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Influence of variable coefficients on global existence of solutions of semilinear heat equations with nonlinear boundary conditions

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Abstract. We consider semilinear parabolic equations with nonlinear boundary conditions. We give conditions which guarantee global existence of solutions as well as blow-up in finite time of all solutions with nontrivial initial data. The results depend on the behavior of variable coefficients as $t \to \infty$.

Keywords: semilinear parabolic equation, nonlinear boundary condition, finite time blow-up.

2020 Mathematics Subject Classification: 35B44, 35K58, 35K61.

1 Introduction

We investigate the global solvability and blow-up in finite time for semilinear heat equation

$$u_t = \Delta u + \alpha(t) f(u) \quad \text{for } x \in \Omega,\ t > 0, \tag{1.1}$$

with nonlinear boundary condition

$$\frac{\partial u(x,t)}{\partial \nu} = \beta(t) g(u) \quad \text{for } x \in \partial\Omega,\ t > 0, \tag{1.2}$$

and initial datum

$$u(x,0) = u_0(x) \quad \text{for } x \in \Omega, \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^n$ for $n \geq 1$ with smooth boundary $\partial\Omega$, ν is the unit exterior normal vector on the boundary $\partial\Omega$. Here $f(u)$ and $g(u)$ are nonnegative continuous functions for $u \geq 0$, $\alpha(t)$ and $\beta(t)$ are nonnegative continuous functions for $t \geq 0$, $u_0(x) \in C^1(\overline{\Omega})$, $u_0(x) \geq 0$ in $\overline{\Omega}$ and satisfies boundary condition (1.2) as $t = 0$. We will consider nonnegative classical solutions of (1.1)–(1.3).

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Blow-up problem for parabolic equations with reaction term in general form were considered in many papers (see, for example, [1,2,8,9,14,21,27] and the references therein). For the global existence and blow-up of solutions for linear parabolic equations with $\beta(t) \equiv 1$ in (1.2), we refer to previous studies [16,17,22,24–26]. In particular, Walter [24] proved that if $g(s)$ and $g'(s)$ are continuous, positive and increasing for large s, a necessary and sufficient condition for global existence is

$$\int^{+\infty} \frac{ds}{g(s)g'(s)} = +\infty.$$

Some papers are devoted to blow-up phenomena in parabolic problems with time-dependent coefficients (see, for example, [4–6,18–20,28]). So, it follows from results of Payne and Philippin [20] blow-up of all nontrivial solutions for (1.1)–(1.3) with $\beta(t) \equiv 0$ under the conditions (2.15) and

$$f(s) \geq z(s) > 0, \qquad s > 0,$$

where z satisfies

$$\int_a^{+\infty} \frac{ds}{z(s)} < +\infty \quad \text{for any } a > 0$$

and Jensen's inequality

$$\frac{1}{|\Omega|} \int_\Omega z(u)\,dx \geq z\left(\frac{1}{|\Omega|} \int_\Omega u\,dx\right). \tag{1.4}$$

In (1.4), $|\Omega|$ is the volume of Ω.

The aim of our paper is study the influence of variable coefficients $\alpha(t)$ and $\beta(t)$ on the global existence and blow-up of classical solutions of (1.1)–(1.3).

This paper is organized as follows. Finite time blow-up of all nontrivial solutions is proved in Section 2. In Section 3, we present the global existence of solutions for small initial data.

2 Finite time blow-up

In this section, we give conditions for blow-up in finite time of all nontrivial solutions of (1.1)–(1.3).

Before giving our main results, we state a comparison principle which has been proved in [7,23] for more general problems. Let $Q_T = \Omega \times (0,T)$, $S_T = \partial\Omega \times (0,T)$, $\Gamma_T = S_T \cup \overline{\Omega} \times \{0\}$, $T > 0$.

Theorem 2.1. *Let $v(x,t), w(x,t) \in C^{2,1}(Q_T) \cap C^{1,0}(Q_T \cup \Gamma_T)$ satisfy the inequalities:*

$$v_t - \Delta v - \alpha(t) f(v) < w_t - \Delta w - \alpha(t) f(w) \quad \text{in } Q_T,$$
$$\frac{\partial v(x,t)}{\partial \nu} - \beta(t) g(v) < \frac{\partial w(x,t)}{\partial \nu} - \beta(t) g(w) \quad \text{on } S_T,$$
$$v(x,0) < w(x,0) \quad \text{in } \overline{\Omega}.$$

Then

$$v(x,t) < w(x,t) \quad \text{in } Q_T.$$

The first our blow-up result is the following.

Theorem 2.2. *Let $g(s)$ be a nondecreasing positive function for $s > 0$ such that*

$$\int^{+\infty} \frac{ds}{g(s)} < +\infty \tag{2.1}$$

and

$$\int_0^{+\infty} \beta(t)\, dt = +\infty. \tag{2.2}$$

Then any nontrivial nonnegative solution of (1.1)–(1.3) *blows up in finite time.*

Proof. We suppose that $u(x,t)$ is a nontrivial nonnegative solution which exists in Q_T for any positive T. Then for some $T > 0$ there exists $(\overline{x}, \overline{t}) \in Q_T$ such that $u(\overline{x}, \overline{t}) > 0$. Since $u_t - \Delta u = \alpha(t) f(u) \geq 0$, by strong maximum principle $u(x,t) > 0$ in $Q_T \setminus \overline{Q_{\overline{t}}}$. Let $u(x_\star, t_\star) = 0$ in some point $(x_\star, t_\star) \in S_T \setminus \overline{S_{\overline{t}}}$. According to Theorem 3.6 of [11] it yields $\partial u(x_\star, t_\star)/\partial \nu < 0$, which contradicts the boundary condition (1.2). Thus, $u(x,t) > 0$ in $Q_T \cup S_T \setminus \overline{Q_{\overline{t}}}$. Then there exists $t_0 > \overline{t}$ such that $\beta(t_0) > 0$ and

$$\min_{\overline{\Omega}} u(x, t_0) > 2\sigma, \tag{2.3}$$

where σ is a positive constant.

Let $G_N(x, y; t-\tau)$ denote the Green's function for the heat equation given by

$$u_t - \Delta u = 0 \quad \text{for } x \in \Omega,\ t > 0$$

with homogeneous Neumann boundary condition. We note that the Green's function has the following properties (see, for example, [12, 13]:

$$\begin{aligned}
&G_N(x,y;t-\tau) \geq 0, && x, y \in \Omega,\ 0 \leq \tau < t, && (2.4)\\
&\int_\Omega G_N(x,y;t-\tau)\, dy = 1, && x \in \Omega,\ 0 \leq \tau < t, && (2.5)\\
&G_N(x,y;t-\tau) \geq c_1, && x,\ y \in \overline{\Omega},\ t-\tau \geq \varepsilon, && (2.6)\\
&|G_N(x,y;t-\tau) - 1/|\Omega|| \leq c_2 \exp[-c_3(t-\tau)], && x,\ y \in \overline{\Omega},\ t-\tau \geq \varepsilon,\\
&\int_{\partial\Omega} G_N(x,y;t-\tau)\, dS_y \leq \frac{c_4}{\sqrt{t-\tau}}, && x \in \overline{\Omega},\ 0 < t-\tau \leq \varepsilon,
\end{aligned}$$

for some small $\varepsilon > 0$. Here by c_i $(i \in \mathbb{N})$ we denote positive constants.

Now we introduce conditions on several auxiliary comparison functions. We suppose that $h(s) \in C^1((0,+\infty)) \cap C([0,+\infty))$, $h(s) > 0$ for $s > 0$, $h'(s) \geq 0$ for $s > 0$, $g(s) \geq h(s)$ and

$$\int^{+\infty} \frac{ds}{h(s)} < +\infty.$$

Let $\xi(t)$ be a positive continuous function for $t \geq t_0$ such that

$$\int_{t_0}^{+\infty} \xi(t)\, dt < \frac{\sigma}{2} \tag{2.7}$$

and $\gamma(t)$ be a positive continuous function for $t \geq t_0$ such that $\gamma(t_0) = \beta(t_0) h(2\sigma)$ and

$$\int_{t_0}^{t} \gamma(\tau) \int_{\partial\Omega} G_N(x,y;t-\tau)\, dS_y\, d\tau < \frac{\sigma}{2} \quad \text{for } x \in \overline{\Omega},\ t \geq t_0. \tag{2.8}$$

We consider the following problem

$$\begin{cases} v_t = \Delta v - \xi(t) \ \text{ for } \ x \in \Omega,\ \ t > t_0,\\ \dfrac{\partial v(x,t)}{\partial \nu} = \beta(t) h(v) - \gamma(t) \ \text{ for } \ x \in \partial\Omega,\ \ t > t_0,\\ v(x,t_0) = 2\sigma \ \text{ for } \ x \in \Omega. \end{cases} \tag{2.9}$$

To find lower bound for $v(x,t)$ we represent (2.9) in equivalent form

$$\begin{aligned} v(x,t) = 2\sigma \int_\Omega G_N(x,y;t)\,dy - \int_{t_0}^t \int_\Omega G_N(x,y;t-\tau)\xi(\tau)\,dy\,d\tau \\ + \int_{t_0}^t \int_{\partial\Omega} G_N(x,y;t-\tau)\,(\beta(\tau)h(v) - \gamma(\tau))\,dS_y\,d\tau. \end{aligned} \tag{2.10}$$

Using (2.7), (2.8) and the properties of the Green's function (2.4), (2.5), we obtain from (2.10)

$$v(x,t) \geq 2\sigma - \int_{t_0}^t \xi(\tau)\,d\tau - \int_{t_0}^t \gamma(\tau) \int_{\partial\Omega} G_N(x,y;t-\tau) dS_y\,d\tau > \sigma. \tag{2.11}$$

As in [22] we put

$$m(t) = \int_\Omega \int_{v(x,t)}^{+\infty} \frac{ds}{h(s)}\,dx.$$

We observe that $m(t)$ is well defined and positive for $t \geq t_0$. Since $v(x,t)$ is the solution of (2.9), we get

$$\begin{aligned} m'(t) &= -\int_\Omega \frac{v_t}{h(v)}\,dx = -\int_\Omega \frac{\Delta v}{h(v)}\,dx + \xi(t)\int_\Omega \frac{dx}{h(v)} \\ &= -\int_\Omega \operatorname{div}\left(\frac{\nabla v}{h(v)}\right)\,dx - \int_\Omega \frac{h'(v)\|\nabla v\|^2}{h^2(v)}\,dx + \xi(t)\int_\Omega \frac{dx}{h(v)}. \end{aligned}$$

Applying the inequality $h'(v) \geq 0$, Gauss theorem, the boundary condition in (2.9) and (2.11), we obtain for $t \geq t_0$

$$m'(t) \leq -\int_{\partial\Omega} \frac{1}{h(v)} \frac{\partial v}{\partial \nu}\,dS + \xi(t)\frac{|\Omega|}{h(\sigma)} \leq -|\partial\Omega|\beta(t) + \frac{|\Omega|\xi(t) + |\partial\Omega|\gamma(t)}{h(\sigma)}. \tag{2.12}$$

Due to (2.2), (2.6)–(2.8) $m(t)$ is negative for large values of t. Hence $v(x,t)$ blows up in finite time T_0. Applying Theorem 2.1 to $v(x,t)$ and $u(x,t)$ in $Q_T \setminus \overline{Q_{t_0}}$ for any $T \in (t_0, T_0)$, we prove the theorem. □

Remark 2.3. If $u_0(x)$ is positive in $\overline{\Omega}$ we can obtain an upper bound for blow-up time of the solution. We put $t_0 = 0$ and $v(x,0) = u_0(x) - \varepsilon$ in (2.9) for $\varepsilon \in (0, \min_{\overline{\Omega}} u_0(x))$. Integrating (2.12) over $[0,T]$, we have

$$m(t) \leq m(0) - |\partial\Omega| \int_0^T \beta(t)\,dt + \int_0^T \frac{|\Omega|\xi(t) + |\partial\Omega|\gamma(t)}{h(\sigma)}\,dt.$$

Since $m(t) > 0$ and ε, $\xi(t)$, $\gamma(t)$ are arbitrary we conclude that the solution of (1.1)–(1.3) blows up in finite time T_b, where $T_b \leq T$ and

$$\int_\Omega \int_{u_0(x)}^{+\infty} \frac{ds}{h(s)}\,dx = |\partial\Omega| \int_0^T \beta(t)\,dt.$$

Remark 2.4. We note that (1.1)–(1.3) with $u_0(x) \equiv 0$ may have trivial and blow-up solutions under the assumptions of Theorem 2.2. Indeed, let the conditions of Theorem 2.2 hold, $\alpha(t) \equiv 0$, $\beta(t) \equiv 1$ and $g(u) = u^p$, $u \in [0,\gamma]$ for some $\gamma > 0$ and $0 < p < 1$. As it was proved in [3], problem (1.1)–(1.3) has trivial and positive for $t > 0$ solutions and last one blows up in finite time by Theorem 2.2.

To prove next blow-up result for (1.1)–(1.3) we need a comparison principle with unstrict inequality in the boundary condition.

Theorem 2.5. *Let $\delta > 0$ and $v(x,t), w(x,t) \in C^{2,1}(Q_T) \cap C^{1,0}(Q_T \cup \Gamma_T)$ satisfy the inequalities:*

$$v_t - \Delta v - \alpha(t) f(v) + \delta < w_t - \Delta w - \alpha(t) f(w) \quad \text{in } Q_T,$$
$$\frac{\partial v(x,t)}{\partial \nu} \le \frac{\partial w(x,t)}{\partial \nu} \quad \text{on } S_T,$$
$$v(x,0) < w(x,0) \quad \text{in } \overline{\Omega}.$$

Then

$$v(x,t) \le w(x,t) \quad \text{in } Q_T.$$

Proof. Let τ be any positive constant such that $\tau < T$ and a positive function $\gamma(x) \in C^2(\overline{\Omega})$ satisfy the following inequality

$$\frac{\partial \gamma(x)}{\partial \nu} > 0 \quad \text{on } \partial\Omega.$$

For positive ε we introduce

$$w_\varepsilon(x,t) = w(x,t) + \varepsilon\gamma(x). \tag{2.13}$$

Obviously,

$$v(x,0) < w_\varepsilon(x,0) \quad \text{in } \overline{\Omega}, \qquad \frac{\partial v(x,t)}{\partial \nu} < \frac{\partial w_\varepsilon(x,t)}{\partial \nu} \quad \text{on } S_\tau.$$

Moreover,

$$v_t - \Delta v - \alpha(t) f(v) < w_{\varepsilon t} - \Delta w_\varepsilon - \alpha(t) f(w_\varepsilon) \quad \text{in } Q_\tau,$$

if we take ε so small that

$$\delta > \varepsilon\Delta\gamma + \alpha(t)[f(w + \varepsilon\gamma) - f(w)] \quad \text{in } Q_\tau.$$

Applying Theorem 2.1 with $\beta(t) \equiv 0$, we obtain

$$v(x,t) < w_\varepsilon(x,t) \quad \text{in } Q_\tau.$$

Passing to the limit as $\varepsilon \to 0$ and $\tau \to T$, we prove the theorem. □

Theorem 2.6. *Let $f(s) > 0$ for $s > 0$,*

$$\int^{+\infty} \frac{ds}{f(s)} < +\infty \tag{2.14}$$

and

$$\int_0^{+\infty} \alpha(t)\, dt = +\infty. \tag{2.15}$$

Then any nontrivial nonnegative solution of (1.1)–(1.3) blows up in finite time.

Proof. We suppose that $u(x,t)$ is a nontrivial nonnegative solution which exists in Q_T for any positive T. In Theorem 2.2 we proved (2.3). Let $\xi(t)$ be a positive continuous function for $t \ge t_0$ such that

$$\max_{[\sigma, 2\sigma]} f(s) \int_{t_0}^{+\infty} \xi(t)\, dt < \sigma. \tag{2.16}$$

We consider the following auxiliary problem

$$\begin{cases} v'(t) = \alpha(t)f(v) - \xi(t)f(v), & t > t_0, \\ v(t_0) = 2\sigma. \end{cases} \tag{2.17}$$

We prove at first that

$$v(t) > \sigma \quad \text{for } t \geq t_0. \tag{2.18}$$

Suppose there exist t_1 and t_2 such that

$$t_2 > t_1 \geq t_0, \qquad v(t_1) = 2\sigma, \qquad v(t_2) = \sigma,$$

and

$$v(t) > \sigma \quad \text{for } t \in [t_0, t_2) \quad \text{and} \quad v(t) \leq 2\sigma \quad \text{for } t \in [t_1, t_2].$$

Integrating the equation in (2.17) over $[t_1, t_2]$, we have due to (2.16)

$$v(t_2) \geq -\max_{[\sigma,2\sigma]} f(s) \int_{t_1}^{t_2} \xi(t)\, dt + v(t_1) > \sigma.$$

A contradiction proves (2.18).

From (2.17) we obtain

$$\int_{2\sigma}^{v(t)} \frac{ds}{f(s)} = \int_{t_0}^{t} [\alpha(\tau) - \xi(\tau)]\, d\tau. \tag{2.19}$$

By (2.14)–(2.16) the left side of (2.19) is finite and the right side of (2.19) tends to infinity as $t \to \infty$. Hence the solution of (2.17) blows up in finite time T_0. Applying Theorem 2.5 to $v(t)$ and $u(x,t)$ in $Q_T \setminus \overline{Q_{t_0}}$ for any $T \in (t_0, T_0)$, we prove the theorem. □

Remark 2.7. If $u_0(x)$ is positive in $\overline{\Omega}$ we can obtain an upper bound for blow-up time of the solution. Taking $t_0 = 0$, we conclude from (2.19) that the solution of (1.1)–(1.3) blows up in finite time T_b, where $T_b \leq T$ and

$$\int_{\min_{\overline{\Omega}} u_0(x)}^{+\infty} \frac{ds}{f(s)} = \int_0^T \alpha(t)\, dt.$$

Remark 2.8. Theorem 2.6 does not hold if $f(s)$ is not positive for $s > 0$. To show this we suppose that $f(u_1) = 0$ for some $u_1 > 0$, $\beta(t) \equiv 0$, $u_0(x) = u_1$. Then problem (1.1)–(1.3) has the solution $u(x,t) = u_1$.

Remark 2.9. We note that (2.14) is necessary condition for blow-up of solutions of (1.1)–(1.3) with $\beta(t) \equiv 0$. Let $f(s) > 0$ for $s > 0$ and

$$\int^{+\infty} \frac{ds}{f(s)} = +\infty.$$

Then any solution of (1.1)–(1.3) is global. Indeed, let $u(x,t)$ be a nontrivial solution of (1.1)–(1.3). Then there exist $t_0 \geq 0$ and $x \in \Omega$ such that $u(x, t_0) > 0$.

We consider the following problem

$$\begin{cases} v'(t) = (\alpha(t) + \xi(t))f(v), \ t > t_0, \\ v(t_0) > \max_{\overline{\Omega}} u(x, t_0) > 0, \end{cases} \tag{2.20}$$

where $\xi(t)$ is some positive continuous function for $t \geq t_0$. Obviously, $v(t)$ is global solution of (2.20). Applying Theorem 2.5 to $u(x,t)$ and $v(t)$ in $Q_T \setminus \overline{Q_{t_0}}$ for any $T > t_0$, we prove the theorem.

Remark 2.10. Problem (1.1)–(1.3) with $u_0(x) \equiv 0$ may have trivial and blow-up solutions under the assumptions of Theorem 2.6. Indeed, let the conditions of Theorem 2.6 hold, $\beta(t) \equiv 0$, $f(s)$ be a nondecreasing Hölder continuous function on $[0,\epsilon]$ for some $\epsilon > 0$ and

$$\int_0^{\epsilon} \frac{ds}{f(s)} < +\infty.$$

As it was proved in [15], problem (1.1)–(1.3) has trivial and positive for $t > 0$ solutions and last one blows up in finite time by Theorem 2.6.

3 Global existence

To formulate global existence result for problem (1.1)–(1.3) we suppose:

$$f(s) \text{ is a nonnegative locally Hölder continuous function for } s \geq 0, \tag{3.1}$$

$$\text{there exists } p > 0 \text{ such that } f(s) \text{ is a positive nondecreasing function for } s \in (0,p), \tag{3.2}$$

$$\int_0 \frac{ds}{f(s)} = +\infty, \qquad \lim_{s\to 0} \frac{g(s)}{s} = 0, \tag{3.3}$$

$$\int_0^{+\infty} (\alpha(t) + \beta(t))\, dt < +\infty \tag{3.4}$$

and there exist positive constants γ, t_0 and K such that $\gamma > t_0$ and

$$\int_{t-t_0}^{t} \frac{\beta(\tau)d\tau}{\sqrt{t-\tau}} \leq K \quad \text{for } t \geq \gamma. \tag{3.5}$$

Theorem 3.1. *Let (3.1)–(3.5) hold. Then problem (1.1)–(1.3) has bounded global solution for small initial datum.*

Proof. It is well known that problem (1.1)–(1.3) has a local nonnegative classical solution $u(x,t)$. Let $y(x,t)$ be a solution of the following problem

$$\begin{cases} y_t = \Delta y, \ x \in \Omega, \ t > 0, \\ \dfrac{\partial y(x,t)}{\partial \nu} = \xi(t) + \beta(t), \ x \in \partial\Omega, \ t > 0, \\ y(x,0) = 1, \ x \in \Omega, \end{cases} \tag{3.6}$$

where $\xi(t)$ is a positive continuous function that satisfies (3.4), (3.5) with $\beta(t) = \xi(t)$. According to Lemma 3.3 of [10] there exists a positive constant Y such that

$$1 \leq y(x,t) \leq Y, \qquad x \in \Omega, \ t > 0.$$

Due to (3.2), (3.3) for any $a \in (0,p)$, there exist $\varepsilon(a)$ and a positive continuous function $\eta(t)$ such that

$$0 < \varepsilon(a) < \frac{a}{Y}, \qquad \int_0^{\infty} \eta(t)\, dt < \infty \quad \text{and} \quad \int_{\varepsilon Y}^{a} \frac{ds}{f(s)} > Y \int_0^{\infty} (\alpha(t) + \eta(t))\, dt$$

for any $\varepsilon \in (0, \varepsilon(a))$. Now for any $T > 0$ we construct a positive supersolution of (1.1)–(1.3) in Q_T in such a form that

$$\overline{u}(x,t) = \varepsilon z(t) y(x,t),$$

where function $z(t)$ is defined in the following way

$$\int_{\varepsilon Y}^{\varepsilon Y z(t)} \frac{ds}{f(s)} = Y \int_0^t (\alpha(\tau) + \eta(\tau))\, d\tau.$$

It is easy to see that $\varepsilon Y z(t) < a$ and $z(t)$ is the solution of the following Cauchy problem

$$z'(t) - \frac{1}{\varepsilon} (\alpha(t) + \eta(t)) f(\varepsilon Y z(t)) = 0, \qquad z(0) = 1.$$

After simple computations it follows that

$$\begin{aligned} \overline{u}_t - \Delta \overline{u} - \alpha(t) f(\overline{u}) &= \varepsilon z' y + \varepsilon z y_t - \varepsilon z \Delta y - \alpha(t) f(\varepsilon z y) \\ &\geq \alpha(t)(f(\varepsilon Y z(t)) - f(\varepsilon z y)) + \eta(t) f(\varepsilon Y z(t)) > 0, \qquad x \in \Omega,\ t > 0, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \overline{u}(x,t)}{\partial \nu} - \beta(t) g(\overline{u}) &= \varepsilon z(t)(\xi(t) + \beta(t)) - \beta(t) g(\varepsilon z(t) y(x,t)) \\ &> \varepsilon z(t) \beta(t) \left[1 - \frac{g(\varepsilon z(t) y(x,t))}{\varepsilon z(t) y(x,t)} y(x,t) \right] \geq 0 \end{aligned}$$

for small values of a. Thus, by Theorem 2.1 there exists bounded global solution of (1.1)–(1.3) for any initial datum satisfying the inequality

$$u_0(x) < \varepsilon. \qquad \square$$

Remark 3.2. We suppose that $g(s)$ is a nondecreasing positive function for $s > 0$, $f(s) > 0$ for $s > 0$ and (2.1), (2.14) hold. Then by Theorem 2.2 and Theorem 2.6 (3.4) is necessary for global existence of solutions of (1.1)–(1.3).

Let for any $a > 0$ $g(s) > \delta(a) > 0$ if $s > a$. Then arguing in the same way as in the proof of Lemma 3.3 of [10] it is easy to show that (3.5) is necessary for the existence of nontrivial bounded global solutions of (1.1)–(1.3).

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References

[1] C. Bandle, H. Brunner Blowup in diffusion equations: A survey, *J. Comput. Appl. Math.* **97**(1998), No. 1–2, 3–22. https://doi.org/10.1016/S0377-0427(98)00100-9; MR1651764; Zbl 0932.65098

[2] J. Bebernes, D. Eberly, *Mathematical problems from combustion theory*, Applied Mathematical Sciences, Vol. 83, Springer-Verlag, New York, 1989. https://doi.org/10.1007/978-1-4612-4546-9; Zbl 0692.35001

[3] C. Cortazar, M. Elgueta, J. D. Rossi, Uniqueness and non-uniqueness for a system of heat equations with nonlinear coupling at the boundary, *Nonlinear Anal.* **37**(1999), No. 2, 257–267. https://doi.org/10.1016/S0362-546X(98)00046-7; MR1689760; Zbl 0932.35123

[4] J. Ding, X. Shen, Blow-up in p-Laplacian heat equations with nonlinear boundary conditions, *Z. Angew. Math. Phys.* **67**(2016), No. 125. https://doi.org/10.1007/s00033-016-0720-5; MR3451372; Zbl 1358.35055

[5] J. Ding, X. Shen, Blow-up analysis for a class of nonlinear reaction diffusion equations with Robin boundary conditions, *Math. Meth. Appl. Sci.* **41**(2018), No. 4, 1683–1696. https://doi.org/10.1002/mma.4697; MR3767261; Zbl 1384.35043

[6] Z. Fang, Y. Wang, Blow-up analysis for a semilinear parabolic equation with time-dependent coefficients under nonlinear boundary flux, *Z. Angew. Math. Phys.* **66**(2015), 2525–2541. https://doi.org/10.1007/s00033-015-0537-7; MR3412310; Zbl 1328.35108

[7] A. Friedman, *Partial differential equations of parabolic type*, Prentice-Hall, Englewood Cliffs, New Jersey, 1964. Zbl 0144.34903

[8] A. Friedman, J. B. McLeod, Blow-up of positive solutions of semilinear heat equations, *Indiana Univ. Math. J.* **34**(1985), No. 2, 425–447. MR783924; Zbl 0576.35068

[9] V. A. Galaktionov, J. L. Vazquez, Necessary and sufficient conditions for complete blow-up and extinction for one-dimensional quasilinear heat equations, *Arch. Ration. Mech. Anal.* **129**(1995), 225–244. https://doi.org/10.1007/BF00383674; MR1328477; Zbl 0827.35055

[10] A. Gladkov, T. Kavitova Blow-up problem for semilinear heat equation with nonlinear nonlocal boundary condition, *Appl. Anal.* **95**(2016), No. 9, 1974–1988. https://doi.org/10.1080/00036811.2015.1080353; MR3515089; Zbl 1362.35163

[11] B. Hu, *Blow-up theories for semilinear parabolic equations*, Lecture Notes in Mathematics, Vol. 2018, Springer-Verlag, Berlin, 2011. Zbl 1226.35001

[12] B. Hu, H. M. Yin, Critical exponents for a system of heat equations coupled in a non-linear boundary condition, *Math. Methods Appl. Sci.* **19**(1996), No. 14, 1099–1120. https://doi.org/10.1002/(SICI)1099-1476(19960925)19:14<1099::AID-MMA780>3.0.CO;2-J; MR1270664

[13] C. S. Kahane, On the asymptotic behavior of solutions of parabolic equations, *Czechoslovak Math. J.* **33**(1983), No. 2, 262–285. MR699025; Zbl 0534.35051

[14] A. V. Lair, M. E. Oxley, A Necessary and sufficient condition for global existence for a degenerate parabolic boundary value problem, *J. Math. Anal. Appl.* **221**(1998), No. 1, 338–348. https://doi.org/10.1006/jmaa.1997.5900; MR1619148; Zbl 0931.35082

[15] R. Laistera, J. C. Robinson, M. Sierżęga, A necessary and sufficient condition for uniqueness of the trivial solution in semilinear parabolic equations, *J. Differential Equations* **262**(2017), No. 10, 4979–4987. https://doi.org/10.1016/j.jde.2017.01.014; Zbl 1372.35156

[16] H. A. Levine, L. E. Payne Nonexistence theorems for the heat equation with nonlinear boundary conditions and for the porous medium equation backward in time, *J. Differential Equations* **16**(1974), No. 2, 319–334. https://doi.org/10.1016/0022-0396(74)90018-7; MR470481; Zbl 0285.35035

[17] J. López-Gómez, V. Márquez, N. Wolanski, Blow up results and localization of blow up points for the heat equation with a nonlinear boundary condition, *J. Differential Equations* **92**(1991), No. 2, 384–401. https://doi.org/10.1016/0022-0396(91)90056-F; MR1120912; Zbl 0735.35016

[18] M. Marras, S. Vernier Piro, Reaction-diffusion problems under non-local boundary conditions with blow-up solutions, *J. Inequal. Appl.* **2014**(2014), No. 167. https://doi.org/10.1186/1029-242X-2014-167; Zbl 06780049

[19] P. Meier, Blow-up of solutions of semilinear parabolic differential equations, *Z. Angew. Math. Phys.* **39**(1988), 135–149. https://doi.org/10.1007/BF00945760; MR937698; Zbl 0661.35051

[20] L. E. Payne, G. A. Philippin, Blow-up phenomena in parabolic problems with time-dependent coefficients under Neumann boundary conditions, *Proc. R. Soc. Edinburgh Sect. A.* **142**(2012), No. 3, 625–631. https://doi.org/10.1017/S0308210511000485; MR2945975; Zbl 1241.35027

[21] L. E. Payne, P. W. Schaefer, Lower bound for blow-up time in parabolic problems under Neumann conditions, *Appl. Anal.* **85**(2006), No. 10, 1301–1311. https://doi.org/10.1080/00036810600915730; MR2263927; Zbl 1110.35032

[22] D. F. Rial, J. D. Rossi, Blow-up results and localization of blow-up points in an N-dimensional smooth domain, *Duke Math. J.* **88**(1997), No. 2, 391–405. https://doi.org/10.1215/S0012-7094-97-08816-5; MR1455526; Zbl 0884.35071

[23] W. Walter, *Differential and integral inequalities*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 55, Springer-Verlag, Berlin, Heidelberg, 1970. https://doi.org/10.1002/zamm.19720520931; Zbl 0252.35005

[24] W. Walter, On existence and nonexistence in the large of solutions of parabolic differential equations with a nonlinear boundary condition, *SIAM J. Math. Anal.* **6**(1975), No. 1, 85–90. https://doi.org/10.1137/0506008; MR364868; Zbl 0268.35052

[25] M. X. Wang, Y. H. Wu, Global existence and blow-up problems for quasilinear parabolic equations with nonlinear boundary conditions, *SIAM J. Math. Anal.* **24**(1993), No. 6, 1515–1521. https://doi.org/10.1137/0524085; MR1241155; Zbl 0790.35042

[26] N. Wolanski, Global behavior of positive solutions to nonlinear diffusion problems with nonlinear absorption through the boundary, *SIAM J. Math. Anal.* **24**(1993), No. 2, 317–326. https://doi.org/10.1137/0524021; MR1205529; Zbl 0778.35047

[27] E. Yanagida, Blow-up of sign-changing solutions for a one-dimensional nonlinear diffusion equation, *Nonlinear Anal.* **185** (2019), 193–205. https://doi.org/10.1016/j.na.2019.03.011; Zbl 1418.35048

[28] J. Zhang, F. Li, Global existence and blow-up phenomena for divergence form parabolic equation with time-dependent coefficient in multidimensional space, *Z. Angew. Math. Phys.* **70**(2019), No. 150. https://doi.org/10.1007/s00033-019-1195-y; MR4010989; Zbl 1423.35175