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On the BMO and $C^{1,\gamma}$-regularity for a weak solution of fully nonlinear elliptic systems in dimension three and four

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Abstract. We consider a nonlinear elliptic system of type

$$-D_\alpha A_i^\alpha(x, Du) = D_\alpha f_i^\alpha$$

and give conditions guaranteeing $C^{1,\gamma}$ interior regularity of weak solutions.

Keywords: nonlinear elliptic systems, regularity, Campanato–Morrey spaces.

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1 Introduction.

In this paper we give conditions guaranteeing that the first derivatives of weak solutions to the Dirichlet problem for a nonlinear elliptic system

$$\begin{cases} -D_\alpha A_i^\alpha(x, Du) = D_\alpha f_i^\alpha, & i = 1, \ldots, N,\ \alpha \in \mathbb{R}^n,\ |\alpha| = 1,\ x \in \Omega, \\ \qquad u(x) = g(x), & x \in \partial\Omega. \end{cases} \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^n$, $n \geq 3$ is a bounded $C^{1,1}$ domain with points $x = (x_1, \ldots, x_n)$, $u : \Omega \to \mathbb{R}^N$, $u(x) = (u^1(x), \ldots, u^N(x))$, $N \geq 2$ is a vector-valued function with gradient $Du = (D_1 u, \ldots, D_n u)$, $D_\alpha = \partial/\partial x_\alpha$ and coefficients A_i^α are continuously differentiable with respect to Du and Hölder continuous with respect to x and in the following we will specify our assumptions imposed on the function (f_i^α) and boundary datum g (throughout the whole text we use the summation convention over repeated indexes).

It is well known that elliptic systems in general do not conserve the regularizing property of Laplace equation and the attempts to find conditions guaranteeing the smoothness of weak

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solutions as well as to construct counterexamples are rich and far reaching. The positive results, i.e. proof that weak solutions of systems of order $2k$ have (under suitable assumptions) continuous partial derivatives of order k, started already with pioneering work of Ch. B. Morrey in 1937 for domains Ω in $\mathbb{R}^2$ (see [16]) and continued by deep results of E. De Giorgi (see [7]) who proved that weak solutions of one equation of second order with linear growth and bounded and measurable coefficients on $\Omega \subset \mathbb{R}^n$ have continuous first derivatives. The case of nonlinear systems on plane domains was solved in paper by J. Stará (see [23]) in 1971 for systems of higher order.

In dimensions $n \geq 3$ analogous results do not hold as was shown by counterexamples of E. De Giorgi (see [8]) and E. Giusti and M. Miranda in 1968 (see [10]), J. Nečas in 1975 (see [19]) and L. Šverák and X. Yan in 2002 (see [24]).

The system (1.1) has been extensively studied in the papers [1, 2, 9, 12, 15, 17, 20] and for detailed and well-arranged informations, see [15]. If $n \geq 3$, it is known that Du can be discontinuous. Campanato in [3] proved for the system (1.1) that $Du \in \mathcal{L}^{2,\theta}_{\text{loc}}(\Omega, \mathbb{R}^{nN})$ with $n-2 < \theta < n$, and also $u \in C^{0,(\theta-n+2)/2}_{\text{loc}}(\Omega, \mathbb{R}^N)$ if $n = 3, 4$. More important for our work is a more general result from Kristensen–Melcher [13].

There are known many conditions on the coefficients which guarantee that solutions of nonlinear elliptic system of equations have required smoothness and, vice versa, counterexamples illustrating that generally such assertions do not hold.

In the present paper, that is extending the articles [4], [5] and [6], we introduce another conditions on coefficients of a nonlinear elliptic system (1.1) and we show that if the first derivatives of weak solutions u to Dirichlet problem for the system satisfy (1.11) with given $\mathcal{M}$ and $\widetilde{\Psi}$ from (1.10) then the gradient of weak solutions are locally BMO or Hölder continuous on domains Ω in $\mathbb{R}^3$ and $\mathbb{R}^4$. The condition (1.11) shows that the our result is applicable to broader class of problems for smaller value of $\mathcal{M}$. Finally, the reality of our theoretical result is illustrated by means of numerical examples.

By a weak solution to the Dirichlet problem for (1.1) we understand $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ such that $u - g \in W^{1,2}_0(\Omega, \mathbb{R}^N)$, $g \in W^{1,2}(\Omega, \mathbb{R}^N)$, $f \in L^2(\Omega, \mathbb{R}^{nN})$ and

$$\int_\Omega A^\alpha_i\,(x, Du(x))\, D_\alpha \varphi^i(x)\, dx = -\int_\Omega f^\alpha_i(x) D_\alpha \varphi^i(x)\, dx, \quad \forall\, \varphi \in W^{1,2}_0(\Omega, \mathbb{R}^N). \tag{1.2}$$

Further the symbol $\Omega_o \subset\subset \Omega$ stands for $\overline{\Omega}_o \subset \Omega$, $d_\Omega = \text{diam}(\Omega)$ and for the sake of simplicity we denote by $|\cdot|$ the norm in $\mathbb{R}^n$ as well as in $\mathbb{R}^N$ and $\mathbb{R}^{nN}$. If $x \in \mathbb{R}^n$ and r is a positive real number, we set $B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$, (i.e., the open ball in $\mathbb{R}^n$), $\Omega_r(x) = \Omega \cap B_r(x)$. Denote by $u_{x,r} = u_r = \int_{\Omega_r(x)} u(y)\, dy / m_n(\Omega_r(x)) = -\!\!\!\!\!\!\int_{\Omega_r(x)} u(y)\, dy$ the mean value of the function $u \in L(\Omega, \mathbb{R}^N)$ over the set $\Omega_r(x)$. Here $m_n(\Omega_r(x))$ is the n-dimensional Lebesgue measure of $\Omega_r(x)$ and we set $U_r(x) = \int_{\Omega_r(x)} |Du(y) - (Du)_{x,r}|^2\, dy / r^n = -\!\!\!\!\!\!\int_{\Omega_r(x)} |Du(y) - (Du)_{x,r}|^2\, dy$, $\phi(x, r) = \int_{\Omega_r(x)} |Du(y) - (Du)_{x,r}|^2\, dy$.

The coefficients $(A^\alpha_i)_{i=1,\dots,N, \alpha=1,\dots,n}$ have linear controlled growth and satisfy strong uniform ellipticity condition. Without loss of generality we can suppose that $A^\alpha_i(x, 0) = 0$. We suppose that $A^\alpha_i(x, p) \in C^1(\mathbb{R}^{nN})$ for all $x \in \Omega$ and

(i) the strong ellipticity condition holds, i.e. there exist ν, $M > 0$ such that for every $x \in \Omega$ and p, $\xi \in \mathbb{R}^{nN}$

$$\nu|\xi|^2 \leq \frac{\partial A^\alpha_i}{\partial p^j_\beta}(x, p)\xi^i_\alpha \xi^j_\beta \leq M|\xi|^2, \tag{1.3}$$

(ii)

$$|A_i^\alpha(x,p)| \le M(1+|p|), \qquad \sum_{i,j,\alpha,\beta} \left| \frac{\partial A_i^\alpha}{\partial p_\beta^j}(x,p) \right| \le M, \tag{1.4}$$

for all $(x,p) \in \Omega \times \mathbb{R}^{nN}$,

(iii) for all $x, y \in \Omega$ and $p \in \mathbb{R}^{nN}$

$$|A_i^\alpha(x,p) - A_i^\alpha(y,p)| \le C_H |x-y|^\chi |p|, \qquad C_H > 0 \tag{1.5}$$

where $\chi = 1$ for $n = 3, 4$,

(iv) there is a real function ω continuous on $[0,\infty)$, which is bounded, nondecreasing, concave, $\omega(0) = 0$ and such that for all $x \in \Omega$ and $p, q \in \mathbb{R}^{nN}$

$$\left| \frac{\partial A_i^\alpha}{\partial p_\beta^j}(x,p) - \frac{\partial A_i^\alpha}{\partial p_\beta^j}(x,q) \right| \le \omega\left(|p-q|\right). \tag{1.6}$$

We denote $\omega_\infty = \lim_{t\to\infty} \omega(t)$ and clearly $\omega(t) \le 2M$.

It is well known (see [9], p.169) that for uniformly continuous $\partial A_i^\alpha / \partial p_\beta^j$ there exists a real function ω satisfying (iv) and, viceversa, (1.6) implies uniform continuity of $\partial A_i^\alpha / \partial p_\beta^j$ and absolute continuity of ω on $[0,\infty)$. By pointwise derivative ω' we will understand the right derivative of ω which is finite on $(0,\infty)$.

Here we will consider the function ω from (1.6) given by the formula

$$\omega(t) = \begin{cases} \omega_o(t), & \text{for } 0 \le t \le t_o,\ t_o > 0 \\ \omega_1(t) = \dfrac{\sqrt{\varepsilon}}{t_o^\gamma} t^\gamma, & \text{for } t_o < t < t_1, \\ \omega_\infty & \text{for } t \ge t_1 \end{cases} \tag{1.7}$$

where ω_o is arbitrary continuous, concave, nondecreasing function such that $\omega_o(0) = 0$ and the constants $0 < \gamma \le 0.44$, $t_o > 0$ are selected in such a way that ω is continuous and concave on $[0,\infty)$.

For example we can choose

$$\omega_o(t) = \frac{2\sqrt{\varepsilon}}{2 + \ln \frac{t_o}{t}} \quad \text{for } 0 < t \le t_o,$$

and this function fail to satisfy Dini condition. It is obvious that in such case the coefficients $\partial A_i^\alpha / \partial p_\beta^j$ are only continuous.

It is well known that on the above assumptions the Dirichlet problem

$$\begin{cases} \operatorname{div}\left(A(x,Du) + f\right) = 0 \quad \text{in } \Omega, \\ \qquad\qquad u - g \in W_0^{1,2}(\Omega, \mathbb{R}^N) \end{cases} \tag{1.8}$$

has for any function $f, g \in W^{1,2}(\Omega, \mathbb{R}^N)$ the unique solution u in the same space.

For the problem (1.8) the following estimate holds

$$\begin{aligned}&\fint_{\Omega}|Du-(Du)_{\Omega}|^2\,dy\\&\quad\le 12\left(\frac{M}{\nu}\right)^2\fint_{\Omega}|Dg-(Dg)_{\Omega}|^2\,dx+\left(\frac{10E}{\nu}\right)^2\left(\frac{E}{\nu}+\frac{M}{\nu}+3\left(\frac{M}{\nu}\right)^2\right)\fint_{\Omega}|Dg|^2\,dx\\&\qquad+20\left(\frac{nN}{\nu}\right)^2\left(1+\left(\frac{4E}{\nu}\right)^2\right)\fint_{\Omega}|f-(f)_{\Omega}|^2\,dx \qquad (1.9)\end{aligned}$$

where $E = nNC_H d_{\Omega}^{\chi}$ (see Appendix A for the proof of (1.9)).

In the following we will use the function $\widetilde{\Psi}(u) = ue^{u^{2/(2\mu-1)}}$, here $u \geq 0$, $\mu \geq 17$ (for detailed information for $\widetilde{\Psi}$, see (2.6)) and we can define the value

$$\mathcal{M} = \sup_{t_o<t<\infty} \frac{\widetilde{\Psi}\left(\frac{\omega^2(t)}{\varepsilon}\right) - \widetilde{\Psi}\left(\frac{\omega^2(t_o)}{\varepsilon}\right)}{t-t_o} < \infty \qquad (1.10)$$

where ω is from (1.7), $\varepsilon = \omega_{\infty}^2/C_{\mu}^{\alpha}$, $\alpha > 1-2/n$ and $C_{\mu} = \left(\frac{(n-2)\mu}{2\mathrm{e}}\right)^{\mu}$.

Now we can formulate the main theorem.

Theorem 1.1. *Let $\Omega_o \subset\subset \Omega \subset \mathbb{R}^n$, $d_o = \operatorname{dist}(\Omega_o, \partial\Omega)/2$, $n = 3,4$. Assume that $g \in W^{1,2}(\Omega, \mathbb{R}^N)$, $Dg \in L^{2,\zeta}(\Omega, \mathbb{R}^{nN})$, $\zeta > 2$, $f \in W^{1,2} \cap \mathcal{L}^{2,\xi}(\Omega, \mathbb{R}^N)$, $n < \xi \leq n+2$, $n \leq \vartheta < \lambda = \min\{2\chi + \zeta, \xi\}$ and moreover $\operatorname{div} f \in L^{\zeta}(\Omega, \mathbb{R}^{nN})$. Let $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a weak solution to the system (1.1) satisfying the conditions*

$$\fint_{\Omega}|Du-(Du)_{\Omega}|^2\,dy < \frac{1}{\mathcal{M}^2}, \qquad (1.11)$$

(1.13) *and*

$$C_M C_H^2 + [f]^2_{\mathcal{L}^{2,\xi}(\Omega,\mathbb{R}^{nN})} \leq \frac{|\Omega|\left(1-(4\epsilon_o)^{\frac{\lambda}{\vartheta}-1}\right)\epsilon_o^3\,\nu^2}{8d_o^n\max\{d_o^{\lambda}, d_o^{\lambda-n}\}\,\mathcal{M}^2} \qquad (1.12)$$

where $\epsilon_o = 1/4(2^{n+5}L)^{\frac{\vartheta}{n+2-\vartheta}}$, the constants L, C_H, C_M come from Lemma 2.5, (1.5) and (3.8), respectively. Then $Du \in C^{0,(\vartheta-n)/2}(\Omega_o, \mathbb{R}^{nN})$ in the case $\vartheta > n$ and $Du \in BMO(\Omega_o, \mathbb{R}^{nN})$ for $\vartheta = n$.

Remark 1.2. In the foregoing formulas the constants $\mu \geq 17$, $\alpha > 1-2/n$ have to be such that

$$C_{\mu}^{\frac{n}{n-2}\alpha-1} \geq 2^{\frac{6}{n-2}}\left(20C_S\frac{M\omega_{\infty}}{\nu^2}\left(\frac{|\Omega|}{(2d_o)^n}\right)^{\frac{1}{2n}}\right)^{\frac{2n}{n-2}}\epsilon_o^{-\frac{n}{n-2}}. \qquad (1.13)$$

Here C_S is the Sobolev embedding constant.

The theorem we formulated above tells that, if coefficients of a nonlinear system satisfy (iv) with some ω given (1.7) and (1.11)–(1.13) are fulfilled, then the gradient of u is Hölder continuous on Ω_o.

In most partial regularity results for the system (1.1) the regular points $x \in \Omega$ of solution u are characterized in such a way that for some $r_x > 0$ the quantity $U_{r_x}(x)$ (for its definition see first section) has to be sufficiently small, but our condition regularity (1.11) allows $U_r(x)$ not to be necessarily small. Moreover, the condition (1.11) is global condition (we do not know an analogous condition from the literature) and has fundamental meaning for domain Ω in which it is possible ensure that the ratio $|\Omega|/(2d_o)^n$ is not extremely great (e.g. for the ball, see (1.13)).

For the function ω from (1.7) the right-hand side of (1.11) can be chosen in the following form

$$\frac{1}{\mathcal{M}^2} = \frac{t_o^2}{4}\left(\min\left\{\frac{1}{3\gamma}, \frac{C_\mu^{\left(-1+\frac{1}{2\gamma}\right)\alpha}}{e^{C_\mu^{\frac{2\alpha}{2\mu-1}}}}\right\}\right)^2. \tag{1.14}$$

(see Appendix B for more information and for μ and α see Remark 1.2).

Remark 1.3. We would like to point that, in the case of (1.8), the left-hand side of (1.11) could be substituted with the right-hand side of (1.9). We can present some consequences of our theorem that follow from estimate (1.9).

$$g = const. \wedge f = const. \implies \fint_\Omega |Du - (Du)_\Omega|^2\, dy = 0 \implies u = P_1$$

$$g = P_1 \wedge f = const. \wedge C_H = 0 \implies \fint_\Omega |Du - (Du)_\Omega|^2\, dy = 0 \implies u = P_1$$

$$g = P_1 \wedge f = const. \wedge d_\Omega \searrow 0 \implies \fint_\Omega |Du - (Du)_\Omega|^2\, dy \searrow 0$$

$$g = P_1 \wedge f \in \mathcal{L}^{2,\xi}(\Omega, \mathbb{R}^{nN}),\, \xi > n \wedge C_H = 0 \wedge d_\Omega \searrow 0 \implies \fint_\Omega |Du - (Du)_\Omega|^2\, dy \searrow 0$$

where P_1 is a polynom of at most first degree. We note that the last mentioned condition involves the data of the problem (1.8) only.

Remark 1.4. It is useful to point out that in the case when the ratio ω_∞/ν is small enough, the regularity of solution to the problem (1.8) is guaranteed by the Proposition 2.4 from [4].

2 Preliminaries and notations

Beside the usually used space $C_0^\infty(\Omega, \mathbb{R}^N)$, Hölder space $C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^N)$ and Sobolev spaces $W^{k,p}(\Omega, \mathbb{R}^N)$, $W^{k,p}_{loc}(\Omega, \mathbb{R}^N)$, $W^{k,p}_0(\Omega, \mathbb{R}^N)$ (see, e.g.[22]) we use the following Campanato and Morrey spaces.

Definition 2.1 (Campanato and Morrey spaces)**.** Let $v \in [0, n]$. The Morrey space $L^{2,v}(\Omega, \mathbb{R}^N)$ is the subspaces of such functions $u \in L^2(\Omega, \mathbb{R}^N)$ for which

$$\|u\|^2_{L^{2,v}(\Omega,\mathbb{R}^N)} = \sup_{r>0, x\in\Omega} r^{-v} \int_{\Omega_r(x)} |u(y)|^2\, dy < \infty.$$

Let $v \in [0, n+2]$. The Campanato space $\mathcal{L}^{2,v}(\Omega, \mathbb{R}^N)$ is the subspace of such functions $u \in L^2(\Omega, \mathbb{R}^N)$ for which

$$[u]^2_{\mathcal{L}^{2,v}(\Omega,\mathbb{R}^N)} = \sup_{r>0, x\in\Omega} r^{-v} \int_{\Omega_r(x)} |u(y) - u_{x,r}|^2\, dy < \infty.$$

Remark 2.2. It is worth recalling the trivial but basic property that $\int_\Omega |u - u_\Omega|^2 dx = \min_{c\in\mathbb{R}^N} \int_\Omega |u - c|^2\, dx$ holds for each $u \in L^2(\Omega, \mathbb{R}^N)$.

For more details see [1], [9] and [22]. In particular, we will use:

Proposition 2.3. *For a bounded domain $\Omega \subset \mathbb{R}^n$ with a Lipschitz boundary we have the following*

(a) $\mathcal{L}^{2,v}(\Omega, \mathbb{R}^N)$ *is isomorphic to the* $C^{0,(v-n)/2}(\overline{\Omega}, \mathbb{R}^N)$*, for* $n < v \le n+2$*,*

(b) $L^{2,v}(\Omega, \mathbb{R}^N)$ *is isomorphic to the* $\mathcal{L}^{2,v}(\Omega, \mathbb{R}^N)$*,* $0 \le v < n$*,*

(c) the imbedding $\mathcal{L}^{2,v_1}(\Omega, \mathbb{R}^N) \subset \mathcal{L}^{2,v_2}(\overline{\Omega}, \mathbb{R}^N)$ *is continuous for all* $0 \le v_2 < v_1 \le n+2$*,*

(d) $L^{2,n}(\Omega, \mathbb{R}^N)$ *is isomorphic to the* $L^\infty(\Omega, \mathbb{R}^N) \subsetneq \mathcal{L}^{2,n}(\Omega, \mathbb{R}^N)$*.*

The following lemma is a modification of a lemma from [5].

Lemma 2.4. *Let* $A > 1$*,* d *be positive numbers,* C*,* B_1*,* $B_2 \ge 0$*,* $n \le \delta < \beta$*,* $\delta < \alpha \le n+2$ *and* $0 < s \le 1$*. Then there exist positive constants* k_1*,* k_2 *so that for any nonnegative nondecreasing function* φ *defined on* $[0, d]$ *and satisfying the inequalities*

$$\varphi(\sigma) \le A\left(\frac{\sigma}{R}\right)^\alpha \varphi(R) + \frac{1}{2}\left(1 + A\left(\frac{\sigma}{R}\right)^\alpha\right)\left[\left(B_1 + B_2 U_{2R}^s\right)\varphi(2R) + CR^\beta\right], \qquad \forall\, 0 < \sigma < R \le \frac{d}{2} \tag{2.1}$$

and

$$B_1 + B_2 U_d^s \le \frac{1}{4}\tau^\delta, \qquad B_2\left(\frac{Cm}{2^\beta \tau^\delta (1 - \tau^{\beta-\delta})}\right)^s \le \frac{1}{4}\tau^\delta \tag{2.2}$$

where $U_R = \phi(R)/R^n$*,* $m = \max\{d^\beta, d^{\beta-n}\}$ *and* $\tau = 1/(2^{\alpha+1}A)^{\frac{1}{\alpha-\delta}}$*. Then it holds*

$$U_\sigma \le \sigma^{\delta-n}(k_1\varphi(d) + k_2), \qquad \forall\, \sigma \in (0, d]\,. \tag{2.3}$$

Proof. I. We will prove by induction that

$$\varphi(\tau^k d) \le \tau^{k\delta}\left(\varphi(d) + \frac{Cm}{2^\beta\tau^\delta}\sum_{j=0}^{k-1}\tau^{(\beta-\delta)j}\right), \qquad U_{\tau^k d} \le \tau^{k(\delta-n)}\left(U_d + \frac{Cm}{2^\beta\tau^\delta}\sum_{j=0}^{k-1}\tau^{(\beta-\delta)j}\right). \tag{2.4}$$

Let $k = 1$. Putting $\sigma = \tau d$, $R = d/2$ in (2.1) we obtain thanks to (2.2) and the assumption on τ

$$\begin{aligned}\varphi(\tau d) &\le 2^\alpha A\tau^\alpha\varphi\left(\tfrac{d}{2}\right) + \tfrac{1}{2}(1 + 2^\alpha A\tau^\alpha)\left[\left(B_1 + B_2U_d^s\right)\varphi(d) + C\left(\tfrac{d}{2}\right)^\beta\right] \\ &\le \left(2^\alpha A\tau^\alpha + B_1 + B_2U_d^s\right)\varphi(d) + C\left(\tfrac{d}{2}\right)^\beta = \tau^\delta\left(\varphi(d) + \tfrac{Cm}{2^\beta\tau^\delta}\right).\end{aligned}$$

Also by means of (2.2) we get

$$U_{\tau d} \le \tau^{\delta-n}\left(U_d + \tfrac{Cm}{2^\beta\tau^\delta}\right), \qquad B_1 + B_2U_{\tau d}^s \le \frac{1}{2}\tau^\delta\,.$$

Next put $\sigma = \tau^{k+1}d$, $R = \tau^k d/2$ into (2.1) we get

$$\begin{aligned}\varphi(\tau^{k+1}d) &\le 2^\alpha A\tau^\alpha\varphi\left(\tfrac{1}{2}\tau^k d\right) + \tfrac{1}{2}(1 + 2^\alpha A\tau^\alpha)\left[\left(B_1 + B_2U_{\tau^k d}^s\right)\varphi(\tau^k d) + \tfrac{Cd^\beta}{2^\beta}\tau^{k\beta}\right] \\ &\le \left(2^\alpha A\tau^\alpha + B_1 + B_2U_{\tau^k d}^s\right)\varphi(\tau^k d) + \tfrac{Cd^\beta}{2^\beta\tau^\delta}\tau^{k\beta+\delta} \le \tau^\delta\varphi(\tau^k d) + \tfrac{Cm}{2^\beta\tau^\delta}\tau^{(k+1)\delta}\end{aligned}$$

because $2^\alpha A\tau^\alpha + B_1 + B_2 U^s_{\tau^k d} \le \tau^\delta$. Using (2.4) we get

$$\varphi(\tau^{k+1}d) \le \tau^\delta \varphi(\tau^k d) + \tfrac{Cd^\beta}{2^\beta \tau^\delta}\tau^{(k+1)\delta} \le \tau^{(k+1)\delta}\left(\varphi(d) + \frac{Cm}{2^\beta\tau^\delta}\sum_{j=0}^{k-1}\tau^{(\beta-\delta)j}\right) + \frac{Cm}{2^\beta\tau^\delta}\tau^{(k+1)\delta}$$
$$= \tau^{(k+1)\delta}\left(\varphi(d) + \frac{Cm}{2^\beta\tau^\delta}\sum_{j=0}^{k}\tau^{(\beta-\delta)j}\right).$$

It immediately implies the estimate of $U_{\tau^{k+1}d}$.

II. Let now σ be an arbitrary positive number less than d. Then there is an integer k such that $\tau^{k+1}d \le \sigma < \tau^k d$. Using monotonicity of φ, this inequality and (2.4) we get

$$\varphi(\sigma) \le \varphi(\tau^k d) \le \tau^{k\delta}\left(\varphi(d) + \frac{Cm}{2^\beta\tau^\delta}\sum_{j=0}^{k-1}\tau^{j(\beta-\delta)}\right) \le \frac{\sigma^\delta}{(\tau d)^\delta}\left(\varphi(d) + \frac{Cm}{2^\beta\tau^\delta(1-\tau^{\beta-\delta})}\right)$$

and this estimate together with the choice of $k_1 = 1/(\tau d)^\delta$, $k_2 = Cm/(2^\beta d^\delta \tau^{2\delta}(1-\tau^{\beta-\delta}))$ completes the proof. □

For the statement of following Lemma see e.g. [1,9,20].

Lemma 2.5. *Consider system of the type* (1.1) *with* $A_i^\alpha(x,p) = A_{ij}^{\alpha\beta} p_\beta^j$, $A_{ij}^{\alpha\beta} \in \mathbb{R}$ *(i.e. linear system with constant coefficients) satisfying (i), (ii) and (iii). Then there exists a constant* $L = L(n,N,M/\nu) \ge 1$ *such that for every weak solution* $v \in W^{1,2}(\Omega,\mathbb{R}^N)$ *and for every* $x \in \Omega$ *and* $0 < \sigma \le R \le \operatorname{dist}(x,\partial\Omega)$ *the following estimate*

$$\int_{B_\sigma(x)} |Dv(y) - (Dv)_{x,\sigma}|^2\, dy \le L\left(\frac{\sigma}{R}\right)^{n+2}\int_{B_R(x)} |Dv(y) - (Dv)_{x,R}|^2\, dy$$

holds.

Remark 2.6. The constant L from the previous lemma can be stated as

$$L = c(n,N)\left(\frac{M}{\nu}\right)^{2\left(2+\left[\frac{n}{2}\right]\right)}$$

and, because of a better presentment, choosing $n = 3$, $N = 2$ we can compute $L < 1.4\cdot 10^8 (M/\nu)^6$.

In the paper [4, p. 108] a system for $n = N = 3$ of type (1.1) was presented for which we can compute $L \approx 10^8$.

Lemma 2.7. *[25, p. 37] Let* $\phi : [0,\infty] \to [0,\infty]$ *be non decreasing function which is absolutely continuous on every closed interval of finite length,* $\phi(0) = 0$. *If* $w \ge 0$ *is measurable and* $E(t) = \{y \in \mathbb{R}^n : w(y) > t\}$ *then*

$$\int_{\mathbb{R}^n} \phi\circ w\, dy = \int_0^\infty m_n(E(t))\,\phi'(t)\, dt\,.$$

In the proof of Theorem 1.1 we will use an inequality which is a consequence of Natanson's lemma (see e.g. [18, p. 262]) and Fatou's lemma. It can be read as follows.

Lemma 2.8. *Let $f : [a, \infty) \to \mathbb{R}$ be a nonnegative function which is integrable on $[a, b]$ for all $a < b < \infty$ and*

$$\mathcal{N} = \sup_{0<h<\infty} \frac{1}{h} \int_a^{a+h} f(t)\, dt < \infty$$

is satisfied. Let $g : [a, \infty) \to \mathbb{R}$ be an arbitrary nonnegative, non-increasing and integrable function. Then

$$\int_a^\infty f(t) g(t)\, dt$$

exists and

$$\int_a^\infty f(t) g(t)\, dt \le \mathcal{N} \int_a^\infty g(t)\, dt$$

holds.

In the proof of Theorem 1.1 we use an inequality which can be read as follows.

Proposition 2.9 (see [4]). *Let $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a weak solution to (1.1) satisfying (i), (ii), (iii) and (iv). Then for every ball $B_{2R}(x) \subset \Omega$ and arbitrary constants $\mu \ge 2$, $b > 0$, $1 < q \le n/(n-2)$ and $c \in \mathbb{R}^{nN}$ we have*

$$\int_{B_R(x)} |Du - (Du)_{x,R}|^2 \ln_+^\mu \left(b|Du - (Du)_{x,R}|^2\right) dy \\ \le 2^{n(q-1)} \left(5C_S \frac{M}{\nu}\right)^{2q} \left(\frac{\mu}{(q-1)\mathrm{e}}\right)^\mu \left(\frac{b}{(2R)^n} \int_{B_{2R}(x)} |Du - c|^2\, dy\right)^{q-1} \int_{B_{2R}(x)} |Du - c|^2\, dy \quad (2.5)$$

where C_S is the Sobolev embedding constant.

Hereafter we shall use conjugate Young functions Φ, Ψ

$$\Phi(u) = u \ln_+^\mu(au) \quad \text{for } u \ge 0, \qquad \Psi(u) \le \overline{\Psi}(u) = \frac{1}{a} u \mathrm{e}^{u^{\frac{2}{2\mu-1}}} \quad \text{for } u \ge 0, \qquad (2.6)$$

where $a > 0$ and $\mu \ge 2$ are constants,

$$\ln_+(au) = \begin{cases} 0 & \text{for } 0 \le u < \frac{1}{a}, \\ \ln(au) & \text{for } u \ge \frac{1}{a}. \end{cases}$$

Then Young inequality for Φ, Ψ reads as

$$xy \le \Phi(x) + \Psi(y), \qquad \forall\, x, y \in \mathbb{R}. \qquad (2.7)$$

3 Proof of Theorem 1.1

Let x_o be any point of $\Omega_o \cap \mathcal{S}$ (it means that $\int_{B_R(x_o)} |Du - (Du)_{x_o,R}|^2\, dx > 0$) and $R \le d_o$. Where no confusion can result, we will use the notation B_R, U_R, $\phi(R)$ and $(Du)_R$ instead of $B_R(x_o)$, $U_R(x_o)$, $\phi(x_o, R)$ and $(Du)_{x_o,R}$. Denoting $A_{ij,0}^{\alpha\beta} = A_{ij}^{\alpha\beta}(x_o, (Du)_R)$,

$$\tilde{A}_{ij}^{\alpha\beta} = \int_0^1 A_{ij}^{\alpha\beta}\left(x_o, (Du)_R + t\left(Du - (Du)_R\right)\right) dt\,,$$

we can rewrite the system (1.1) as

$$-D_\alpha \left(A_{ij,0}^{\alpha\beta} D_\beta u^j\right) = -D_\alpha \left(\left(A_{ij,0}^{\alpha\beta} - \tilde{A}_{ij}^{\alpha\beta}\right)\left(D_\beta u^j - \left(D_\beta u^j\right)_R\right)\right) \\ - D_\alpha \left(A_i^\alpha(x_o, Du) - A_i^\alpha(x, Du)\right) + D_\alpha \left(f_i^\alpha(x) - (f_i^\alpha)_R\right).$$

Split u as $v + w$ where v is the solution of the Dirichlet problem

$$-D_\alpha\left(A^{\alpha\beta}_{ij,0}D_\beta v^j\right) = 0 \quad \text{in } B(R)$$
$$v - u \in W^{1,2}_0\left(B_R, \mathbb{R}^N\right).$$

and $w \in W^{1,2}_0(B_R, \mathbb{R}^N)$ is the weak solution of the system

$$-D_\alpha\left(A^{\alpha\beta}_{ij,0}D_\beta w^j\right) = - D_\alpha\left(\left(A^{\alpha\beta}_{ij,0} - \tilde{A}^{\alpha\beta}_{ij}\right)\left(D_\beta u^j - \left(D_\beta u^j\right)_R\right)\right)$$
$$- D_\alpha\left(A^\alpha_i(x_o, Du) - A^\alpha_i(x, Du)\right) + D_\alpha\left(f^\alpha_i(x) - (f^\alpha_i)_R\right).$$

For every $0 < \sigma \leq R$ from Lemma 2.5 it follows

$$\int_{B_\sigma} |Dv - (Dv)_\sigma|^2\, dx \leq L\left(\frac{\sigma}{R}\right)^{n+2}\int_{B_R} |Dv - (Dv)_R|^2\, dx$$

hence

$$\begin{aligned}\int_{B_\sigma} |Du - (Du)_\sigma|^2\, dx &\leq 2L\left(\frac{\sigma}{R}\right)^{n+2}\int_{B_R} |Dv - (Dv)_R|^2\, dx + 4\int_{B_R} |Dw|^2\, dx\\ &\leq 4L\left(\frac{\sigma}{R}\right)^{n+2}\int_{B_R} |Du - (Du)_R|^2\, dx + 4\left(1 + 2L\left(\frac{\sigma}{R}\right)^{n+2}\right)\int_{B_R} |Dw|^2\, dx. \qquad (3.1)\end{aligned}$$

Now $w \in W^{1,2}_0(B_R, \mathbb{R}^N)$ satisfies

$$\begin{aligned}\int_{B_R} A^{\alpha\beta}_{ij,0}D_\beta w^j D_\alpha\varphi^i\, dx \leq &\int_{B_R}\left|A^{\alpha\beta}_{ij,0} - \tilde{A}^{\alpha\beta}_{ij}\right|\left|D_\beta u^j - (D_\beta u^j)_R||D_\alpha\varphi^i\right|\, dx\\ &+ \int_{B_R}|A^\alpha_i(x_o, Du) - A^\alpha_i(x, Du)|\left|D_\alpha\varphi^i\right|\, dx\\ \leq &\left(\int_{B_R}\omega^2\left(|Du - (Du)_R|\right)|Du - (Du)_R|^2\, dx\right)^{1/2}\left(\int_{B_R}|D\varphi|^2\, dx\right)^{1/2}\\ &+ \left(\int_{B_R}|A^\alpha_i(x_o, Du) - A^\alpha_i(x, Du)|^2\, dx\right)^{1/2}\left(\int_{B_R}|D\varphi|^2\, dx\right)^{1/2}\\ &+ \left(\int_{B_R}|f - f_R|^2\, dx\right)^{1/2}\left(\int_{B_R}|D\varphi|^2\, dx\right)^{1/2}\end{aligned}$$

for any $\varphi \in W^{1,2}_0(B_R, \mathbb{R}^N)$. Hence, choosing $\varphi = w$, we get

$$\begin{aligned}\nu^2\int_{B_R}|Dw|^2\, dx \leq\, &2\int_{B_R}\omega^2\left(|Du - (Du)_R|\right)|Du - (Du)_R|^2\, dx\\ &+ 4\int_{B_R}|A^\alpha_i(x_o, Du) - A^\alpha_i(x, Du)|^2\, dx + 4\int_{B_R}|f - f_R|^2\, dx. \qquad (3.2)\end{aligned}$$

From (3.1) and (3.2) we have

$$\begin{aligned}\phi(\sigma) = &\int_{B_\sigma}|Du - (Du)_\sigma|^2\, dx \leq 4L\left(\frac{\sigma}{R}\right)^{n+2}\int_{B_R}|Du - (Du)_R|^2\, dx\\ &+ \frac{8\left(1 + 2L\left(\frac{\sigma}{R}\right)^{n+2}\right)}{\nu^2}\left[\int_{B_R}\omega^2\left(|Du - (Du)_R|\right)|Du - (Du)_R|^2\, dx\right.\\ &\left. + 2\int_{B_R}|A^\alpha_i(x_o, Du) - A^\alpha_i(x, Du)|^2\, dx + 2\int_{B_R}|f - f_R|^2\, dx\right]\\ = &\,4L\left(\frac{\sigma}{R}\right)^{n+2}\phi(R) + \frac{8\left(1 + 2L\left(\frac{\sigma}{R}\right)^{n+2}\right)}{\nu^2}\left(I_1 + 2I_2 + 2I_3\right) \qquad (3.3)\end{aligned}$$

We use the Young inequality (2.7) (here complementary functions are defined through (2.6)) and for any $0 < \varepsilon < \omega_\infty^2$ we obtain

$$\begin{aligned} I_1 &= \int_{B_R} \omega^2 \left(|Du - (Du)_R|\right) |Du - (Du)_R|^2 \, dx \\ &\leq \varepsilon \int_{B_R} |Du - (Du)_R|^2 \ln_+ \left(a\varepsilon \, |Du - (Du)_R|^2 \right) dx + \int_{B_R} \Psi \left(\frac{\omega_R^2}{\varepsilon} \right) dx = \varepsilon J_1 + J_2 \end{aligned} \tag{3.4}$$

where $\omega_R^2(x) = \omega^2 \left(|Du(x) - (Du)_R|\right)$.

The term J_1 can be estimated by means of Proposition 2.9 (here $q = n/(n-2)$) and we get

$$J_1 \leq C C_\mu \left(a\varepsilon U_{2R} \right)^{q-1} \phi(2R) \tag{3.5}$$

where

$$C = 2^{(q-1)n} \left(5 C_S \frac{M}{\nu} \right)^{2q}, \qquad C_\mu = \left(\frac{n-2}{2\mathrm{e}} \mu \right)^{\mu}.$$

Taking in Lemma 2.7 $w(y) = |v(y) - v_{x,R}|$ on $B_R(x)$ and $w = 0$ otherwise, we have $E_R(t) = \{y \in B_R(x) \, : \, |v(y) - v_{x,R}| > t\}$ and for the the second integral J_2 we get

$$J_2 = \frac{1}{a} \int_0^\infty \frac{d}{dt} \widetilde{\Psi} \left(\frac{\omega^2(t)}{\varepsilon} \right) m_n \left(E_R(t) \right) dt \tag{3.6}$$

where $\widetilde{\Psi} = a \overline{\Psi}$.

We have (we use Lemma 2.8) for $\forall \, \varepsilon > 0$

$$\begin{aligned} &\int_0^\infty \frac{d}{dt} \widetilde{\Psi} \left(\frac{\omega^2(t)}{\varepsilon} \right) m_n \left(E_R(t) \right) dt \\ &\leq \int_0^{t_o} \frac{d}{dt} \widetilde{\Psi} \left(\frac{\omega^2(t)}{\varepsilon} \right) m_n \left(E_R(t) \right) dt + \int_{t_o}^\infty \frac{d}{dt} \widetilde{\Psi} \left(\frac{\omega^2(t)}{\varepsilon} \right) m_n \left(E_R(t) \right) dt \\ &\leq \kappa_n R^n \int_0^{t_o} \frac{d}{dt} \widetilde{\Psi} \left(\frac{\omega^2(t)}{\varepsilon} \right) dt + \sup_{t_o < t < \infty} \left(\frac{1}{t - t_o} \int_{t_o}^t \frac{d}{ds} \widetilde{\Psi} \left(\frac{\omega^2(s)}{\varepsilon} \right) ds \right) \int_{t_o}^\infty m_n \left(E_R(s) \right) ds \\ &\leq \kappa_n \widetilde{\Psi} \left(\frac{\omega^2(t_o)}{\varepsilon} \right) R^n + \sup_{t_o < t < \infty} \left[\frac{\widetilde{\Psi} \left(\frac{\omega^2(t)}{\varepsilon} \right) - \widetilde{\Psi} \left(\frac{\omega^2(t_o)}{\varepsilon} \right)}{t - t_o} \right] \int_{B_R} |Du - (Du)_R| \, dx \\ &\leq \kappa_n \widetilde{\Psi} \left(\frac{\omega^2(t_o)}{\varepsilon} \right) R^n + \mathcal{M} \kappa_n^{1/2} R^{n/2} \phi^{1/2}(R) \\ &\leq \left[\frac{\kappa_n}{2^n} \frac{\widetilde{\Psi} \left(\frac{\omega^2(t_o)}{\varepsilon} \right)}{U_{2R}} + \left(\frac{\kappa_n}{2^n} \right)^{1/2} \frac{\mathcal{M}}{\sqrt{U_{2R}}} \right] \phi(2R) \leq \left[\frac{\widetilde{\Psi} \left(\frac{\omega^2(t_o)}{\varepsilon} \right)}{U_{2R}} + \frac{\mathcal{M}}{\sqrt{U_{2R}}} \right] \phi(2R) \, . \end{aligned} \tag{3.7}$$

If for some $R > 0$ the average $U_R = 0$ then it is clear that x_o is the regular point. So next we can suppose U_R is positive for all $R > 0$.

From [2] and [13] we have that $Du \in L^{2,\zeta}(\Omega, \mathbb{R}^{nN})$, $\zeta \in (2,3)$ and also

$$\begin{aligned} \int_{B_R} |Du|^2 \, dx &\leq \frac{c^2(\zeta, M/\nu, C_H, \chi, \Omega)}{\nu^2} \left(\|f\|^2_{L^{2,\zeta}(\Omega, \mathbb{R}^{nN})} + \|Dg\|^2_{L^{2,\zeta}(\Omega, \mathbb{R}^{nN})} \right) R^\zeta \\ &= C_M \, R^\zeta, \qquad \forall \, 0 < R \leq d_o \, . \end{aligned} \tag{3.8}$$

From the assumptions (iii) follows

$$I_2 \leq C_M C_H^2 R^{2\chi} \int_{B_R} |Du|^2 \, dx \leq C_M C_H^2 R^{2\chi+\zeta} \tag{3.9}$$

and

$$I_3 \leq [f]^2_{\mathcal{L}^{2,\xi}(\Omega_o,\mathbb{R}^{nN})} R^{\xi} \, . \tag{3.10}$$

We get from (3.3) and (3.4) by means of (3.5), (3.7), (3.9) and (3.10)

$$\begin{aligned}
\phi(\sigma) &\leq 4L\left(\frac{\sigma}{R}\right)^{n+2}\phi(2R) \\
&\quad + 8\left(1+2L\left(\frac{\sigma}{R}\right)^{n+2}\right)\left\{\left[\frac{CC_\mu\,\varepsilon}{\nu^2}\left(a\varepsilon U_{2R}\right)^{q-1} + \frac{1}{a\nu^2}\left(\widetilde{\Psi}\left(\frac{\omega^2(t_o)}{\varepsilon}\right)\frac{1}{U_{2R}} + \frac{\mathcal{M}}{\sqrt{U_{2R}}}\right)\right]\phi(2R)\right. \\
&\quad \left. + 2C_M\left(\frac{C_H}{\nu}\right)^2 R^{2+\zeta} + \frac{2}{\nu^2}[f]^2_{\mathcal{L}^{2,\xi}(\Omega_o,\mathbb{R}^{nN})}R^{\xi}\right\} \\
&\leq 4L\left(\frac{\sigma}{R}\right)^{n+2}\phi(2R) + 8\left(1+2L\left(\frac{\sigma}{R}\right)^{n+2}\right)\times \\
&\quad \times\left\{\left[\frac{CC_\mu\varepsilon}{\nu^2}\left(a\varepsilon U_{2R}\right)^{q-1} + \frac{1}{a\nu^2}\left(\widetilde{\Psi}\left(\frac{\omega^2(t_o)}{\varepsilon}\right)\frac{1}{U_{2R}} + \frac{\mathcal{M}}{\sqrt{U_{2R}}}\right)\right]\phi(2R)\right. \\
&\quad \left. + \frac{2}{\nu^2}\left(C_M C_H^2 + [f]^2_{\mathcal{L}^{2,\xi}(\Omega_o,\mathbb{R}^{nN})}\right)R^{\lambda}\right\}
\end{aligned} \tag{3.11}$$

where $\lambda = \min\{2\chi+\zeta, \xi\}$.

In (3.11) we can choose

$$\varepsilon = \frac{\omega_\infty^2}{C_\mu^\alpha}, \qquad a = \frac{128|\Omega|^{1/2}}{(2d_o)^{n/2}\,\nu^2\epsilon_o\,U_{2R}} \quad \text{for } U(2R) > 0 \tag{3.12}$$

where $\epsilon_o = \frac{1}{4(2^{n+5}L)^{\vartheta/(n+2-\vartheta)}}$ and $\mu \geq 17$, $\alpha > 1 - 2/n$ are suitable constants.

We set $P = \omega_\infty/\nu$. Then we obtain for $U_{2R} > 0$

$$\begin{aligned}
\phi(\sigma) &\leq 4L\left(\frac{\sigma}{R}\right)^{n+2}\phi(R) + \frac{1}{2}\left(1+2L\left(\frac{\sigma}{R}\right)^{n+2}\right) \\
&\quad \times\left\{\left[\frac{16CP^2}{C_\mu^{\alpha-1}}\left(\frac{128|\Omega|^{1/2}P^2}{(2d_o)^{n/2}\,C_\mu^\alpha\,\epsilon_o}\right)^{q-1} + \frac{(2d_o)^{n/2}\nu^2}{8|\Omega|^{1/2}}\left(\widetilde{\Psi}\left(\frac{\omega^2(t_o)}{\varepsilon}\right) + \mathcal{M}\sqrt{U_{2R}}\right)\epsilon_o\right]\phi(2R)\right. \\
&\quad \left. + \frac{32}{\nu^2}\left(C_M C_H^2 + [f]^2_{\mathcal{L}^{2,\xi}(\Omega_o,\mathbb{R}^{nN})}\right)R^{\lambda}\right\} \\
&\leq 4L\left(\frac{\sigma}{R}\right)^{n+2}\phi(R) + \frac{1}{2}\left(1+2L\left(\frac{\sigma}{R}\right)^{n+2}\right) \\
&\quad \times\left\{\left[\frac{2^{7q-3}CP^{2q}}{C_\mu^{q\alpha-1}\,\epsilon_o^{q-1}}\left(\frac{|\Omega|}{(2d_o)^n}\right)^{\frac{q-1}{2}} + \frac{1}{8}\left(3 + \frac{(2d_o)^{n/2}}{|\Omega|^{1/2}}\mathcal{M}\sqrt{U_{2R}}\right)\epsilon_0\right]\phi(2R)\right. \\
&\quad \left. + \frac{32}{\nu^2}\left(C_M C_H^2 + [f]^2_{\mathcal{L}^{2,\xi}(\Omega_o,\mathbb{R}^{nN})}\right)R^{\lambda}\right\}
\end{aligned}$$

for all $0 < \sigma \le R \le d_o$ (for the estimate $\widetilde{\Psi}(\omega^2(t_o)/\varepsilon) \le 3$, see Appendix).

In the last term of the foregoing inequality we employed the estimate from (1.11). The constants $\alpha > 1 - 2/n$ and $\mu \ge 17$ can be always chosen in such a way that

$$\frac{2^{7q-3} C P^{2q}}{C_\mu^{q\alpha-1}\,\epsilon_o^{q-1}} \left(\frac{|\Omega|}{(2d_o)^n}\right)^{\frac{q-1}{2}} \le \frac{1}{2}\epsilon_o \iff C_\mu^{q\alpha-1} \ge \frac{2^{7q-3} C P^{2q}}{\epsilon_o^q} \left(\frac{|\Omega|}{(2d_o)^n}\right)^{\frac{q-1}{2}} \tag{3.13}$$

and we get

$$\begin{aligned} \phi(\sigma) \le 4L\left(\frac{\sigma}{R}\right)^{n+2}\phi(R) &+ \frac{1}{2}\left(1 + 2L\left(\frac{\sigma}{R}\right)^{n+2}\right) \\ &\times \left\{\left[\frac{7}{8}\epsilon_o + \frac{1}{8}\epsilon_o \mathcal{M}\,\frac{(2d_o)^{n/2}}{|\Omega|^{1/2}}\sqrt{U_{2R}}\right]\phi(2R) + \frac{32\left(C_M C_H^2 + [f]^2_{\mathcal{L}^{2,\xi}(\Omega,\mathbb{R}^{nN})}\right)}{\nu^2} R^\lambda\right\} \end{aligned}$$

for all $0 < \sigma \le R \le d_o$.

We can put $A = 4L$, $\alpha = n + 2$,

$$B_1 = \frac{7}{8}\epsilon_o, \qquad B_2 = \frac{\mathcal{M}\,(2d_o)^{n/2}}{8|\Omega|^{1/2}}\epsilon_o, \qquad C = \frac{32\left(C_M C_H^2 + [f]^2_{\mathcal{L}^{2,\xi}(\Omega,\mathbb{R}^{nN})}\right)}{\nu^2},$$

$s = 1/2$, $\beta = \lambda$, $\delta = \vartheta$, $\tau^\delta/4 = \epsilon_o$ and $d = d_o$. Now from (1.11) follows that $B_2\sqrt{U_{2d_o}(x)} \le \epsilon_o/8$ and if (1.12) is satisfied we can using Lemma 2.4. In conclusion we get

$$\phi(\sigma) \le \sigma^\vartheta(k_1\phi(2d_o) + k_2), \qquad \forall\, 0 < \sigma \le d_o,\ \ n \le \vartheta < \lambda\,.$$

□

4 Illustrating examples and comments

Example 4.1. We will consider the system (1.1) with ω from Example 1.7 for $\Omega = B_R(0)$, $\Omega_o = B_{R/2}(0)$ and also $d_o = R/4$. Supposing $n = 3$, $N = 2$, $q = 3$, $\vartheta = 3.1$, $\omega_\infty = \nu$, $M/\nu = 10$, $C_S = 10$, $\epsilon_o \approx 10^{-28}$ (the value ϵ_o seems to be realistic, see Remark 2.6, here $L \approx 10^{14}$), $\chi = 1$ and $\lambda = 4$ we can get as follows:

ω_∞	$=$	10^{30}	10^{50}	10^{70}	10^{90}	10^{110}	10^{130}
t_o	$=$	10^{3}	10^{11}	10^{18}	10^{24}	10^{30}	10^{36}
$\omega(t_o)$	$\approx$	10^{8}	10^{28}	10^{48}	10^{68}	10^{88}	10^{108}
t_1	$\approx$	10^{58}	10^{67}	10^{73}	10^{79}	10^{85}	10^{91}
$\omega(\omega_\infty)$	$\approx$	10^{19}	10^{44}	10^{69}	10^{90}	10^{110}	10^{130}
real value $\frac{1}{\mathcal{M}^2}$	$\approx$	10^{5}	10^{21}	10^{35}	10^{47}	10^{59}	10^{71}
estimate $\frac{1}{\mathcal{M}^2}$ by means of (1.14)	$\approx$	10^{5}	10^{21}	10^{35}	10^{47}	10^{59}	10^{71}
$\omega\left(\frac{1}{\mathcal{M}^2}\right)$	$\approx$	10^{10}	10^{32}	10^{55}	10^{78}	10^{100}	10^{122}
α	$=$	1.9	1.92	1.92	1.92	1.91	1.9
γ	$=$	0.39	0.39	0.39	0.39	0.39	0.39
μ	$=$	30.3	30.1	30.1	30.1	30.2	30.3
the right-hand side of (1.12)	$\approx$	$\frac{10^{-16}}{Z_R}$	$\frac{10^{40}}{Z_R}$	$\frac{10^{94}}{Z_R}$	$\frac{10^{146}}{Z_R}$	$\frac{10^{198}}{Z_R}$	$\frac{10^{250}}{Z_R}$

Here t_1 is the point for which $\omega(t_1) = \omega_\infty$ and $\mathcal{Z}_R = \max\{(R/4)^4, R/4\}$. It is necessary to remember that the condition (1.13) from the main Theorem is satisfied for the above-mentioned parameters.

In conclusion is possible to say, that the theorem gives good results if $\nu \geq 1/\epsilon_o = 4(2^{n+5}L)^{\frac{\vartheta}{n+2-\vartheta}}$.

Appendix A

First we have to estimate of $\int_\Omega |Du|^2\,dx$. We can rewrite the system (1.1) as

$$\int_\Omega \left[\widetilde{A}_{ij}^{\alpha\beta}\left(D_\beta u^j - D_\beta g^j\right) + \left(A_i^\alpha(x,Du) - A_i^\alpha(x_o,Du)\right) + \widetilde{\widetilde{A}}_{ij}^{\alpha\beta} D_\beta g^j\right] D_\alpha \varphi^i\,dx = \int_\Omega \left(f_i^\alpha - (f_i^\alpha)_\Omega\right) D_\alpha \varphi^i\,dx \quad \text{(A.1)}$$

where $\widetilde{A}_{ij}^{\alpha\beta} = \int_0^1 A_{ij}^{\alpha\beta}\left(x_o, Dg + t(Du - Dg)\right)dt$ and $\widetilde{\widetilde{A}}_{ij}^{\alpha\beta} = \int_0^1 A_{ij}^{\alpha\beta}(x_o, tDg)\,dt$.

We put in (A.1) $\varphi^i = u^i - g^i$ and we get as follows

$$\begin{aligned}
&\int_\Omega \widetilde{A}_{ij}^{\alpha\beta} D_\beta u^j D_\alpha u^i\,dx + \int_\Omega \widetilde{A}_{ij}^{\alpha\beta} D_\beta g^j D_\alpha g^i\,dx \\
&\quad = \int_\Omega \widetilde{A}_{ij}^{\alpha\beta}\left(D_\beta u^j D_\alpha g^i + D_\beta g^j D_\alpha u^i\right)dx - \int_\Omega \widetilde{\widetilde{A}}_{ij}^{\alpha\beta} D_\beta g^j D_\alpha u^i\,dx + \int_\Omega \widetilde{\widetilde{A}}_{ij}^{\alpha\beta} D_\beta g^j D_\alpha g^i\,dx \\
&\qquad - \int_\Omega \left(A_i^\alpha(x,Dg) - A_i^\alpha(x_o,Dg)\right)\left(D_\alpha u^i - D_\alpha g^i\right)dx \\
&\qquad + \int_\Omega \left(f_i^\alpha - (f_i^\alpha)_\Omega\right) D_\alpha u^i\,dx - \int_\Omega \left(f_i^\alpha - (f_i^\alpha)_\Omega\right) D_\alpha g^i\,dx
\end{aligned}$$

From ellipticity (1.3) we have

$$\begin{aligned}
&\nu \int_\Omega |Du|^2\,dx + \nu \int_\Omega |Dg|^2\,dx \\
&\quad \leq \int_\Omega \left|\widetilde{A}_{ij}^{\alpha\beta}\right| \left(|D_\beta u^j||D_\alpha g^i| + |D_\beta g^j||D_\alpha u^i|\right)dx \\
&\qquad + \int_\Omega \left|\widetilde{\widetilde{A}}_{ij}^{\alpha\beta}\right| |D_\beta g^j||D_\alpha u^i|\,dx + \int_\Omega \left|\widetilde{\widetilde{A}}_{ij}^{\alpha\beta}\right| |D_\beta g^j||D_\alpha g^i|\,dx \\
&\qquad + \int_\Omega |A_i^\alpha(x,Dg) - A_i^\alpha(x_o,Dg)|\,|D_\alpha g^i|\,dx \\
&\qquad + \int_\Omega |f_i^\alpha - (f_i^\alpha)_\Omega|\,|D_\alpha u^i|\,dx + \int_\Omega |f_i^\alpha - (f_i^\alpha)_\Omega|\,|D_\alpha g^i|\,dx \\
&\quad = I_1 + I_2 + I_3 + I_4 + I_5 + I_6\,. \qquad \text{(A.2)}
\end{aligned}$$

By means of Young's inequality we get by choosing $\varepsilon = \nu/2M$

$$\begin{aligned}
I_1 &\leq 2\int_\Omega |Du||Dg| \sum \left|\widetilde{A}_{ij}^{\alpha\beta}\right| dx \leq \frac{1}{2}\nu \int_\Omega |Du|^2\,dx + \frac{2M^2}{\nu}\int_\Omega |Dg|^2\,dx\,, \\
I_2 &\leq \int_\Omega |Du||Dg| \sum \left|\widetilde{\widetilde{A}}_{ij}^{\alpha\beta}\right| dx \leq \frac{1}{4}\nu \int_\Omega |Du|^2\,dx + \frac{M^2}{\nu}\int_\Omega |Dg|^2\,dx\,, \\
I_3 &\leq \int_\Omega |Dg||Dg| \sum \left|\widetilde{\widetilde{A}}_{ij}^{\alpha\beta}\right| dx \leq M\int_\Omega |Dg|^2\,dx\,, \\
I_4 &\leq C_H \int_\Omega |x - x_o|^\chi |Dg| \sum |D_\alpha g^i|\,dx \leq nNC_H d_\Omega^\chi \int_\Omega |Dg|^2\,dx\,,
\end{aligned}$$

$\varepsilon = \nu/4$

$$I_5 = \int_\Omega |f_i^\alpha - (f_i^\alpha)_\Omega||D_\alpha u^i|\,dx \le \int_\Omega |Du| \sum |f_i^\alpha - (f_i^\alpha)_\Omega|\,dx$$
$$\le \frac{1}{8}\nu \int_\Omega |Du|^2\,dx + \frac{2n^2N^2}{\nu} \int_\Omega |f-(f)_\Omega|^2\,dx\,,$$

$\varepsilon = 2\nu$

$$I_6 = \int_\Omega \sum |f_i^\alpha - (f_i^\alpha)_\Omega||D_\alpha g^i|\,dx \le \int_\Omega |Dg| \sum |f_i^\alpha - (f_i^\alpha)_\Omega|\,dx$$
$$\le 2\nu \int_\Omega |Dg|^2\,dx + \frac{n^2N^2}{8\nu} \int_\Omega |f-(f)_\Omega|^2\,dx\,.$$

Together from (A.2) we have

$$\int_\Omega |Du|^2\,dx \le 8\left(\frac{nNC_H d_\Omega^\chi}{\nu} + \frac{M}{\nu} + 3\left(\frac{M}{\nu}\right)^2\right)\int_\Omega |Dg|^2\,dx + \frac{18n^2N^2}{\nu^2}\int_\Omega |f-(f)_\Omega|^2\,dx\,. \tag{A.3}$$

Now we can rewrite the system (1.1) as

$$\int_\Omega \left[\widetilde{A}_{ij}^{\alpha\beta}\left(D_\beta u^j - (D_\beta g^j)_\Omega\right) + \left(A_i^\alpha(x,Du) - A_i^\alpha(x_o,Du)\right)\right] D_\alpha\varphi^i\,dx$$
$$= \int_\Omega \left(f_i^\alpha - (f_i^\alpha)_\Omega\right) D_\alpha\varphi^i\,dx \tag{A.4}$$

where $\widetilde{A}_{ij}^{\alpha\beta} = \int_0^1 A_{ij}^{\alpha\beta}\left(x_o, (Dg)_\Omega + t\,(Du - (Dg)_\Omega)\right)\,dt$.

We put in (A.4) $\varphi^i = \left(u^i - (D_\alpha g^i)_\Omega x_\alpha\right) - \left(g^i - (D_\alpha g^i)_\Omega x_\alpha\right)$ and we get as follows

$$\int_\Omega \widetilde{A}_{ij}^{\alpha\beta}\left(D_\beta u^j - (D_\beta g^j)_\Omega\right)\left(D_\alpha u^i - (D_\alpha g^i)_\Omega\right)$$
$$- \int_\Omega \widetilde{A}_{ij}^{\alpha\beta}\left(D_\beta u^j - (D_\beta g^j)_\Omega\right)\left(D_\alpha g^i - (D_\alpha g^i)_\Omega\right)\,dx$$
$$+ \int_\Omega \left(A_i^\alpha(x,Du) - A_i^\alpha(x_o,Du)\right)\left(D_\alpha u^i - (D_\alpha g^i)_\Omega\right)\,dx$$
$$- \int_\Omega \left(A_i^\alpha(x,Du) - A_i^\alpha(x_o,Du)\right)\left(D_\alpha g^i - (D_\alpha g^i)_\Omega\right)\,dx$$
$$= \int_\Omega \left(f_i^\alpha(x) - (f_i^\alpha)_\Omega\right)\left(D_\alpha u^i - (D_\alpha g^i)_\Omega\right)\,dx - \int_\Omega \left(f_i^\alpha(x) - (f_i^\alpha)_\Omega\right)\left(D_\alpha g^i - (D_\alpha g^i)_\Omega\right)\,dx\,.$$

From ellipticity (1.3) we have

$$\nu \int_\Omega |Du - (Dg)_\Omega|^2\,dx \le \int_\Omega \left|\widetilde{A}_{ij}^{\alpha\beta}\right|\left|D_\alpha u^i - (D_\alpha g^i)_\Omega\right|\left|D_\alpha g^i - (D_\alpha g^i)_\Omega\right|\,dx$$
$$+ \int_\Omega |A_i^\alpha(x,Du) - A_i^\alpha(x_o,Du)|\left|D_\alpha u^i - (D_\alpha g^i)_\Omega\right|\,dx$$
$$+ \int_\Omega |A_i^\alpha(x,Du) - A_i^\alpha(x_o,Du)|\left|D_\alpha g^i - (D_\alpha g^i)_\Omega\right|\,dx$$
$$+ \int_\Omega |f_i^\alpha(x) - (f_i^\alpha)_\Omega|\left|D_\alpha u^i - (D_\alpha g^i)_\Omega\right|\,dx$$
$$+ \int_\Omega |f_i^\alpha(x) - (f_i^\alpha)_\Omega|\left|D_\alpha g^i - (D_\alpha g^i)_\Omega\right|\,dx$$
$$= I_1 + I_2 + I_3 + I_4 + I_5\,. \tag{A.5}$$

By means of Young inequality we get by choosing $\varepsilon = \nu/M$

$$\begin{aligned} I_1 &\le \int_\Omega |Du-(Dg)_\Omega|\,|Dg-(Dg)_\Omega| \sum \left|\widetilde{A}_{ij}^{\alpha\beta}\right| dx \\ &\le \frac{1}{2}\nu \int_\Omega |Du-(Dg)_\Omega|^2\,dx + \frac{M^2}{2\nu}\int_\Omega |Dg-(Dg)_\Omega|^2\,dx\,, \end{aligned}$$

$\varepsilon = \nu/2$

$$\begin{aligned} I_2 &\le \int_\Omega |Du-(Dg)_\Omega| \sum |A_i^\alpha(x,Du)-A_i^\alpha(x_o,Du)|\,dx \\ &\le \frac{1}{4}\nu\int_\Omega |Du-(Dg)_\Omega|^2\,dx + \frac{n^2N^2}{\nu}\int_\Omega |A(x,Du)-A(x_o,Du)|^2\,dx \\ &\le \frac{1}{4}\nu\int_\Omega |Du-(Dg)_\Omega|^2\,dx + \frac{n^2N^2C_H^2 d_\Omega^{2\chi}}{\nu}\int_\Omega |Du|^2\,dx\,, \end{aligned}$$

$\varepsilon = \nu$

$$\begin{aligned} I_3 &\le \int_\Omega |Dg-(Dg)_\Omega| \sum |A_i^\alpha(x,Du)-A_i^\alpha(x_o,Du)|\,dx \\ &\le \frac{1}{2}\nu\int_\Omega |Dg-(Dg)_\Omega|^2\,dx + \frac{n^2N^2}{2\nu}\int_\Omega |A(x,Du)-A(x_o,Du)|^2\,dx \\ &\le \frac{1}{2}\nu\int_\Omega |Dg-(Dg)_\Omega|^2\,dx + \frac{n^2N^2C_H^2 d_\Omega^{2\chi}}{2\nu}\int_\Omega |Du|^2\,dx\,, \end{aligned}$$

$$\begin{aligned} I_5 &\le \int_\Omega |f_i^\alpha(x)-(f_i^\alpha)_\Omega| \left|D_\alpha g^i-(D_\alpha g^i)_\Omega\right| dx \\ &\le \frac{1}{2}\nu\int_\Omega |Dg-(Dg)_\Omega|^2\,dx + \frac{n^2N^2}{2\nu}\int_\Omega |f-(f)_\Omega|^2\,dx\,, \end{aligned}$$

$\varepsilon = \nu/4$

$$\begin{aligned} I_4 &\le \int_\Omega |f_i^\alpha(x)-(f_i^\alpha)_\Omega| \left|D_\alpha u^i-(D_\alpha g^i)_\Omega\right| dx \\ &\le \frac{1}{8}\nu\int_\Omega |Du-(Dg)_\Omega|^2\,dx + \frac{2n^2N^2}{\nu}\int_\Omega |f-(f)_\Omega|^2\,dx\,. \end{aligned}$$

Together we get

$$\begin{aligned} \int_\Omega |Du-(Du)_\Omega|^2\,dx \le \int_\Omega |Du-(Dg)_\Omega|^2\,dx &\le 4\left(2+\left(\frac{M}{\nu}\right)^2\right)\int_\Omega |Dg-(Dg)_\Omega|^2\,dx \\ &+ \frac{20n^2N^2}{\nu^2}\int_\Omega |f-(f)_\Omega|^2\,dx + \frac{12n^2N^2C_H^2 d_\Omega^{2\chi}}{\nu^2}\int_\Omega |Du|^2\,dx\,. \end{aligned} \tag{A.6}$$

By means of (A.3) we are getting from (A.6) final estimate

$$\begin{aligned} \int_\Omega |Du-(Du)_\Omega|^2\,dx &\le 4\left(2+\left(\frac{M}{\nu}\right)^2\right)\int_\Omega |Dg-(Dg)_\Omega|^2\,dx \\ &+ \frac{96n^2N^2C_H^2 d_\Omega^{2\chi}}{\nu^2}\left[\frac{nNC_H d_\Omega^\chi}{\nu}+\frac{M}{\nu}+3\left(\frac{M}{\nu}\right)^2\right]\int_\Omega |Dg|^2\,dx \\ &+ \frac{4n^2N^2}{\nu^2}\left[5+\frac{54n^2N^2C_H^2 d_\Omega^{2\chi}}{\nu^2}\right]\int_\Omega |f-(f)_\Omega|^2\,dx\,. \end{aligned}$$

Appendix B

We give estimates of the constant $\mathcal{M}$ defined by (1.10) where ω is defined by (1.7). We consider $t_o > 0$, $\alpha > 1 - 2/n$, $\mu \geq 17$ and $0 < \gamma \leq 0.44$.

$$h'(t) = \left(\frac{\widetilde{\Psi}\left(\frac{\omega^2(t)}{\varepsilon}\right) - \widetilde{\Psi}\left(\frac{\omega^2(t_o)}{\varepsilon}\right)}{t - t_o} \right)'$$

$$= \frac{\omega(t)\left[2\omega'(t)\left(1+\frac{2}{2\mu-1}\left(\frac{\omega^2(t)}{\varepsilon}\right)^{\frac{2}{2\mu-1}}\right)(t-t_o) - \omega(t)\right] \mathrm{e}^{\left(\frac{\omega^2(t)}{\varepsilon}\right)^{\frac{2}{2\mu-1}}} + \omega^2(t_o)\mathrm{e}^{\left(\frac{\omega^2(t_0)}{\varepsilon}\right)^{\frac{2}{2\mu-1}}}}{\varepsilon(t-t_o)^2}$$

$$= \frac{\omega(t)\left[2\omega'(t)\left(1+a\left(\frac{\omega^2(t)}{\varepsilon}\right)^{a}\right)(t-t_o) - \omega(t)\right] \mathrm{e}^{\left(\frac{\omega^2(t)}{\varepsilon}\right)^{a}} + \omega^2(t_o)\mathrm{e}^{\left(\frac{\omega^2(t_0)}{\varepsilon}\right)^{a}}}{\varepsilon(t-t_o)^2}, \quad 0 < t_o < t < t_1.$$

where $a = 2/(2\mu - 1) < 0.06$ ($\mu \geq 17$).

For $\omega(t) = \omega_1(t) = kt^\gamma$, $k = \sqrt{\varepsilon}/t_o^\gamma$ we get

$$h'(t) = \frac{k^2\left\{t^{2\gamma}\left[2\gamma\left(1+a\left(\frac{k^2t^{2\gamma}}{\varepsilon}\right)^a\right)\left(1-\frac{t_o}{t}\right)-1\right]\mathrm{e}^{\left(\frac{k^2t^{2\gamma}}{\varepsilon}\right)^a} + t_o^{2\gamma}\mathrm{e}^{\left(\frac{k^2t_o^{2\gamma}}{\varepsilon}\right)^a}\right\}}{\varepsilon(t-t_o)^2}$$

$$= \frac{2\gamma\left(1+a\left(\frac{t}{t_o}\right)^{2a\gamma}\right)\left(1-\frac{t_o}{t}\right) + \left(\frac{t_o}{t}\right)^{2\gamma}\mathrm{e}^{\left(1-\left(\frac{t}{t_o}\right)^{2a\gamma}\right)} - 1}{(t-t_o)^2}\left(\frac{t}{t_o}\right)^{2\gamma}\mathrm{e}^{\left(\frac{t}{t_o}\right)^{2a\gamma}}$$

$$= \frac{g_1(t)+g_2(t)-1}{(t-t_o)^2}\left(\frac{t}{t_o}\right)^{2\gamma}\mathrm{e}^{\left(\frac{t}{t_o}\right)^{2a\gamma}}, \qquad 0 < t_o < t < t_1\,. \tag{B.1}$$

We prove that there exists at most one point $t_o < t_m \leq t_1$ such that $h'(t) < 0$ on (t_o, t_m) and $h'(t) > 0$ on (t_m, ∞). For the proof, that $h'(t) < 0$ on (t_o, t_m) is sufficiently show, that

$$g_1(t) + g_2(t) - 1 < 0, \qquad \forall\ t_o < t < t_m.$$

If we put $t = t_o + h$, $h > 0$ and $\xi = 2a\gamma$ we have

$$g_1(t_o+h) + g_2(t_o+h) = 2\gamma\left(1 + a\left(1+\frac{h}{t_o}\right)^\xi\right)\frac{h}{t_o+h} + \left(1-\frac{h}{t_o+h}\right)^{2\gamma}\mathrm{e}^{1-\left(1+\frac{h}{t_o}\right)^\xi} < 1.$$

Now we development the functions $(1+h/t_o)^\xi$, $(1-h/(t_o+h))^{2\gamma}$ and $\mathrm{e}^{1-(1+h/t_o)^\xi}$ to power series we can rewrite the previous ones into the form

$$\frac{2\gamma+\xi}{t_o+h}h + \frac{\xi^2}{t_o(t_o+h)}h^2 + o_1(h^2) + \left(1 - \frac{2\gamma}{t_o+h}h - \frac{\gamma(1-2\gamma)}{(t_o+h)^2}h^2 + o_2(h^2)\right)$$
$$\times\left(1 - \frac{\xi}{t_o}h + \frac{\xi(1-\xi)}{2t_o^2}h^2 + o_1(h^2) + \frac{1}{2}\left(-\frac{\xi}{t_o}h + \frac{\xi(1-\xi)}{2t_o^2}h^2 + o_1(h^2)\right)^2 + o_3(h^2)\right) < 1\,.$$

After some adjustment and if we suppose that $h \leq t_o$ we can write

$$\left(\frac{\xi^2+2\gamma\xi}{t_o(t_o+h)} - \frac{\gamma(1-2\gamma)}{(t_o+h)^2}\right)h^2 + \frac{c(a,\gamma)}{t_o^3}h^3 < 0\,.$$

It is also sufficient to prove

$$\frac{\gamma}{t_o+h}\left(\frac{4a^2\gamma+4a\gamma}{t_o}-\frac{1-2\gamma}{t_o+h}\right)h^2+\frac{c(a,\gamma)}{t_o^3}h^3<0$$
$$\iff \quad \left[2(1+2a+2a^2)\gamma-1\right]t_o+c_1(a,\gamma)h<0$$

and because $\lim_{h\to 0+} c_1(a,\gamma)h = 0$ we can rewrite for sufficiently small $0 < h \le t_o$ preceding inequality as follows

$$\gamma < \frac{1}{2\left(1+\frac{4}{2\mu-1}+\frac{8}{(2\mu-1)^2}\right)} > 0.44, \qquad \forall\, \mu \ge 17\,. \tag{B.2}$$

From this consideration we have

$$\begin{aligned}
\mathcal{M} &= \sup_{t_o<t<t_1}\frac{\widetilde{\Psi}\left(\frac{\omega^2(t)}{\varepsilon}\right)-\widetilde{\Psi}\left(\frac{\omega^2(t_o)}{\varepsilon}\right)}{t-t_o} = \max\left\{\left(\frac{d}{dt}\widetilde{\Psi}\left(\frac{\omega^2(t)}{\varepsilon}\right)\right)_{t=t_o}, \frac{\widetilde{\Psi}\left(\frac{\omega^2(t_1)}{\varepsilon}\right)-\widetilde{\Psi}\left(\frac{\omega^2(t_o)}{\varepsilon}\right)}{t_1-t_o}\right\}\\
&= \max\left\{\frac{6\gamma}{t_o}, \frac{C_\mu^\alpha e^{C_\mu^{\frac{2\alpha}{2\mu-1}}}-e}{t_o(C_\mu^{\frac{\alpha}{2\gamma}}-1)}\right\} \le \frac{1}{t_o}\max\left\{6\gamma, \frac{C_\mu^\alpha e^{C_\mu^{\frac{2\alpha}{2\mu-1}}}-e}{C_\mu^{\frac{\alpha}{2\gamma}}-1}\right\}\\
&\le \frac{2}{t_o}\max\left\{3\gamma, \frac{e^{C_\mu^{\frac{2\alpha}{2\mu-1}}}}{C_\mu^{\left(-1+\frac{1}{2\gamma}\right)\alpha}}\right\}.
\end{aligned} \tag{B.3}$$

For the term $\widetilde{\Psi}\left(\frac{\omega^2(t_o)}{\varepsilon}\right)$ from the definition of $\mathcal{M}$ we get

$$\widetilde{\Psi}\left(\frac{\omega^2(t_o)}{\varepsilon}\right) = \frac{\omega^2(t_o)}{\varepsilon}e^{\left(\frac{\omega^2(t_o)}{\varepsilon}\right)^{2/(2\mu-1)}} = e, \qquad \forall\, t_o > 0\,.$$

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