p\theta = 1 \ (p\theta = 2m\pi), \qquad \cos q\theta = -1 \ (q\theta = (2n+1)\pi); \qquad m, n \in \mathbb{N}. \tag{4.45}$$

From (4.45) it follows that $\theta = 2m\pi/p = (2n+1)\pi/q$. The first conclusion is that $p/q = (2m)(2n+1)^{-1}$. Therefore the modulus equation of (4.43) has a solution iff p/q is such that p is even and q is odd. Now, since $0 \leq \theta \leq 2\pi/q$, it follows that $m \leq p/q$ and $2n+1 \leq 2$ i.e. $n = 0$. *Therefore $\theta = \pi/q$ is the only possible solution of* (4.43). Observe that for this value – corresponding to $\varphi = \pi$ – the phase equation in (4.43) is automatically fulfilled. An elementary computation shows that *this root is simple*. The other roots have their modulus less than 1 and their number is finite and among them there is one whose modulus is maximal hence $\rho_k \leq \rho_0 < 1$ for all roots inside the unit disk.

Otherwise i.e. iff both p and q are *odd* (4.43) has no solution that is (4.41) has no roots with modulus 1; therefore in this case *all* roots of (4.41) satisfy $\rho_k \leq \rho_0 < 1$. The aforementioned properties are thus valid for equation (4.40) hence for (4.39) in the case of $\nu \in \mathbb{Q}$.

Consider now the case of irrational ν that is of rationally independent delays in (4.35). This case can be tackled *via* Theorem 4.4 dealing with strong stability of (4.35). Since there are two delays the computation of the spectral radius reduces to its computation for $A_1 + A_2e^{\imath\theta}$ where $\theta \in [0, 2\pi)$. The matrix being of dimension 2×2, it has two eigenvalues which both have to be inside the unit disk for *all* $\theta \in [0, 2\pi)$. The characteristic equation of $A_1 + A_2e^{\imath\theta}$ with A_i given by (4.36) results as

$$(z - \rho_1)(z + \rho_2 e^{\imath\theta}) + (1-\rho_1)(1-\rho_2)e^{\imath\theta} = 0. \tag{4.46}$$

Location of the roots of (4.46) – a second degree equation *with complex coefficients* – inside the unit disk can be checked with the Schur–Cohn criterion. However, since (4.46) is very much alike to (4.41), we can take the same approach. Let $z = re^{\imath\varphi}$, $r > 1$, $\varphi \in [0,2\pi)$. We shall have after a straightforward computation

$$\begin{aligned}|z-\rho_1|^2 \cdot |z+\rho_2 e^{\imath\theta}|^2 &= |re^{\imath\varphi}-\rho_1|^2 \cdot |re^{\imath\varphi}+\rho_2 e^{\imath\theta}|^2 \\ &= (r^2 - 2r\rho_1\cos\varphi + \rho_1)^2 (r^2 + 2r\rho_2\cos\varphi\cos\theta + \rho_2)^2 \\ &= [(r-\rho_1)^2 + 2\rho_1 r(1-\cos\varphi)][(r-\rho_2)^2 + 2\rho_2 r(1-\cos\varphi)\cos\theta] \\ &> (r-\rho_1)^2(r-\rho_2)^2 > (1-\rho_1)^2(1-\rho_2)^2.\end{aligned}$$

Therefore equation (4.46) has no roots of modulus larger than 1. Let now $r = 1$. We have to check the equality

$$(e^{\imath\varphi}-\rho_1)(e^{\imath\varphi}+\rho_2 e^{\imath\theta}) + (1-\rho_1)(1-\rho_2)e^{\imath\theta} = 0. \tag{4.47}$$

We can proceed as in the case of (4.42) but here the problem is simpler. Let $\varphi = 0$, $\theta = \pi$: obviously (4.47) is fulfilled. Therefore $z = 1$ is one of the two roots of (4.46) for $\theta = \pi$, the other one being $(\rho_1 + \rho_2 - 1) \in (-1,1)$. This result is sufficient to obtain $\gamma_0(A) = 1$ and, therefore, that statement (ii) of Theorem 4.4 is not fulfilled. Since $(ii) \Leftrightarrow (i)$, we have also non$(i) \Leftrightarrow$ non(ii). It follows that that there will be *no (asymptotic) stability of the difference system* (4.35) *for irrational* ν.

We are now in position to summarize the results concerning (asymptotic) stability of the difference system (4.35). This system is stable *in the sense of Lyapunov*: the result was obtained using the Lyapunov functional (4.24) whose derivative, also given in (4.24) is non-positive definite; stability should be viewed in the sense of the Lyapunov functional itself.

For the sake of completeness, it should be mentioned that the difference system (4.35) is obtained from (4.21) for $z(\tau) \equiv 0$. In this case the derivative functional of (4.24) is subject to $\mathcal{W}(\eta_1^-(\cdot,\tau),\eta_1^+(\cdot,\tau)) \equiv 0$, system (4.35) resulting conservative – in the metrics of the Lyapunov functional $\mathcal{V}$ of (4.24), also restricted to $z(\tau) \equiv 0$. Therefore system (4.35) is stable in the sense of the metrics of $\mathcal{V}$.

If this aspect is viewed from the point of view of the characteristic equation (4.39), an elementary computation of the derivative of its right hand side will show that for any real ν the possible roots of modulus 1 will be *simple*. We reiterated here that system (4.35) as well as system (4.21) are Lyapunov stable in the sense of the metrics induced by the Lyapunov functional (4.24). Based on Theorem 4.1 and on the representation formulae (4.13), Lyapunov stability is ensured for system (4.15) – in the sense of the metrics defined by the Lyapunov functional (4.18).

Return now to the problem of asymptotic stability. We showed in the previous development, based on Theorems 4.4 and 4.6 that the asymptotic stability of system (4.35) is true only in a single case of rationally dependent delays – when their ratio $\nu = p/q \in \mathbb{Q}$ in the particular case when both p and q are odd numbers. *Only in this case* the invariance principle Barbashin–Krasovskii–LaSalle (Theorem 9.8.2 of [18], page 293) can be applied to system (4.27) to obtain its asymptotic stability and, *via* Theorem 4.1 and the representation formulae (4.13), of system (4.15).

Summarizing, Lyapunov stability is ensured for (4.21) hence for (4.15) but the *asymptotic stability is fragile*: it holds for a countable set of rational ratios of propagation time constants – those rational ν having both odd numerator and denominator. The fragility appears from the fact that the set of irrationals is dense and a small uncertainty in the delays will modify ν from rational to irrational.

5 Some conclusions

The examined stability analysis concerns a model arising from hydraulic engineering, largely accepted among engineers, as it might be seen from the cited references. As it appeared from the stability analysis of the water hammer, the only stabilizing device of this phenomenon is the surge tank. Since the engineering philosophy states that a stabilizing device should be stable itself (display inherent stability), the paper contains a standard stability analysis of the surge tank. Its result is the dimension of the equivalent cross-section area of the surge tank which must be larger than the so called *Thoma cross-section area*. The analysis was made using a suitable Lyapunov function giving an asymptotic stability result combined with an estimate of the attraction domain. The aforementioned analysis is valid for the physically accepted equilibrium. Other steady states may be foreseen, corresponding to rather abnormal situations (from the engineering point of view). The analysis might point out instabilities, limit cycles, hidden attractors.

On the other hand, the model for the water hammer itself is described by a nonstandard (i.e. with derivative boundary conditions) boundary value problem for hyperbolic $1D$ equations. We applied here a well established method, coming from the paper of A. D. Myshkis [1], to associate a system of functional differential equations (in most cases, of neutral type) to a nonstandard initial boundary value problem for hyperbolic partial differential equations. A one to one correspondence between the solutions of the two mathematical objects being established e.g. [28], all results obtained for one mathematical object are thus projected back on the other one.

Consider here stability obtained *via* "weak" (in the sense of N. G. Četaev) Lyapunov function(al)s i.e. having the derivative function(al) only non-positive (the best known are the energy type function(al)s). In this case the main instrument for the asymptotic stability is the Barbashin–Krasovskii–LaSalle invariance principle. For neutral functional differential equations this principle is established as a theorem for equations with stable difference operator (Theorem 9.8.2 of [18]). However this stability is *robust (non-fragile)* with respect to delay uncertainties only if the difference operator is strongly stable. As an example, for a single delay case – $p = 1$ in (4.28) – the strong stability follows from the location of the eigenvalues of A_1 inside the unit disk of $\mathbb{C}$; moreover, in this case stability and strong stability are the same thing.

In our opinion, this assumption, occurring for the first time in the paper [9], turned to be capital for stability studies. This was also due to the fact that many applications leading to neutral functional differential equations displayed conditions for the fulfilment of the (strong) stability assumptions of the difference operator.

In the last years there were however exposed applications (mainly from Mechanics and Mechanical Engineering) with matrix A_1 – again the case $p = 1$ in (4.28) – having its eigenvalues on the unit circle i.e. in a critical case (a list of such applications is available in [28]; other applications, dealing with synchronization of mechanical oscillators, can be found in [29]); these cases were not yet seriously tackled.

The case described in this paper looks different: displaying two delays, it displays also a *fragile asymptotic stability* – valid for rationally dependent delays, but only in one case of two possible. The fragility of the asymptotic stability with respect to the delays is confirmed by practical measurement (in-site), displaying some oscillatory modes. Such aspects arising from practice should stimulate revival of some "old" studies which have been obscured by the Cruz-Hale assumptions: the book [11] and its reference list are a good starting point to meet this challenge.

There exists however another challenge, arising from the fact that stability of the difference operator is a premise to apply the Barbashin–Krasovskii–LaSalle invariance principle. As pointed out in [31], the assumption of stability for the difference operator is necessary to obtain pre-compactness of the positive orbits whenever the solution is bounded (Chapter VI, p. 341 and next). The cited reference gives an alternative in its Chapter V (Section 4). Interesting enough, the case considered in Chapter V is a boundary value problem for hyperbolic partial differential equations. With the aforementioned one to one correspondence between the solutions of the boundary value problem for hyperbolic partial differential equations and those of the associated system of neutral functional differential equations, the things become clearer. Our point of view is that all this is a question of choosing the state space for neutral functional differential equations (other than $\mathcal{C}$) – see [17]. In any case there is plenty of motivation to follow this line of research.

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It is more than 40 years since our paths crossed with László Hatvani on the "field" of the mathematical research in differential equations and dynamical systems. His deep mathematical knowledge doubled by the highest human qualities made me feel proud and lucky of meeting him and being almost side by side in our scientific interest. At this cross road moment I wish him good health and open mind to continue – for the good of our science and for the joy of his friends.

Appendix

We shall give in this Appendix (reproduced after [10]) the principal notations of the paper – in fact the notations which are usual in the field of hydroelectric engineering and can be met in field's references, in particular in those of the present paper.

The notations of the state variables are as follows

- $V_i(x,t)$, $Q_i(x,t)$, $H_i(x,t)$, $i = 1,2$ – water flow velocity, water flow and piezometric head at $(x,t) \in \{(x,t) \mid 0 \le x \le L_i, t \in \mathbb{R}\}$, x being the coordinates along the conduits ($i = 1$ accounts for the tunnel and $i = 2$ for the penstock);
- H_0 – piezometric head of the lake;
- $Z(t)$ – water level in the surge tank;
- $\Omega(t)$ – turbine rotating speed; Ω_c – the synchronous speed.

Also the notations for system's parameters are as follows

- F_i, D_i, L_i ($i = 1,2$) – the cross section areas, the hydraulic diameters and the lengths of the conduits, respectively;
- F_s, F_θ – equivalent cross section areas of the surge tank and regulated flow area of the turbine wicket gates, respectively;
- J, η_θ – moment of inertia and efficiency of the hydraulic turbine, respectively;

- γ, g – specific weight of the water and gravity acceleration, respectively;
- $N_g = \Omega_c M_g$ – the mechanical power supplied to the hydrogenerator, where M_g is the torque;
- λ_s – coefficient of losses of the throttling of the surge tank;
- λ_i, a_i $(i = 1,2)$ – coefficients of the Darcy–Weisbach losses and the propagation speeds of the water hammer along the conduits respectively;
- α_q – a flow coefficient;
- k – a corrective coefficient for the flow through the wicket gates of the turbine.

We list further the following time constants of the conduits

- T_{wi} – the starting time constant: $T_{wi} = (L_i\bar{Q})(F_i H_0 g)^{-1}$ $(i = 1,2)$;
- T_i – the fill up time constant: $T_i = (L_i F_i)/\bar{Q}$ $(i = 1,2)$;
- T_{pi} – the wave propagation time: $T_{pi} = L_i/a_i$ $(i = 1,2)$,

and also

- T_s – the fill up time constant of the surge tank: $T_s = F_s H_0/\bar{Q}$;
- T_a – the starting time constant of the turbine:

$$T_a = \frac{J\Omega_c^2}{\eta_\theta \frac{\gamma}{2g}\bar{Q}H_0}.$$

The following rated state variables and parameters are listed below

- q_i – rated water flow along the conduit i defined by $q_i = Q_i/\bar{Q}$, $i = 1,2$;
- h_i – rated piezometric head along the conduit i defined by $h_i = H_i/H_0$, $i = 1,2$;
- z – rated piezometric head at the surge tank defined by $z = Z/H_0$;
- φ – rated rotating speed of the hydraulic turbine, defined by $\varphi = \Omega/\Omega_c$;
- ν_g – rated load mechanical power of the hydraulic turbine, defined by

$$\nu_g = \frac{N_g}{\eta_\theta \frac{\gamma}{2g}\bar{Q}H_0}.$$

We list finally the rated (to T_1) time constants as follows

- θ_{wi} – the rated starting time constant $\theta_{wi} = T_{wi}/T_1$ $(i = 1,2)$;
- θ_i – the rated fill up time constant $\theta_i = T_i/T_1$ $(i = 1,2, \theta_1 = 1)$;
- θ_{pi} – the rated wave propagation time $\theta_{pi} = T_{pi}/T_1$ $(i = 1,2)$;
- θ_s – the rated fill up time constant of the surge tank $\theta_s = T_s/T_1$;
- θ_a – the rated starting time constant of the turbine $\theta_a = T_a/T_1$

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