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Blow-up analysis for a porous media equation with nonlinear sink and nonlinear boundary condition

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Abstract. In this paper, we study porous media equation $u_t = \Delta u^m - u^p$ with nonlinear boundary condition $\frac{\partial u}{\partial \nu} = ku^q$. We determine some sufficient conditions for the occurrence of finite time blow-up or global existence. Moreover, lower and upper bounds for blow-up time are also derived by using various inequality techniques.

Keywords: porous media equation, nonlinear boundary condition, bounds for blow-up time.

2020 Mathematics Subject Classification: 35B44, 35K40.

1 Introduction

In this paper, we are interested in investigating the blow-up phenomena of the following porous media equation with nonlinear sink and nonlinear boundary condition:

$$\begin{cases} u_t = \Delta u^m - u^p, & (x,t) \in \Omega \times (0, t^*), \\ \frac{\partial u}{\partial \nu} = ku^q, & (x,t) \in \partial\Omega \times (0, t^*), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $m > 1$ and $p > 1, q \geq 1$, k is a positive constant, $\Omega \subset \mathbb{R}^3$ is a star-shaped domain with smooth boundary. ν is the unit outward normal vector on $\partial\Omega$, $u_0(x) > 0$ is the initial value. t^* is the blow-up time if the solutions blow up. It is well known that the data m, p, q may greatly affect the behavior of $u(x,t)$ as time evolves.

The mathematical investigation of the phenomenon of blow-up of solutions to parabolic equations and systems has received much attention in the recent literature. We refer to the readers the books of Straughan [14] and Quittner and Souplet [13], as well as papers of Weissler [15, 16], and so on. The determination of sufficient conditions for blow-up and the existence or nonexistence of global solution to problem, as well as bounds for the blow-up time have been the focus of some of these studies [1, 5–7, 18].

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For the initial-boundary value problem of porous media equation

$$u_t = \Delta u^m + f(u) \tag{1.2}$$

where $f(u) \geq 0$, Wu and Gao [17] established the blow-up criterion of equation (1.2) by using the method of energy. Besides, there are many references for the blow-up behavior of its solutions [4,8]. The methods used in the study of blow-up often lead to upper bound for the blow-up time when blow-up occur. However, in applied problems, because of the explosive nature of the solution, a lower bound on blow-up time is more important. Then, there are many papers giving the estimate of the lower bound of blow-up time [2,3,9,10,12]. In [9], the authors gave the estimations of the lower bound for blow-up time for problem (1.2) under Robin boundary conditions, by using various inequalities. When $m = 1$, Payne and Philippin etc. [10] studied blow-up phenomena of the classical solution of the following initial-boundary problem

$$u_t = \Delta u - f(u) \tag{1.3}$$

under the help of energy method and Sobolev type inequality, they gave the lower bound of blow-up time when condition for blow-up holds.

However, to our best knowledge, there is no paper where the blow-up phenomenon is studied with $m > 1$ and nonlinear sink as a reaction term. So, it is natural to consider problem (1.1). Methods used in this paper are motivated by the aforementioned papers. Because of the difference between the diffusion term and the reaction term, we will study the blow-up phenomena of (1.1) by modifying their techniques.

In Sections 2 and 3, by using energy method and various inequality techniques, we determine a criterion which implies blow-up, and drive upper and lower bounds for t^*; in Section 4, a criterion for boundedness of the solution in all time $t > 0$ is determined; In the last section, a relevant example will be listed to illustrate applications of our results.

2 Blow-up and upper bound estimation of t^*

In this section we establish a blow-up criterion for problem (1.1) and derive an upper bound for blow-up time, by using the auxiliary function method.

Theorem 2.1. *Let $u(x,t)$ be a nonnegative classical solution of problem (1.1) and assume $m + q - 1 \geq p$. Then $u(x,t)$ will blow-up in finite time t^*, and*

$$t^* \leq \frac{2m\varphi_0^{-a}}{(m+1)^2 a(1+a)M},$$

where a, M and φ_0^{-a} are some constants which will be given in the later proof.

Proof. Denote

$$\varphi(t) = \int_\Omega u^{m+1} dx. \tag{2.1}$$

Taking the derivative of (2.1), we have

$$\begin{aligned}\varphi'(t) &= (m+1)\int_{\Omega} u^m u_t dx \\ &= (m+1)\int_{\Omega} u^m(\Delta u^m - u^p)dx \\ &= (m+1)\int_{\Omega} u^m \Delta u^m dx - (m+1)\int_{\Omega} u^{m+p}dx \\ &= (m+1)mk\int_{\partial\Omega} u^{2m+q-1}ds - (m+1)\int_{\Omega}|\nabla u^m|^2 dx - (m+1)\int_{\Omega} u^{m+p}dx.\end{aligned} \tag{2.2}$$

Moreover, by using the notation

$$\psi(t) = \frac{2m^2k}{2m+q-1}\int_{\partial\Omega} u^{2m+q-1}ds - \int_{\Omega}|\nabla u^m|^2 dx - \frac{2m}{m+p}\int_{\Omega} u^{m+p}dx, \tag{2.3}$$

since $m+q-1 \geq p$, we have

$$\varphi'(t) \geq \frac{(m+1)(2m+q-1)}{2m}\psi(t). \tag{2.4}$$

From (2.3) one obtains

$$\psi'(t) = 2m^2k\int_{\partial\Omega} u^{2m+q-2}u_t ds - \int_{\Omega}|\nabla u^m|_t^2 dx - 2m\int_{\Omega} u^{m+p-1}u_t dx, \tag{2.5}$$

because

$$\nabla(u_t^m \nabla u^m) = u_t^m \Delta u^m + \frac{1}{2}|\nabla u^m|_t^2. \tag{2.6}$$

Integrate both sides of (2.6), then we obtain

$$\begin{aligned}\int_{\Omega}|\nabla u^m|_t^2 dx &= 2\int_{\partial\Omega} u_t^m \frac{\partial u^m}{\partial \nu}ds - 2\int_{\Omega} u_t^m \Delta u^m dx \\ &= 2m^2k\int_{\partial\Omega} u^{2m+q-2}u_t ds - 2m\int_{\Omega} u^{m-1}\Delta u^m u_t dx.\end{aligned} \tag{2.7}$$

Substituting (2.7) into (2.5), we have

$$\begin{aligned}\psi'(t) &= 2m\int_{\Omega} u^{m-1}\Delta u^m u_t dx - 2m\int_{\Omega} u^{m+p-1}u_t dx \\ &= 2m\int_{\Omega} u^{m-1}u_t^2 dx > 0.\end{aligned} \tag{2.8}$$

Using Hölder's inequality, we obtain

$$\begin{aligned}(\varphi'(t))^2 &= \left[(m+1)\int_{\Omega} u^m u_t dx\right]^2 \\ &\leq (m+1)^2\int_{\Omega} u^{m+1}dx\int_{\Omega} u^{m-1}u_t^2 dx \\ &= \frac{(m+1)^2}{2m}\varphi(t)\psi'(t).\end{aligned} \tag{2.9}$$

Thus (2.4) implies

$$(\varphi'(t))^2 \geq \frac{(m+1)(2m+q-1)}{2m}\varphi'(t)\psi(t). \tag{2.10}$$

We get from (2.9) and (2.10)

$$\varphi(t)\psi'(t) \geq \frac{(2m+q-1)}{m+1}\varphi'(t)\psi(t),$$

by using the notation $\frac{2m+q-1}{m+1} = 1+a$, we find

$$\varphi(t)\psi'(t) \geq (1+a)\varphi'(t)\psi(t).$$

From the above inequality, we obtain

$$(\varphi^{-(1+a)}\psi)' \geq 0,$$

hence

$$\varphi^{-(1+a)}\psi \geq \varphi_0^{-(1+a)}\psi_0 = M,$$

where $\varphi_0 = \varphi(0)$ and $\psi_0 = \psi(0)$. Combining the above formula with (2.10), we find

$$\varphi'(t) \geq \frac{(m+1)(2m+q-1)}{2m}\psi(t) \geq \frac{(m+1)^2}{2m}(1+a)M\varphi^{1+a}, \tag{2.11}$$

then we have

$$\varphi^{-a}(t) \leq \varphi_0^{-a} - \frac{(m+1)^2}{2m}a(1+a)Mt. \tag{2.12}$$

Therefore

$$t^* \leq \frac{2m\varphi_0^{-a}}{(m+1)^2 a(1+a)M}. \tag{2.13}$$

□

3 Lower bound for the blow-up time

In this section, we estimate the lower bound of the blow-up time by constructing some auxiliary functions and using different inequality techniques, such as Sobolev type inequality and Hölder inequality etc. Our theorem is given as follows.

Theorem 3.1. *Assume that $u(x,t)$ is a nonnegative classical solution of problem (1.1), further it blows up at finite time t^*. Then*

$$t^* \geq \int_{\phi(0)}^{\infty} \frac{d\eta}{k_1\eta^{\frac{3}{2}} + k_2\eta^{\frac{3(n-m+1)}{n}} + k_3\eta - k_4\eta^{\frac{n+p-1}{n}}},$$

where $n = 2(m+2q-3)$ and $\phi(0), k_1, k_2, k_3, k_4$ are constants, defined in the proof later.

Proof. We define

$$\phi(t) = \int_\Omega u^{2(m+2q-3)}dx = \int_\Omega u^n dx. \tag{3.1}$$

The derivative of (3.1) w.r.t. t can be written as follows

$$\begin{aligned}
\phi'(t) &= n\int_\Omega u^{n-1}u_t dx \\
&= n\int_\Omega u^{n-1}(\Delta u^m - u^p)dx \\
&= n\int_\Omega u^{n-1}\Delta u^m dx - n\int_\Omega u^{n+p-1}dx \\
&= nmk\int_{\partial\Omega} u^{n+m+q-2}ds - n(n-1)m\int_\Omega u^{n+m-3}|\nabla u|^2 dx - n\int_\Omega u^{n+p-1}dx.
\end{aligned} \tag{3.2}$$

To estimate $\int_{\partial\Omega} u^{n+m+q-2} ds$, we can refer to the Lemma A.1 in [11], and obtain

$$\int_{\partial\Omega} u^{n+m+q-2} ds \leq \frac{N}{\rho_0} \int_{\Omega} u^{n+m+q-2} dx + \frac{(n+m+q-2)d}{\rho_0} \int_{\Omega} u^{n+m+q-3} |\nabla u| dx, \tag{3.3}$$

where

$$\rho_0 = \min_{\partial\Omega} (x \cdot \nu), \qquad d = \max_{\partial\Omega} |x|.$$

Note that ρ_0 is positive since Ω is star-shaped by assumption.

Applying the Hölder inequality, we have

$$\begin{aligned}
\int_{\Omega} u^{n+m+q-2} dx &\leq \left(\int_{\Omega} u^n dx \right)^{\frac{q-1}{m+2q-3}} \left(\int_{\Omega} u^{n+m+2q-3} dx \right)^{\frac{m+q-2}{m+2q-3}} \\
&= \left(\int_{\Omega} u^n dx \right)^{\frac{2(q-1)}{n}} \left(\int_{\Omega} u^{\frac{3n}{2}} dx \right)^{\frac{2(m+q-2)}{n}} \\
&\leq \frac{2(q-1)}{n} \int_{\Omega} u^n dx + \frac{2(m+q-2)}{n} \int_{\Omega} u^{\frac{3n}{2}} dx.
\end{aligned} \tag{3.4}$$

Using Cauchy's inequality with ϵ and inverse Hölder inequality, we get

$$\int_{\Omega} u^{n+m+q-3} |\nabla u| dx \leq \frac{1}{4\epsilon} \int_{\Omega} u^{n+m+2q-3} dx + \epsilon \int_{\Omega} u^{n+m-3} |\nabla u|^2 dx, \tag{3.5}$$

and

$$\int_{\Omega} u^{n+p-1} dx \geq |\Omega|^{\frac{1-p}{n}} \left(\int_{\Omega} u^n dx \right)^{\frac{n+p-1}{n}}. \tag{3.6}$$

First taking (3.4) and (3.5) into (3.3), then taking (3.3) and (3.6) into (3.2), (3.2) becomes

$$\begin{aligned}
\phi'(t) \leq & \left[\frac{kmd\epsilon(n+m+q-2)}{\rho_0} - m(n-1) \right] n \int_{\Omega} u^{n+m-3} |\nabla u|^2 dx \\
&+ \left[\frac{kmnd(n+m+q-2)}{4\epsilon\rho_0} + \frac{2kmN(m+q-2)}{\rho_0} \right] \int_{\Omega} u^{\frac{3n}{2}} dx \\
&+ \frac{2mkN(q-1)}{\rho_0} \int_{\Omega} u^n dx - n|\Omega|^{\frac{1-p}{n}} \left(\int_{\Omega} u^n dx \right)^{\frac{n+p-1}{n}}.
\end{aligned} \tag{3.7}$$

Now we estimate $\int_{\Omega} u^{\frac{3n}{2}} dx$, using Sobolev type inequality (see (A.5) in [11]) which holds if $N = 3$ and obtain for arbitrary $\mu > 0$

$$\begin{aligned}
\int_{\Omega} u^{\frac{3n}{2}} dx \leq & \frac{1}{3^{\frac{3}{4}}} \left[\frac{3}{2\rho_0} \int_{\Omega} u^n dx + \frac{n}{2} \left(1 + \frac{d}{\rho_0} \right) \int_{\Omega} u^{n-1} |\nabla u| dx \right]^{\frac{3}{2}} \\
\leq & \frac{2^{\frac{1}{2}}}{3^{\frac{3}{4}}} \Bigg\{ \left(\frac{3}{2\rho_0} \right)^{\frac{3}{2}} \left(\int_{\Omega} u^n dx \right)^{\frac{3}{2}} + \left[\frac{n}{2} (1 + \frac{d}{\rho_0}) \right]^{\frac{3}{2}} \frac{1}{4\mu^3} |\Omega|^{\frac{3(m-1)}{n}} \left(\int_{\Omega} u^n dx \right)^{\frac{3(n-m+1)}{n}} \\
&+ \left[\frac{n}{2} \left(1 + \frac{d}{\rho_0} \right) \right]^{\frac{3}{2}} \frac{3\mu}{4} \int_{\Omega} u^{n+m-3} |\nabla u|^2 dx \Bigg\}.
\end{aligned} \tag{3.8}$$

By substituting (3.8) into (3.7), we obtain

$$\begin{aligned}\phi'(t) \le & \left\{ \left[\frac{kmnd\epsilon(n+m+q-2)}{\rho_0} - mn(n-1) \right] \right. \\ & + \left[\frac{kmnd(n+m+q-2)}{4\epsilon\rho_0} + \frac{2kmN(m+q-2)}{\rho_0} \right] \\ & \left. \times \left[\frac{n}{2}(1+\frac{d}{\rho_0}) \right]^{\frac{3}{2}} \frac{3\mu}{4} \frac{2^{\frac{1}{2}}}{3^{\frac{3}{4}}} \right\} \int_\Omega u^{n+m-3} |\nabla u|^2 dx \\ & + k_1 \phi^{\frac{3}{2}} + k_2 \phi^{\frac{3(n-m+1)}{n}} + k_3 \phi - k_4 \phi^{\frac{n+p-1}{n}}. \end{aligned} \tag{3.9}$$

For $\epsilon > 0$ small enough, choosing an appropriate $\mu > 0$ such that $k_0 \le 0$, this leads to

$$\phi'(t) \le k_1 \phi^{\frac{3}{2}} + k_2 \phi^{\frac{3(n-m+1)}{n}} + k_3 \phi - k_4 \phi^{\frac{n+p-1}{n}}, \tag{3.10}$$

where

$$\begin{aligned} k_0 = & \left\{ \left[\frac{kmnd\epsilon(n+m+q-2)}{\rho_0} - mn(n-1) \right] + \left[\frac{kmnd(n+m+q-2)}{4\epsilon\rho_0} + \frac{2kmN(m+q-2)}{\rho_0} \right] \right. \\ & \left. \times \left[\frac{n}{2} \left(1+\frac{d}{\rho_0}\right) \right]^{\frac{3}{2}} \frac{3\mu}{4} \frac{2^{\frac{1}{2}}}{3^{\frac{3}{4}}} \right\}, \\ k_1 = & \left[\frac{kmnd(n+m+q-2)}{4\epsilon\rho_0} + \frac{2kmN(m+q-2)}{\rho_0} \right] \frac{2^{\frac{1}{2}}}{3^{\frac{3}{4}}} \left(\frac{3}{2\rho_0} \right)^{\frac{3}{2}}, \\ k_2 = & \left[\frac{kmnd(n+m+q-2)}{4\epsilon\rho_0} + \frac{2kmN(m+q-2)}{\rho_0} \right] \frac{2^{\frac{1}{2}}}{3^{\frac{3}{4}}} \left[\frac{n}{2} \left(1+\frac{d}{\rho_0}\right) \right]^{\frac{3}{2}} \frac{1}{4\mu^3} |\Omega|^{\frac{3(m-1)}{n}}, \\ k_3 = & \frac{2kmN(q-1)}{\rho_0}, \\ k_4 = & n|\Omega|^{\frac{1-p}{n}}. \end{aligned}$$

Integrating (3.10) from 0 to t^*, we obtain

$$t^* \ge \int_{\phi(0)}^{\infty} \frac{d\eta}{k_1 \eta^{\frac{3}{2}} + k_2 \eta^{\frac{3(n-m+1)}{n}} + k_3 \eta - k_4 \eta^{\frac{n+p-1}{n}}}. \tag{3.11}$$

□

4 Non-existence of blow-up

In this section we show that if the classical solution exists then it may not blow-up when the exponents satisfy $p > m + 2(q-1)$. We define

$$\varphi(t) = \int_\Omega u^{m+1} dx. \tag{4.1}$$

We establish the following theorem.

Theorem 4.1. *Let $p > m + 2(q-1)$, if $u(x,t)$ is a classical solution of (1.1) for $t < t^* \le \infty$ then $\varphi(t)$ is bounded for all $t < t^*$.*

Proof. Assume that $u(x,t)$ is a classical solution of (1.1) for $t < t^* \le \infty$. Taking the derivative of (4.1), by (2.2) we have

$$\varphi'(t) = (m+1)mk \int_{\partial\Omega} u^{2m+q-1} ds - (m+1) \int_\Omega |\nabla u^m|^2 dx - (m+1) \int_\Omega u^{m+p} dx. \tag{4.2}$$

To estimate $\int_{\partial\Omega} u^{2m+q-1} ds$, we obtain

$$\begin{aligned} \int_{\partial\Omega} u^{2m+q-1} ds &\le \frac{N}{\rho_0} \int_\Omega u^{2m+q-1} dx + \frac{(2m+q-1)d}{\rho_0} \int_\Omega u^{2m+q-2} |\nabla u| dx \\ &= \frac{N}{\rho_0} \int_\Omega u^{2m+q-1} dx + \frac{(2m+q-1)d}{\rho_0 m} \int_\Omega u^{m+q-1} |\nabla u^m| dx. \end{aligned} \tag{4.3}$$

Applying Cauchy's inequality with β, we have

$$\int_\Omega u^{m+q-1} |\nabla u^m| dx \le \beta \int_\Omega u^{2(m+q-1)} dx + \frac{1}{4\beta} \int_\Omega |\nabla u^m|^2 dx. \tag{4.4}$$

Choosing $\beta = \frac{(2m+q-1)kd}{4\rho_0}$, and inserting (4.4) into (4.3), then inserting (4.3) into (4.2), we get

$$\varphi'(t) \le (m+1) \left[\frac{kmN}{\rho_0} \int_\Omega u^{2m+q-1} dx + \frac{kd\beta(2m+q-1)}{\rho_0} \int_\Omega u^{2(m+q-1)} dx - \int_\Omega u^{m+p} dx \right]. \tag{4.5}$$

Using Hölder's inequality and Young's inequality with ε, we obtain

$$\begin{aligned} \int_\Omega u^{2(m+q-1)} dx &\le \left(\int_\Omega u^{m+p} dx \right)^\alpha \left(\int_\Omega u^{2m+q-1} dx \right)^{1-\alpha} \\ &\le \alpha \varepsilon^{\frac{1}{\alpha}} \int_\Omega u^{m+p} dx + (1-\alpha) \varepsilon^{\frac{1}{\alpha-1}} \int_\Omega u^{2m+q-1} dx, \end{aligned} \tag{4.6}$$

where $\alpha = \frac{q-1}{p-(m+q-1)}$ and $0 < \alpha < 1$ by the assumption of the theorem.

Combining (4.6) with (4.5), we find

$$\varphi'(t) \le (m+1) \left[H \int_\Omega u^{2m+q-1} dx - W \int_\Omega u^{m+p} dx \right], \tag{4.7}$$

where

$$H = \left[(1-\alpha) \varepsilon^{\frac{1}{\alpha-1}} \frac{kd\beta(2m+q-1)}{\rho_0} + \frac{kmN}{\rho_0} \right],$$

$$W = \left[1 - \alpha \varepsilon^{\frac{1}{\alpha}} \frac{kd\beta(2m+q-1)}{\rho_0} \right],$$

we may choose ε so small that $W > 0$ holds.

Using Hölder's inequality again

$$\int_\Omega u^{2m+q-1} dx \le |\Omega|^{\frac{p-(m+q-1)}{m+p}} \left(\int_\Omega u^{m+p} dx \right)^{\frac{2m+q-1}{m+p}}, \tag{4.8}$$

thus

$$\int_\Omega u^{m+p}dx \geq |\Omega|^{\frac{-p+(m+q-1)}{2m+q-1}} \left(\int_\Omega u^{2m+q-1}dx\right)^{\frac{m+p}{2m+q-1}}, \tag{4.9}$$

where $|\Omega|$ denotes the measure of Ω. Inserting (4.9) into (4.7), we have

$$\varphi'(t) \leq (m+1)\int_\Omega u^{2m+q-1}dx \left[H - W|\Omega|^{\frac{-p+(m+q-1)}{2m+q-1}} \left(\int_\Omega u^{2m+q-1}dx\right)^{\frac{p-(m+q-1)}{2m+q-1}}\right]. \tag{4.10}$$

Application of Hölder's inequality leads to

$$\varphi(t) = \int_\Omega u^{m+1}dx \leq \left(\int_\Omega u^{2m+q-1}dx\right)^{\frac{m+1}{2m+q-1}} |\Omega|^{\frac{m+q-2}{2m+q-1}}. \tag{4.11}$$

From the above equation, we obtain

$$\left(|\Omega|^{\frac{-(m+q-2)}{2m+q-1}} \int_\Omega u^{m+1}dx\right)^{\frac{2m+q-1}{m+1}} \leq \int_\Omega u^{2m+q-1}dx.$$

Thus from (4.10) we derive

$$\varphi'(t) \leq (m+1)\int_\Omega u^{2m+q-1}dx \left[H - W|\Omega|^{\frac{m+q-1-p}{m+1}} \varphi(t)^{\frac{p-(m+q-1)}{m+1}}\right]. \tag{4.12}$$

Since $p > m + 2(q-1) \geq m + q - 1$, from (4.12) one can conclude that $\varphi(t)$ is bounded for $t < t^* \leq +\infty$. In fact, if for some $t_0 < t^*$, $\varphi(t_0)$ is so large that $\left[H - W|\Omega|^{\frac{m+q-1-p}{m+1}} \varphi(t_0)^{\frac{p-(m+q-1)}{m+1}}\right]$ is negative, then $\varphi'(t) < 0$ for all $t_0 < t < t^*$ with the property $\varphi(t) > \varphi(t_0)$ since the exponent of $\varphi(t)$ is positive. Consequently, the continuously differentiable function $\varphi(t)$ is (strictly) monotone decreasing in $[t_0, t^*)$, thus $\varphi(t) \leq \varphi(t_0)$ if $t_0 < t < t^*$. □

Remark 4.2. For $q = 1$, we can see, $p = m$ is the blow-up exponent. But for $q > 1$ and $m + q - 1 < p < m + 2(q-1)$, we do not assert whether the solutions blow-up in finite time with nonlinear boundary condition. Due to technical reasons up to now, we can not give a positive or negative answer.

5 Example and applications

In this part, we give an example to illustrate applications of Theorem 2.1 and Theorem 3.1.

Example 5.1. Let $u(x,t)$ is a solution of the following problem

$$\begin{cases} u_t = \Delta u^3 - u^3, & (x,t) \in \Omega \times (0, t^*), \\ \frac{\partial u}{\partial \nu} = u^2, & (x,t) \in \partial\Omega \times (0,t^*), \\ u(x,0) = u_0(x) = 0.5 - |x|^2 > 0, & x \in \Omega, \end{cases}$$

where $\Omega = \{x \in \mathbb{R}^3 \mid |x|^2 = \sum_{i=1}^3 x_i^2 < 0.0001\}$ is a ball in $\mathbb{R}^3$. Now $m = 3, q = 2$, $p = 3$, $k = 1$, $u_0 = 0.5 - |x|^2, N = 3, \rho_0 = 0.01, d = 0.01, |\Omega| = 4.1888 \times 10^{-6}$.

First, we get the upper bound of blow-up time through the following calculations

$$\begin{aligned}\varphi(0) &= \int_\Omega u_0^{m+1}dx \\ &= \int_0^{2\pi} d\theta \int_0^\pi \sin\varphi d\varphi \int_0^{0.01} (0.5-|r|^2)^4 r^2 dr \\ &= 4\pi \int_0^{0.01} (0.5-|r|^2)^4 r^2 dr = 2.6167\times 10^{-7},\end{aligned}$$

$$\begin{aligned}\psi(0) &= \frac{2m^2k}{2m+q-1}\int_{\partial\Omega} u_0^{2m+q-1}ds - \int_\Omega |\nabla u_0^m|^2 dx - \frac{2m}{m+p}\int_\Omega u_0^{m+p}dx \\ &= \frac{18}{7}\int_0^{2\pi} d\theta \int_0^\pi \sin\varphi d\varphi \int_0^{0.01} (0.5-|r|^2)^7 r^2 dr \\ &\quad - 9\int_0^{2\pi} d\theta \int_0^\pi \sin\varphi d\varphi \int_0^{0.01} (0.5-|r|^2)^4 |\nabla(0.5-|r|^2)|^2 r^2 dr \\ &\quad - \int_0^{2\pi} d\theta \int_0^\pi \sin\varphi d\varphi \int_0^{0.01} (0.5-|r|^2)^6 r^2 dr \\ &= \frac{72}{7}\pi \int_0^{0.01} (0.5-|r|^2)^7 r^2 dr - 144\pi \int_0^{0.01} (0.5-|r|^2)^4 r^4 dr \\ &\quad - 4\pi \int_0^{0.01} (0.5-|r|^2)^6 r^2 dr = 1.8111\times 10^{-8}.\end{aligned}$$

Taking M and a into (2.13), then

$$t^* \le \frac{2m\varphi_0}{(m+1)^2 a(1+a)\psi(0)} = 4.1280. \tag{5.1}$$

Next, we obtain the lower bound of blow-up time by the following calculations

$$\begin{aligned}\phi(0) &= \int_\Omega u_0^{2(m+2q-3)}dx \\ &= \int_0^{2\pi} d\theta \int_0^\pi sin\varphi d\varphi \int_0^{0.01} (0.5-|r|^2)^8 r^2 dr \\ &= 4\pi \int_0^{0.01} (0.5-|r|^2)^8 r^2 dr = 1.6347\times 10^{-8}.\end{aligned}$$

We choose $\epsilon = 0.1$, $\mu = 0.0022$, and calculate that

$$k_1 = 6.9069\times 10^6, \quad k_2 = 1.8015\times 10^8, \quad k_3 = 1800, \quad k_4 = 176.8348.$$

Then

$$t^* \ge \int_{\phi(0)}^\infty \frac{d\eta}{k_1\eta^{\frac{3}{2}} + k_2\eta^{\frac{3(n-m+1)}{n}} + k_3\eta - k_4\eta^{\frac{n+p-1}{n}}} = 0.0012. \tag{5.2}$$

Therefore, combining (5.1) with (5.2), we get

$$0.0012 \le t^* \le 4.1280.$$

References

[1] J. T. Ding, H. J. Hu, Blow-up and global solutions for a class of nonlinear reaction diffusion equations under Dirichlet boundary conditions, *J. Math. Anal. Appl.* **433**(2016), No. 2, 1718–1735. https://doi.org/10.1016/j.jmaa.2015.08.046; MR3398787; Zbl 1327.35177

[2] Z. B. Fang, Y. X. Wang, Blow-up analysis for a semilinear parabolic equation with time dependent coefficients under nonlinear boundary flux, *Z. Angew. Math. Phys.* **66**(2015), No. 5, 2525–2541. https://doi.org/10.1007/s00033-015-0537-7; MR3412310; Zbl 1328.35108

[3] F. S. Li, J. L. Li, Global existence and blow-up phenomena for p-Laplacian heat equation with inhomogeneous Neumann boundary conditions, *Bound. Value Probl.* **2014**, 2014:219, 14 pp. https://doi.org/10.1186/s13661-014-0219-y; MR3286131; Zbl 1304.35387

[4] D. M. Liu, C. L. Mu, Q. Qin, Lower bounds estimate for the blow-up time of a nonlinear nonlocal porous medium equation, *Acta Math. Sci. Ser. B (Engl. Ed.)* **32**(2012), No. 3, 1206–1212. https://doi.org/10.1016/S0252-9602(12)60092-7; MR2921952; Zbl 1274.35168

[5] L. W. Ma, Z. B. Fang, Blow-up analysis for a reaction-diffusion equation with weighted nonlocal inner absorptions under nonlinear boundary flux, *Nonlinear Anal. Real World Appl.* **32**(2016), 338–354. https://doi.org/10.1016/j.nonrwa.2016.05.005; MR3514929; Zbl 1382.35136

[6] L. W. Ma, Z. B. Fang, Blow-up analysis for a nonlocal reaction-diffusion equation with Robin boundary conditions, *Taiwanese J. Math.* **21**(2017), No. 1, 131–150. https://doi.org/10.11650/tjm.21.2017.7380; MR3613977; Zbl 1357.35191

[7] M. Marras, Bounds for blow-up time in nonlinear parabolic systems under various boundary conditions, *Numer. Funct. Anal. Optim.* **32**(2011), No. 4, 453–468. https://doi.org/10.1080/01630563.2011.554949; MR2780924; Zbl 1233.35048

[8] N. Mizoguchi, F. Quirós, J. L. Vázquez, Multiple blow-up for a porous medium equation with reaction, *Math. Ann.* **350**(2011), No. 4, 801–827. https://doi.org/10.1007/s00208-010-0584-5; MR2818715; Zbl 1233.35049

[9] B. P. Ouyang, Y. W. Lin, Y. Liu, Z. H. Cai, Lower bound for the blow-up time for a general nonlinear nonlocal porous medium equation under nonlinear boundary condition, *Bound. Value Probl.* **2020**, Paper No. 76, 12 pp. https://doi.org/10.1186/s13661-020-01372-x; MR4083192

[10] L. E. Payne, G. A. Philippin, S. V. Piro, Blow-up phenomena for a semilinear heat equation with nonlinear boundary condition, I., *Z. Angew. Math. Phys* **61**(2010), No. 6, 999–1007. https://doi.org/10.1007/s00033-010-0071-6; MR2738300; Zbl 1227.35173

[11] L. E. Payne, G. A. Philippin, S. V. Piro, Blow-up phenomena for a semilinear heat equation with nonlinear boundary condition. II., *Nonlinear Anal.* **73**(2010), No. 4, 971–978. https://doi.org/10.1016/j.na.2010.04.023; MRMR2653764; Zbl 1198.35131

[12] L. E. Payne, G. A. Philippin, P. W. Schaefer, Blow-up phenomena for some nonlinear parabolic problems, *Nonlinear Anal.* **69**(2008), No. 10, 3495–3502. https://doi.org/10.1016/j.na.2007.09.035; MR2450554; Zbl 1159.35382

[13] P. Quittner, P. Souplet, *Superlinear parabolic problems*, Birkhäuser/Springer, Cham, 2019. https://doi.org/10.1007/978-3-030-18222-9; MR3967048

[14] B. Straughan, *Explosive instabilities in mechanics*, Springer Science Business Media, 2012. https://doi.org/10.1007/978-3-642-58807-5; MR1637904

[15] F. B. Weissler, Local existence and nonexistence for semilinear parabolic equations in L^p, *Indiana Univ. Math. J.* **29**(1980), No. 1, 79–102. https://doi.org/10.1512/iumj.1980.29.29007; MR554819

[16] F. B. Weissler, Existence and nonexistence of global solutions for a semilinear heat equation, *Israel J. Math.* **38**(1981), No. 1–2, 29–40. https://doi.org/10.1007/bf02761845; MR0599472

[17] X. L. Wu, W. J. Gao, Blow-up of the solution for a class of porous medium equation with positive initial energy, *Acta Math. Sci. Ser. B (Engl. Ed.)* **33**(2013), No. 4, 1024–1030. https://doi.org/10.1016/S0252-9602(13)60060-0; MR3072138

[18] Y. Z. Zhao, X. R. Ma, Blow-up analysis of the solution of nonlocal reaction-diffusion equation with variable diffusion coefficient (in Chinese), *Acta Math. Sci. Ser. A (Chin. Ed.)* **38**(2018), No. 4, 750–769. https://doi.org/10.3969/j.issn.1003-3998.2018.04.013; MR3857708