Electronic Journal of Qualitative Theory of Differential Equations
2020, No. 44, 1–32; https://doi.org/10.14232/ejqtde.2020.1.44 www.math.u-szeged.hu/ejqtde/

Antiprincipal solutions at infinity for symplectic systems on time scales

Iva Dřímalová and **Roman Šimon Hilscher**[✉]

Department of Mathematics and Statistics, Faculty of Science, Masaryk University, Kotlářská 2, Brno, CZ-61137, Czech Republic

Received 15 May 2020, appeared 30 June 2020
Communicated by Mihály Pituk

Abstract. In this paper we introduce a new concept of antiprincipal solutions at infinity for symplectic systems on time scales. This concept complements the earlier notion of principal solutions at infinity for these systems by the second author and Šepitka (2016). We derive main properties of antiprincipal solutions at infinity, including their existence for all ranks in a given range and a construction from a certain minimal antiprincipal solution at infinity. We apply our new theory of antiprincipal solutions at infinity in the study of principal solutions, and in particular in the Reid construction of the minimal principal solution at infinity. In this work we do not assume any normality condition on the system, and we unify and extend to arbitrary time scales the theory of antiprincipal solutions at infinity of linear Hamiltonian differential systems and the theory of dominant solutions at infinity of symplectic difference systems.

Keywords: symplectic system on time scale, antiprincipal solution at infinity, principal solution at infinity, nonoscillation, linear Hamiltonian system, normality.

2020 Mathematics Subject Classification: 34N05, 34C10, 39A12, 39A21.

1 Introduction

In this paper we focus on symplectic dynamic system

$$x^{\Delta} = \mathcal{A}(t)x + \mathcal{B}(t)u; \quad u^{\Delta} = \mathcal{C}(t)x + \mathcal{D}(t)u, \quad t \in [a,\infty)_{\mathbb{T}}, \tag{S}$$

where $\mathbb{T}$ is a time scale, that is, $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. We assume that $\mathbb{T}$ is unbounded from above and bounded from below with $a := \min \mathbb{T}$ and $[a,\infty)_{\mathbb{T}} := [a,\infty) \cap \mathbb{T}$. The coefficients $\mathcal{A}(t)$, $\mathcal{B}(t)$, $\mathcal{C}(t)$, $\mathcal{D}(t)$ of system (S) are real piecewise rd-continuous $n \times n$ matrices on $[a,\infty)_{\mathbb{T}}$ such that the $2n \times 2n$ matrices

$$\mathcal{S}(t) := \begin{pmatrix} \mathcal{A}(t) & \mathcal{B}(t) \\ \mathcal{C}(t) & \mathcal{D}(t) \end{pmatrix}, \quad \mathcal{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

[✉]Corresponding author. Email: hilscher@math.muni.cz

satisfy the identity

$$\mathcal{S}^T(t)\,\mathcal{J} + \mathcal{J}\mathcal{S}(t) + \mu(t)\,\mathcal{S}^T(t)\,\mathcal{J}\mathcal{S}(t) = 0, \quad t \in [a,\infty)_{\mathbb{T}},$$

where $\mu(t)$ is the graininess function of $\mathbb{T}$. Solutions of (S) are piecewise rd-continuously Δ-differentiable functions, i.e., they are continuous on $[a,\infty)_{\mathbb{T}}$ and their Δ-derivative is piecewise rd-continuous on $[a,\infty)_{\mathbb{T}}$. Basic theory of dynamic equations on time scales, including the theory of symplectic dynamic systems, are covered for example in the monographs [6,7]. Advanced topics about symplectic systems on time scales, such as the theory of Riccati matrix dynamic equations, quadratic functionals, oscillation theorems, Rayleigh principle and their applications e.g. in the optimal control theory can be found in the references [1,11,13,15–17,28–30]. Our particular interest is connected with the theory of principal and antiprincipal solutions of (S) at infinity, which was initiated by Došlý in [9] for system (S) satisfying a certain eventual normality or controllability assumption. In 2016 the second author and Šepitka provided in [22] a generalization of the concept of the principal solution at infinity to a possibly abnormal (or uncontrollable) system (S), see also [18–21].

In the present paper we continue in this investigation by introducing the corresponding theory of antiprincipal solutions of (S) at infinity in the absence of the eventual normality or controllability assumption (Definition 4.1). Note that these solutions are also called nonprincipal solutions at infinity in the context of the reference [9], or dominant solutions at infinity in the context of the references [2,10,24,25]. We present three sets of results about the antiprincipal solutions of (S) at infinity. The first set of results is devoted to their basic properties, such as the invariance with respect to the considered interval (Theorem 4.3), a characterization in terms of the limit of the associated S-matrix (Theorem 4.4), and the invariance with respect to a certain relation between conjoined bases (Theorems 4.6 and 4.7). The second set of results is devoted to the existence of antiprincipal solutions of (S) at infinity (Theorem 5.3), which requires to derive as main tools an important characterization of minimal conjoined bases of (S) on a given interval (Theorem 5.1) and a characterization of the T-matrices associated with conjoined bases of (S) (Theorem 5.2). The third set of results is devoted to applications of antiprincipal solutions of (S) at infinity, in particular in the connection with the so-called minimal antiprincipal solutions of (S) at infinity (Theorems 6.3, 6.4, and 6.6) and maximal antiprincipal solutions of (S) at infinity (Theorems 6.5 and 6.6). These are, respectively, the antiprincipal solutions of (S) at infinity with the smallest and the largest possible rank (see Section 4).

The main condition on system (S) is the assumption of its nonoscillation, i.e., every conjoined basis (X,U) of (S) is assumed to be nonoscillatory. This means that for every (X,U) there exists a point $\alpha \in [a,\infty)_{\mathbb{T}}$ such that (X,U) has no focal points in the real interval (α,∞), which is according to [14, Definition 4.1] formulated as

$$\operatorname{Ker} X(s) \subseteq \operatorname{Ker} X(t) \quad \text{for all } t,s \in [\alpha,\infty)_{\mathbb{T}} \text{ with } t \le s, \tag{1.1}$$

$$X(t)\,[X^{\sigma}(t)]^{\dagger}\,\mathcal{B}(t) \ge 0 \quad \text{for all } t \in [\alpha,\infty)_{\mathbb{T}}. \tag{1.2}$$

Condition (1.1) means that the kernel of $X(t)$ is nonincreasing on the time scale interval $[\alpha,\infty)_{\mathbb{T}}$. Hence, the point α can be chosen large enough, so that the set $\operatorname{Ker} X(t)$ is constant on $[\alpha,\infty)_{\mathbb{T}}$. Noninvertible matrix functions, such as $X(t)$ above or $S(t)$ defined in (3.1) below, then naturally occur in our theory. For this reason we utilize the Moore–Penrose pseudoinverse matrices as the principal tool for their investigation (see Remark 2.1).

The theory of antiprincipal solutions at infinity for linear Hamiltonian differential systems and the theory of dominant solutions at infinity for symplectic difference systems were developed in [20,23] and [10,24], respectively. The present work is not a mere unification of those, however. Working on arbitrary time scales we also provide a clarification of incomplete or missing arguments in several results compared with the corresponding original continuous time or discrete time statements (see the proofs of Proposition 3.18 and Theorems 3.4, 5.1, and 6.5). This paper together with [22] can be regarded as a starting point for a unified Sturmian theory for Hamiltonian and symplectic dynamic systems on time scales, whose first steps were taken in [5,9] about twenty years ago. Recent progress in the continuous and discrete Sturmian theory, where antiprincipal solutions (or dominant solutions) at infinity play a fundamental role, is documented in the papers [25–27]. We strongly believe that future development in this unified Sturmian theory will benefit from the results obtained in the presented work (see also Section 7).

The paper is organized as follows. In Section 2 we briefly recall some results from matrix analysis, which we directly use later in this paper. In Section 3 we provide basic results about symplectic systems on time scales, which form the base for the definition of an antiprincipal solution of ($\mathbb{S}$) at infinity. In Section 4 we introduce the notion of an antiprincipal solution at infinity for system ($\mathbb{S}$) and include its main properties, which are connected to the relation being contained for conjoined bases of ($\mathbb{S}$). In Section 5 we derive the existence of antiprincipal solutions at infinity for a nonoscillatory system ($\mathbb{S}$), including the existence of antiprincipal solutions at infinity with arbitrary given rank and pointing out the essential role played by the minimal antiprincipal solutions of ($\mathbb{S}$) at infinity. In Section 6 we focus on applications of the presented theory of antiprincipal solutions at infinity, in particular in the theory of principal solutions of ($\mathbb{S}$) and in the Reid construction of the minimal principal solution of ($\mathbb{S}$) at infinity. Finally, in Section 7 we comment about the results of this paper in the context of some open problems.

2 Notation and matrix analysis

In this section we introduce basic notation and recall some properties of the Moore–Penrose pseudoinverse matrices, which we will use later. For a real matrix M we denote by $\operatorname{Im} M$, $\operatorname{Ker} M$, $\operatorname{rank} M$, M^T, M^{-1}, $M^\dagger$ the image, kernel, rank (i.e., the dimension of the image), transpose, inverse (if M is a square invertible matrix), and the Moore–Penrose pseudoinverse of M (see its definition below), respectively. For a symmetric matrix $M \in \mathbb{R}^{n\times n}$ we write $M \geq 0$ or $M > 0$ if M is positive semidefinite or positive definite, respectively. If M_1 and M_2 are two real symmetric $n \times n$ matrices, then we write $M_1 \leq M_2$ when $M_2 - M_1 \geq 0$, respectively we write $M_1 < M_2$ when $M_2 - M_1 > 0$. The identity matrix will be denoted by I.

Furthermore, let V and W be linear subspaces in $\mathbb{R}^n$. We denote by $V \oplus W$ the direct sum of the subspaces V and W, and by $V^\perp$ the orthogonal complement of the subspace V in $\mathbb{R}^n$. By $\mathcal{P}_V$ we denote the orthogonal projector onto the subspace V. Then the $n \times n$ matrix $\mathcal{P}_V$ is symmetric, idempotent, and positive semidefinite.

In our approach it is essential to use the properties of the Moore–Penrose pseudoinverse. First we recall its definition via the following four properties, which will often be used in our calculations. Let M be a real $m \times n$ matrix. A real $n \times m$ matrix $M^\dagger$ satisfying the equalities

$$MM^\dagger M = M, \quad M^\dagger MM^\dagger = M^\dagger, \quad M^\dagger M = (M^\dagger M)^T, \quad MM^\dagger = (MM^\dagger)^T \tag{2.1}$$

is called the Moore–Penrose pseudoinverse of the matrix M. We will use the following properties of the Moore–Penrose pseudoinverse, which can be found e.g. in [3,4,8] and [14, Lemma 2.1]. These properties play an essential role in our theory.

Remark 2.1. For any matrix $M \in \mathbb{R}^{m\times n}$ there exists a unique matrix $M^\dagger \in \mathbb{R}^{n\times m}$ satisfying the identities in (2.1). Moreover, the following properties hold.

(i) $(M^\dagger)^T = (M^T)^\dagger$, $(M^\dagger)^\dagger = M$, and $\operatorname{Im} M^\dagger = \operatorname{Im} M^T$, $\operatorname{Ker} M^\dagger = \operatorname{Ker} M^T$.

(ii) The matrix $MM^\dagger$ is the orthogonal projector onto $\operatorname{Im} M$, and the matrix $M^\dagger M$ is the orthogonal projector onto $\operatorname{Im} M^T$.

(iii) Let $\{M_j\}_{j=1}^\infty$ be a sequence of $m\times n$ matrices such that $M_j \to M$ for $j\to\infty$. Then the limit of $M_j^\dagger$ for $j\to\infty$ exists if and only if there exists an index $j_0\in\mathbb{N}$ such that $\operatorname{rank} M_j = \operatorname{rank} M$ for all $j\ge j_0$. In this case $\lim_{j\to\infty} M_j^\dagger = M^\dagger$.

(iv) Let $M(t)$ be an $m\times n$ matrix function defined on the interval $[a,\infty)_\mathbb{T}$ such that $M(t)\to M$ for $t\to\infty$. Then the limit of $M^\dagger(t)$ for $t\to\infty$ exists if and only if there exists a point $t_0\in[a,\infty)_\mathbb{T}$ such that $\operatorname{rank} M(t) = \operatorname{rank} M$ for all $t\in[t_0,\infty)_\mathbb{T}$. In this case $\lim_{t\to\infty} M^\dagger(t) = M^\dagger$.

(v) Let M_1 and M_2 be symmetric and positive semidefinite matrices such that $M_1\le M_2$. Then inequality $M_2^\dagger \le M_1^\dagger$ holds if and only if $\operatorname{Im} M_1 = \operatorname{Im} M_2$, or equivalently if and only if $\operatorname{rank} M_1 = \operatorname{rank} M_2$.

(vi) If M is symmetric positive and semidefinite, then also $M^\dagger$ is symmetric and positive semidefinite. That is, if $M\ge 0$, then also $M^\dagger\ge 0$.

(vii) For any matrices M and N with suitable dimensions, the pseudoinverse of their product is given by

$$(MN)^\dagger = (\mathcal{P}_{\operatorname{Im} M^T} N)^\dagger\, (M\,\mathcal{P}_{\operatorname{Im} N})^\dagger = (M^\dagger MN)^\dagger\, (MNN^\dagger)^\dagger. \tag{2.2}$$

(viii) Let $M(t)$ be an $m\times n$ piecewise rd-continuously Δ-differentiable matrix function defined on $[a,\infty)_\mathbb{T}$ such that the kernel of $M(t)$ is constant on $[a,\infty)_\mathbb{T}$. Then the matrix function $M^\dagger(t)$ is also piecewise rd-continuously Δ-differentiable on $[a,\infty)_\mathbb{T}$ and

$$[M^\dagger(t)]^\Delta M(t) = -[M^\dagger(t)]^\sigma\, M^\Delta(t) = -[M^\sigma(t)]^\dagger\, M^\Delta(t), \quad t\in[a,\infty)_\mathbb{T}. \tag{2.3}$$

The following proposition covers a special property of orthogonal projectors, which we will use later, see the proof of Theorem 5.2 and [19, Theorem 9.2] for details.

Proposition 2.2. *Let $P_*, P, \tilde P \in \mathbb{R}^{n\times n}$ be arbitrary orthogonal projectors satisfying*

$$\operatorname{Im} P_* \subseteq \operatorname{Im} P, \quad \operatorname{Im} P_* \subseteq \operatorname{Im} \tilde P, \quad \operatorname{rank} P = \operatorname{rank} \tilde P.$$

Then there exists an invertible matrix $E\in\mathbb{R}^{n\times n}$ such that $EP_* = P_*$ and $\operatorname{Im} EP = \operatorname{Im}\tilde P$.

According to Remark 2.1(ii), the Moore–Penrose pseudoinverse can be conveniently used for the construction of the orthogonal projectors onto the image of $X^T(t)$ or onto the image of $X(t)$ of a matrix function $X: [a,\infty)_\mathbb{T}\to\mathbb{R}^{n\times n}$. In particular, the following two orthogonal projectors play important role in our theory. Define

$$P(t) := \mathcal{P}_{\operatorname{Im} X^T(t)} = X^\dagger(t)\, X(t), \quad R(t) := \mathcal{P}_{\operatorname{Im} X(t)} = X(t)\, X^\dagger(t), \quad t\in[a,\infty)_\mathbb{T}. \tag{2.4}$$

Note that from the defining properties of Moore–Penrose pseudoinverse in (2.1) we get

$$P(t)\, X^\dagger(t) = X^\dagger(t), \quad X^\dagger(t)\, R(t) = X^\dagger(t), \quad t\in[a,\infty)_\mathbb{T}. \tag{2.5}$$

Remark 2.3. We will often work with matrix functions $X : [a,\infty)_{\mathbb{T}} \to \mathbb{R}^{n\times n}$ with constant kernel on some interval $[\alpha,\infty)_{\mathbb{T}}$. In this case the associated orthogonal projector $P(t)$ defined in (2.4) is constant on $[\alpha,\infty)_{\mathbb{T}}$, since $\mathbb{R}^n = [\operatorname{Ker} X(t)]^{\perp} \oplus \operatorname{Ker} X(t)$, where the subspace $[\operatorname{Ker} X(t)]^{\perp} = \operatorname{Im} X^T(t)$ is constant on $[\alpha,\infty)_{\mathbb{T}}$. In this case we denote by P the corresponding constant orthogonal projector in (2.4), i.e., we define

$$P := P(t) \quad \text{for } t \in [\alpha,\infty)_{\mathbb{T}}, \text{ where } \operatorname{Ker} X(t) \text{ is constant.} \tag{2.6}$$

3 Results on symplectic systems on time scales

In this section we collect basic information about symplectic systems on time scales and their conjoined bases. We split this section into three subsections, separating the introductory part and two slightly more advanced (yet still preparatory) parts.

3.1 Basic preparatory results

In this subsection we recall the facts, which need to be understood for the definition of an antiprincipal solution at infinity. The results in this subsection are not new, most of them can be found in [22], where they were presented in a slightly different logical order and, in some cases, with incomplete arguments. In particular, we present full details about the monotonicity of the S-matrices for conjoined bases of (S), which yield a correct definition of the associated T-matrix. The latter matrix is the cornerstone of our investigation of antiprincipal solutions of (S) at infinity.

A solution (X,U) of (S) is a *conjoined basis*, if $X^T(t)\,U(t)$ is a symmetric matrix and $\operatorname{rank}(X^T(t),\, U^T(t))^T = n$ at some and hence at any $t \in [a,\infty)_{\mathbb{T}}$. For any two solutions (X,U) and $(\bar{X},\bar{U})$ of (S) their *Wronskian matrix* $N := X^T(t)\,\bar{U}(t) - U^T(t)\,\bar{X}(t)$ is constant on $[a,\infty)_{\mathbb{T}}$. Two conjoined bases (X,U) and $(\bar{X},\bar{U})$ are called *normalized*, if their constant Wronskian matrix N satisfies $N = I$. A conjoined basis (X,U) of (S) is called *nonoscillatory*, if there exists $\alpha \in [a,\infty)_{\mathbb{T}}$ such that (X,U) has no focal points in the real interval (α,∞), i.e., if conditions (1.1) and (1.2) hold. We say that the system (S) is *nonoscillatory* if every conjoined basis of (S) is nonoscillatory.

Let (X,U) be a conjoined basis of system (S). For simplicity we say that (X,U) has *constant kernel* on an interval $[\alpha,\infty)_{\mathbb{T}}$ if the matrix $X(t)$ has constant kernel on this interval. Similarly, we say that (X,U) has rank r on $[\alpha,\infty)_{\mathbb{T}}$, if the matrix $X(t)$ has rank r on this interval. Note that if the system (S) is nonoscillatory, then the kernel (and hence also the rank) of any of its conjoined bases is eventually constant. If (X,U) is a conjoined basis of (S) with constant kernel on some interval $[\alpha,\infty)_{\mathbb{T}}$, then it is convenient to work with the so-called S-matrix corresponding to (X,U) on $[\alpha,\infty)_{\mathbb{T}}$. It is defined by

$$S(t) := \int_{\alpha}^{t} [X^{\sigma}(s)]^{\dagger}\,\mathcal{B}(s)\,[X^{\dagger}(s)]^T\,\Delta s, \quad t \in [\alpha,\infty)_{\mathbb{T}}. \tag{3.1}$$

Note that the definition of the matrix $S(t)$ is correct, since according to Remark 2.1(viii) the matrix function $X^{\dagger}$ is piecewise rd-continuously Δ-differentiable on $[\alpha,\infty)_{\mathbb{T}}$, so that $X^{\dagger}$ is continuous on $[\alpha,\infty)_{\mathbb{T}}$ and $(X^{\sigma})^{\dagger} = (X^{\dagger})^{\sigma}$ is rd-continuous on $[\alpha,\infty)_{\mathbb{T}}$. This implies that the function $(X^{\sigma})^{\dagger}\,\mathcal{B}\,(X^{\dagger})^T$ is piecewise rd-continuous (hence Δ-integrable) on $[\alpha,\infty)_{\mathbb{T}}$. Moreover, according to [14, Lemma 3.1] the matrix

$$X(t)\,[X^{\sigma}(t)]^{\dagger}\,\mathcal{B}(t) \text{ is symmetric on } [\alpha,\infty)_{\mathbb{T}}. \tag{3.2}$$

Proposition 3.1. *Let (X, U) be a conjoined basis of the system (S) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$. Then the corresponding S-matrix given by (3.1) is symmetric.*

Proof. Directly from the definition of $S(t)$ and using the fact that $P(t)$ is constant and hence $P(t) = P = P^\sigma(t)$ on $[\alpha, \infty)_{\mathbb{T}}$, we get for $t \in [\alpha, \infty)_{\mathbb{T}}$

$$\begin{aligned} S(t) &= \int_\alpha^t [X^\sigma(s)]^\dagger \mathcal{B}(s)\, [X^\dagger(s)]^T \Delta s \overset{(2.5)}{=} \int_\alpha^t P^\sigma(s)\, [X^\sigma(s)]^\dagger \mathcal{B}(s)\, [X^\dagger(s)]^T \Delta s \\ &= \int_\alpha^t P(s)\, [X^\sigma(s)]^\dagger \mathcal{B}(s)\, [X^\dagger(s)]^T \Delta s \overset{(2.4)}{=} \int_\alpha^t X^\dagger(s)\, X(s)\, [X^\sigma(s)]^\dagger \mathcal{B}(s)\, [X^\dagger(s)]^T \Delta s. \end{aligned} \tag{3.3}$$

The latter expression together with (3.2) proves the result. □

Next we present a statement about the relation between $\operatorname{Im} S(t)$ and $\operatorname{Im} P$.

Lemma 3.2. *Let (X, U) be a conjoined basis of the system (S) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and let the matrices P and $S(t)$ be defined by (2.6) and (3.1). Then*

$$\operatorname{Im} S(t) \subseteq \operatorname{Im} P \quad \text{for all } t \in [\alpha, \infty)_{\mathbb{T}}. \tag{3.4}$$

Proof. Fix $t \in [\alpha, \infty)_{\mathbb{T}}$ and let $u \in \operatorname{Im} S(t)$. Then there exists $v \in \mathbb{R}^n$ such that $S(t)\, v = u$. From (3.3) we get $S(t) = PS(t)$. Then $u = PS(t)\, v$ and hence $u \in \operatorname{Im} P$. □

Remark 3.3. Let (X, U) be a conjoined basis of (S) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$. If we use the symmetry of $S(t)$ on $[\alpha, \infty)_{\mathbb{T}}$ and Remark 2.1(v), then the inclusion of the sets in (3.4) from the previous lemma can be equivalently written as

$$PS(t) = S(t) = S(t)\, P \quad \text{or} \quad PS^\dagger(t) = S^\dagger(t) = S^\dagger(t)\, P, \quad t \in [\alpha, \infty)_{\mathbb{T}}. \tag{3.5}$$

The next theorem is fundamental for the definition of an antiprincipal solution of (S) at infinity and we display its proof with full details. In the theorem the so-called T-matrix corresponding to the conjoined basis (X, U) is introduced.

Theorem 3.4. *Let (X, U) be a conjoined basis of (S) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞) and let the matrix $S(t)$ given by (3.1). Then the limit of $S^\dagger(t)$ as $t \to \infty$ exists. Moreover, the matrix T defined by*

$$T := \lim_{t \to \infty} S^\dagger(t) \tag{3.6}$$

is symmetric, positive semidefinite, i.e., $T \geq 0$, and there exists $\beta \in [\alpha, \infty)_{\mathbb{T}}$ such that

$$\operatorname{rank} T \leq \operatorname{rank} S(t) \leq \operatorname{rank} X(t) \quad \text{for all } t \in [\beta, \infty)_{\mathbb{T}}. \tag{3.7}$$

Proof. First we show that the limit of $S^\dagger(t)$ exists. According to Proposition 3.1, the matrix $S(t)$ is symmetric. The constant kernel of the conjoined basis (X, U) on $[\alpha, \infty)_{\mathbb{T}}$ guarantees that $P(t) = P$ is constant on $[\alpha, \infty)_{\mathbb{T}}$. Since the conjoined basis (X, U) has no focal points in (α, ∞), we get for $t \in [\alpha, \infty)_{\mathbb{T}}$

$$S^\Delta(t) = [X^\sigma(t)]^\dagger \mathcal{B}(t)\, [X^T(t)]^\dagger \overset{(3.3)}{=} X^\dagger(t)\, X(t)\, [X^\sigma(t)]^\dagger \mathcal{B}(t)\, [X^T(t)]^\dagger \overset{(1.2)}{\geq} 0.$$

This means that matrix $S(t)$ is nondecreasing on $[\alpha, \infty)_{\mathbb{T}}$, i.e.,

$$S(t_1) \leq S(t_2) \quad \text{for all } t_1, t_2 \in [\alpha, \infty)_{\mathbb{T}} \text{ such that } t_1 < t_2. \tag{3.8}$$

Since $S(\alpha) = 0$, we get

$$S(t) \geq 0, \quad t \in [\alpha, \infty)_{\mathbb{T}}. \tag{3.9}$$

This implies that $\operatorname{Im} S(t)$ is eventually constant, i.e., there exists $\beta \in [\alpha, \infty)_{\mathbb{T}}$ such that

$$\operatorname{Im} S(t_1) = \operatorname{Im} S(t_2) \quad \text{for all } t_1, t_2 \in [\beta, \infty)_{\mathbb{T}}. \tag{3.10}$$

Now we use Remark 2.1(v), where we put $M_1 := S(t_1)$ and $M_2 := S(t_2)$ for $t_1, t_2 \in [\beta, \infty)_{\mathbb{T}}$. The symmetry of $S(t)$ on $[\alpha, \infty)_{\mathbb{T}}$ and conditions (3.8), (3.9) and (3.10) on $[\beta, \infty)_{\mathbb{T}}$ imply that

$$S^{\dagger}(t_2) \leq S^{\dagger}(t_1) \quad \text{for all } t_1, t_2 \in [\beta, \infty)_{\mathbb{T}} \text{ such that } t_1 < t_2, \tag{3.11}$$

i.e., the matrix $S^{\dagger}(t)$ is nonincreasing on $[\beta, \infty)_{\mathbb{T}}$. By (3.9) and Remark 2.1(vi) we then get

$$S^{\dagger}(t) \geq 0 \quad \text{for all } t \in [\beta, \infty)_{\mathbb{T}}. \tag{3.12}$$

This implies that the limit of $S^{\dagger}(t)$ for $t \to \infty$ exists and the matrix T in (3.6) is correctly defined. Finally, matrix T is symmetric and positive semidefinite as the limit of matrices with the same properties. Condition (3.7) then follows from inclusion (3.4) and from the inclusion $\operatorname{Im} T \subseteq \operatorname{Im} S^{\dagger}(t) = \operatorname{Im} S(t)$ on $[\beta, \infty)_{\mathbb{T}}$ derived from (3.11) together with (3.12). $\square$

Remark 3.5. The proof of the Theorem 3.4 reveals some properties of the S-matrix corresponding to a conjoined basis (X, U) of (S) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞). Namely,

(i) the matrix $S(t)$ is nondecreasing on $[\alpha, \infty)_{\mathbb{T}}$,

(ii) the set $\operatorname{Im} S(t)$ is nondecreasing on $[\alpha, \infty)_{\mathbb{T}}$ and eventually constant, i.e., there exists $\beta \in [\alpha, \infty)_{\mathbb{T}}$ such that $\operatorname{Im} S(t)$ is constant on $[\beta, \infty)_{\mathbb{T}}$,

(iii) the set $\operatorname{Ker} S(t) = [\operatorname{Im} S(t)]^{\perp}$ is nonincreasing on $[\alpha, \infty)_{\mathbb{T}}$ and eventually constant.

In the next part we define the order of abnormality of system (S) in the same way as in [14, 22]. For any $\alpha \in [a, \infty)_{\mathbb{T}}$ denote by $\Lambda[\alpha, \infty)_{\mathbb{T}}$ the linear space of n-vector functions $u : [\alpha, \infty)_{\mathbb{T}} \to \mathbb{R}^n$ such that $\mathcal{B}(t)\, u(t) = 0$ and $u^{\Delta} = \mathcal{D}(t)\, u(t)$ on $[\alpha, \infty)_{\mathbb{T}}$. The number $d[\alpha, \infty)_{\mathbb{T}} := \dim \Lambda[\alpha, \infty)_{\mathbb{T}}$ is called the *order of abnormality* of system (S) on the interval $[\alpha, \infty)_{\mathbb{T}}$. The limit

$$d_{\infty} := \lim_{t \to \infty} d[t, \infty)_{\mathbb{T}} \tag{3.13}$$

is then called the *maximal order of abnormality* of the system (S). Note that this definition is correct since limit in (3.13) exists and equals to $\max\{d[t, \infty)_{\mathbb{T}},\ t \in [\alpha, \infty)_{\mathbb{T}}\}$. This can be seen from the fact that a solution $(x \equiv 0, u)$ of (S) on $[\alpha, \infty)_{\mathbb{T}}$ is also the solution of (S) on $[\beta, \infty)_{\mathbb{T}}$ for any $\beta \in (\alpha, \infty)_{\mathbb{T}}$. Then $\Lambda[\alpha, \infty)_{\mathbb{T}} \subseteq \Lambda[\beta, \infty)_{\mathbb{T}}$ for $\alpha, \beta \in [a, \infty)_{\mathbb{T}}$ with $\alpha < \beta$ and hence, the function $d[t, \infty)_{\mathbb{T}}$ as a function of t is nondecreasing on $[a, \infty)_{\mathbb{T}}$. Then the integer values $d[t, \infty)_{\mathbb{T}}$ and d_{∞} satisfy

$$0 \leq d[t, \infty)_{\mathbb{T}} \leq d_{\infty} \leq n, \quad t \in [a, \infty)_{\mathbb{T}}. \tag{3.14}$$

In a similar way we define the order of abnormality $d[\alpha, t]_{\mathbb{T}}$ of system (S) on the interval $[\alpha, t]_{\mathbb{T}}$. Then, obviously, the relation $d[\alpha, \infty)_{\mathbb{T}} = \lim_{t \to \infty} d[\alpha, t]_{\mathbb{T}}$ holds.

In addition, denote by $\Lambda_0[\alpha, \infty)_{\mathbb{T}}$ the subspace of $\mathbb{R}^n$ of the initial values $u(\alpha)$ of the elements $u \in \Lambda[\alpha, \infty)_{\mathbb{T}}$. Then $\dim \Lambda_0[\alpha, \infty)_{\mathbb{T}} = \dim \Lambda[\alpha, \infty)_{\mathbb{T}} = d[t, \infty)_{\mathbb{T}}$. This auxiliary subspace will be used e.g. in Proposition 3.18 when dealing with minimal conjoined bases of (S).

3.2 Additional preparatory results

In this subsection we recall several results, which we will use in order to derive the properties of antiprincipal solutions of ($\mathbb{S}$) at infinity. We consider a conjoined basis (X, U) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞). We also consider the associated matrix $S(t)$ defined in (3.1), for which $\operatorname{Im} S(t)$ is constant on some interval $[\beta, \infty)_{\mathbb{T}}$ with $\beta \in [\alpha, \infty)_{\mathbb{T}}$, see Remark 3.5. The following additional properties of the matrices $S(t)$ and T are proven in [22, Theorem 3.2].

Proposition 3.6. *Let (X, U) be a conjoined basis of ($\mathbb{S}$) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and let matrices P, $R(t)$, $S(t)$ be defined in (2.6), (2.4), and (3.1). Then*

(i) $\operatorname{Im}[U(t)(I-P)] = \operatorname{Ker} R(t)$ *and hence* $R(t)\,U(t) = R(t)\,U(t)P$ *on* $[\alpha, \infty)_{\mathbb{T}}$,

(ii) $R^{\sigma}(t)\,\mathcal{B}(t) = \mathcal{B}(t)$ *and* $\mathcal{B}(t)\,R(t) = \mathcal{B}(t)$ *on* $[\alpha, \infty)_{\mathbb{T}}$.

If in addition (X, U) has no focal points in (α, ∞) and if T is defined in (3.6), then

(iii) $PT = T = TP$ *and* $PT^{\dagger} = T^{\dagger} = T^{\dagger}P$.

On the intervals $[\beta, \infty)_{\mathbb{T}}$, where the subspace $\operatorname{Im} S(t)$ is constant, we can define the associated constant orthogonal projector

$$P_{S\infty} := P_S(t), \quad t \in [\beta, \infty)_{\mathbb{T}}, \quad P_S(t) := \mathcal{P}_{\operatorname{Im} S(t)} = S(t)\,S^{\dagger}(t) = S^{\dagger}(t)\,S(t). \tag{3.15}$$

From (3.5) we can see that the following inclusions

$$\operatorname{Im} S(t) \subseteq \operatorname{Im} P_{S\infty} \subseteq \operatorname{Im} P, \quad t \in [\beta, \infty)_{\mathbb{T}}, \tag{3.16}$$

hold. By using the symmetry of $S(t)$, the inclusions in (3.16) can be written as

$$P_{S\infty}\,S(t) = S(t) = S(t)\,P_{S\infty}, \quad t \in [\beta, \infty)_{\mathbb{T}}, \quad PP_{S\infty} = P_{S\infty} = P_{S\infty}P. \tag{3.17}$$

Finally, using the definition of Moore–Penrose pseudoinverse in (2.1) and observing the limit

$$T = \lim_{t\to\infty} S^{\dagger}(t) = \lim_{t\to\infty} S^{\dagger}(t)\,S(t)\,S^{\dagger}(t) = P_{S\infty}\left(\lim_{t\to\infty} S^{\dagger}(t)\right) = P_{S\infty}\,T,$$

we obtain the equalities

$$P_{S\infty}\,T = T = TP_{S\infty}, \quad \text{i.e.,} \quad \operatorname{Im} T \subseteq \operatorname{Im} P_{S\infty}. \tag{3.18}$$

By the principal solution of ($\mathbb{S}$) at the point $\alpha \in [a, \infty)_{\mathbb{T}}$, denoted by $(\hat{X}^{[\alpha]}, \hat{U}^{[\alpha]})$, we mean the conjoined basis of ($\mathbb{S}$) satisfying the initial conditions $\hat{X}^{[\alpha]}(\alpha) = 0$ and $\hat{U}^{[\alpha]}(\alpha) = I$. The following important result provides an information about any conjoined basis of ($\mathbb{S}$) through the properties of the principal solution $(\hat{X}^{[\alpha]}, \hat{U}^{[\alpha]})$. It is proven as a part of [22, Proposition 3.9].

Lemma 3.7. *Let (X, U) be a conjoined basis of ($\mathbb{S}$) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞). Let the matrices P, $R(t)$, $S(t)$, $P_{S\infty}$ be defined by (2.6), (2.4), (3.1), (3.15). Then the principal solution $(\hat{X}^{[\alpha]}, \hat{U}^{[\alpha]})$ satisfies for all $t \in [\alpha, \infty)_{\mathbb{T}}$ the following properties:*

$$\hat{X}^{[\alpha]}(t) = X(t)\,S(t)\,X^{T}(\alpha), \tag{3.19}$$

$$\operatorname{rank} S(t) = \operatorname{rank} \hat{X}^{[\alpha]}(t) = n - d[\alpha, t]_{\mathbb{T}}, \tag{3.20}$$

$$\operatorname{rank} P_{S\infty} = n - d[\alpha, \infty)_{\mathbb{T}}, \tag{3.21}$$

$$\Lambda_0[\alpha, \infty)_{\mathbb{T}} = \operatorname{Im}[X^{\dagger T}(\alpha)\,(I - P_{S\infty})] \oplus \operatorname{Im}[U(\alpha)\,(I-P)], \tag{3.22}$$

$$n - d[\alpha, \infty)_{\mathbb{T}} \le \operatorname{rank} X(t) \le n. \tag{3.23}$$

Remark 3.8. From Theorem 3.4 and (3.20) it follows that $0 \leq \operatorname{rank} T \leq n - d_\infty$ for the T-matrix associated with an arbitrary conjoined basis (X, U) of (S) with constant kernel on $[\alpha, \infty)_\mathbb{T}$ and no focal points in (α, ∞).

The next two results contain additional properties of the matrices $S(t)$ and T, which are proven in [22, Propositions 5.5 and 5.6] and, with a slightly different formulation, in [22, Remark 5.7] (see also the proof of [10, Proposition 6.105] in the discrete case).

Proposition 3.9. *Let (X, U) be a conjoined basis of (S) with constant kernel on $[\alpha, \infty)_\mathbb{T}$ and no focal points in (α, ∞). Let $S(t)$ and T be defined in (3.1) and (3.6). If $d[\alpha, \infty)_\mathbb{T} = d_\infty$, then there exists $\beta \in [\alpha, \infty)_\mathbb{T}$ such that*

$$S^\dagger(t) \geq T \geq 0 \quad \text{and} \quad \operatorname{rank}[S^\dagger(t) - T] = n - d_\infty \quad \text{on } [\beta, \infty)_\mathbb{T}. \tag{3.24}$$

Proposition 3.10. *Let (X, U) be a conjoined basis of (S) with constant kernel on $[\alpha, \infty)_\mathbb{T}$ and no focal points in (α, ∞), where the point $\alpha \in [a, \infty)_\mathbb{T}$ is that $d[\alpha, \infty)_\mathbb{T} = d_\infty$. Then*

$$\operatorname{Im}[P_{S\infty} - S(\beta)\, S^\dagger(t)] = \operatorname{Im} P_{S\infty} = \operatorname{Im}[P_{S\infty} - S(\beta)\, S^\dagger(t)]^T, \quad \beta, t \in [\alpha, \infty)_\mathbb{T},\ t \geq \beta, \tag{3.25}$$

$$\operatorname{Im}[P_{S\infty} - S(t)\, T] = \operatorname{Im} P_{S\infty} = \operatorname{Im}[P_{S\infty} - S(t)\, T]^T, \quad t \in [\alpha, \infty)_\mathbb{T}. \tag{3.26}$$

The following relation is a useful tool for the construction of conjoined bases of (S) with certain desired properties from a conjoined basis of (S), which the same properties already has. This relation is studied in [22, Section 4] in more details. For this purpose we also recall the concept of *equivalent solutions* (X_1, U_1) and (X_2, U_2) of (S) on some interval $[\alpha, \infty)_\mathbb{T}$, which is defined by the property $X_1(t) = X_2(t)$ on $[\alpha, \infty)_\mathbb{T}$.

Definition 3.11. Let (X, U) be a conjoined basis of (S) with constant kernel on $[\alpha, \infty)_\mathbb{T}$ and no focal points in (α, ∞) and let the matrices P and $P_{S\infty}$ be defined by (2.6) and (3.15). Consider an orthogonal projector P_* satisfying

$$\operatorname{Im} P_{S\infty} \subseteq \operatorname{Im} P_* \subseteq \operatorname{Im} P. \tag{3.27}$$

We say that a conjoined basis (X_*, U_*) of (S) is *contained* in (X, U) on $[\alpha, \infty)_\mathbb{T}$ with respect to P_*, or that (X, U) *contains* (X_*, U_*) on $[\alpha, \infty)_\mathbb{T}$ with respect to P_*, if the solutions (X_*, U_*) and (XP_*, UP_*) are equivalent, that is, if $X_*(t) = X(t)\, P_*$ on $[\alpha, \infty)_\mathbb{T}$.

It should be stressed that the relation in Definition 3.11 is between a conjoined basis (X, U) of (S) with constant kernel on $[\alpha, \infty)_\mathbb{T}$ and no focal points in (α, ∞) and an arbitrary conjoined basis (X_*, U_*). This means that if we say that a conjoined basis (X, U) contains a conjoined basis (X_*, U_*) on $[\alpha, \infty)_\mathbb{T}$, then we automatically suppose that (X, U) has constant kernel on $[\alpha, \infty)_\mathbb{T}$ and no focal points in (α, ∞). The following proposition is proven in [22, Proposition 4.2] and it shows that the conjoined basis (X_*, U_*) from Definition 3.11 inherits the properties of (X, U) on the interval $[\alpha, \infty)_\mathbb{T}$.

Proposition 3.12. *Let (X, U) be a conjoined basis of (S) with constant kernel on $[\alpha, \infty)_\mathbb{T}$ and no focal points in (α, ∞) and assume that a conjoined basis (X_*, U_*) of (S) is contained in (X, U) on $[\alpha, \infty)_\mathbb{T}$ with respect to an orthogonal projector P_* satisfying (3.27).*

(i) Then (X_*, U_*) has also constant kernel on $[\alpha, \infty)_\mathbb{T}$ and no focal points in (α, ∞). Moreover, the matrix P_* is then the associated orthogonal projector defined in (2.6) for (X_*, U_*), i.e., $P_* = \mathcal{P}_{\operatorname{Im} X_*^T(t)} = X_*^\dagger(t)\, X_*(t)$ on $[\alpha, \infty)_\mathbb{T}$.

(ii) If $S(t)$ and $S_*(t)$ are the S-matrices corresponding to the conjoined bases (X,U) and (X_*,U_*) on $[\alpha,\infty)_{\mathbb{T}}$, then $S(t)=S_*(t)$ on $[\alpha,\infty)_{\mathbb{T}}$.

The next proposition from [22, Theorem 5.1] guarantees the existence of a conjoined basis of ($\mathbb{S}$) with constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and no focal points in (α,∞), which has any given rank between the numbers $n-d_\infty$ and n. We will use it later in the construction of antiprincipal solutions of ($\mathbb{S}$) with a desired rank. Note that the conjoined bases with the given rank r are constructed by the relation being contained in Definition 3.11.

Proposition 3.13. *Assume that there exists a conjoined basis of ($\mathbb{S}$) with constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and no focal points in (α,∞). Then for any integer r between $n-d_\infty$ and n there exists a conjoined basis (X,U) of ($\mathbb{S}$), which has constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and no focal points in (α,∞) too, such that* $\operatorname{rank} X(t)=r$ *on* $[\alpha,\infty)_{\mathbb{T}}$.

For the investigation of all solutions (or conjoined bases) of ($\mathbb{S}$) it is important to choose a suitable fundamental matrix of system ($\mathbb{S}$). When one conjoined basis (X,U) of ($\mathbb{S}$) is given, it turns out that it is possible to complete it to a fundamental matrix of ($\mathbb{S}$) by a specific conjoined basis $(\bar X,\bar U)$. Some of the properties of this conjoined basis $(\bar X,\bar U)$ were presented in [22, Proposition 3.3 and Remarks 3.4 and 3.5]. We include some additional properties based on the discrete time results in [25, Proposition 3.5] or in [10, Proposition 6.67].

Proposition 3.14. *Let (X,U) be a conjoined basis of ($\mathbb{S}$) with constant kernel on $[\alpha,\infty)_{\mathbb{T}}$, let the matrices P and $S(t)$ defined by (2.6) and (3.1). Then there exists a conjoined basis $(\bar X,\bar U)$ of ($\mathbb{S}$) such that (X,U) and $(\bar X,\bar U)$ satisfy*

(i) the Wronskian $N:=X^T(t)\,\bar U(t)-U^T(t)\,\bar X(t)\equiv I$ on $[a,\infty)_{\mathbb{T}}$, and

(ii) $X^\dagger(\alpha)\,\bar X(\alpha)=0$.

Moreover, such a conjoined basis $(\bar X,\bar U)$ then satisfies

(iii) $X^\dagger(t)\,\bar X(t)\,P=S(t)$ for all $t\in[\alpha,\infty)_{\mathbb{T}}$,

(iv) $\bar X(t)\,P=X(t)\,S(t)$ and $\bar U(t)\,P=U(t)\,S(t)+X^{\dagger T}(t)+U(t)\,(I-P)\,\bar X^T(t)\,X^{\dagger T}(t)$ for all $t\in[\alpha,\infty)_{\mathbb{T}}$ (in particular $\bar X(\alpha)\,P=0$), and the solution $(\bar XP,\bar UP)$ of ($\mathbb{S}$) is uniquely determined by (X,U),

(v) $\operatorname{Ker}\bar X(t)=\operatorname{Im}[P-P_S(t)]=\operatorname{Im}P\cap\operatorname{Ker}S(t)$ for all $t\in[\alpha,\infty)_{\mathbb{T}}$,

(vi) the function $\bar X(t)$ is uniquely determined by (X,U),

(vii) $\bar P(t)=I-P+P_S(t)$ for all $t\in[\alpha,\infty)_{\mathbb{T}}$, where $\bar P(t):=\bar X^\dagger(t)\,\bar X(t)$,

(viii) $S^\dagger(t)=\bar X^\dagger(t)\,X(t)\,P_S(t)=\bar X^\dagger(t)\,X(t)\,\bar P(t)$ for all $t\in[\alpha,\infty)_{\mathbb{T}}$,

(ix) $\operatorname{Im}\bar X(\alpha)=\operatorname{Im}[I-R(\alpha)]$ and $\operatorname{Im}\bar X^T(\alpha)=\operatorname{Im}(I-P)$,

(x) the matrix $X(\alpha)-\bar X(\alpha)$ is invertible with $[X(\alpha)-\bar X(\alpha)]^{-1}=X^\dagger(\alpha)-\bar X^\dagger(\alpha)$,

(xi) $\bar X^\dagger(\alpha)=-(I-P)\,U^T(\alpha)$.

If in addition the conjoined basis (X,U) has no focal points in (α,∞), then

(xii) $X(t)\,\bar X^T(t)\geq 0$ for all $t\in[\alpha,\infty)_{\mathbb{T}}$.

Proof. The properties (i)–(iii) and (vi) are shown in [22, Proposition 3.3], property (iv) is shown in [22, Remark 3.5]. The remaining properties (v) and (vii)–(xii) can be proven analogously to the discrete case, see the proof of [22, Proposition 6.67]. □

The following result from [22, Proposition 3.6] shows important properties of two conjoined bases of ($\mathbb{S}$), which are mutually representable in terms of symplectic fundamental matrices involving the conjoined bases from Proposition 3.14.

Proposition 3.15. *Let (X_1, U_1) and (X_2, U_2) be conjoined bases of ($\mathbb{S}$) with constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and no focal points in (α,∞) and let P_1 and P_2 be the constant orthogonal projectors defined in (2.6) through the functions X_1 and X_2, respectively. Let the conjoined basis (X_2, U_2) be expressed in terms of (X_1, U_1) via the matrices M_1 and N_1, and let the conjoined basis (X_1, U_1) be expressed in terms of (X_2, U_2) via the matrices M_2 and N_2, i.e.,*

$$\begin{pmatrix} X_2(t) \\ U_2(t) \end{pmatrix} = \begin{pmatrix} X_1(t) & \bar{X}_1(t) \\ U_1(t) & \bar{U}_1(t) \end{pmatrix} \begin{pmatrix} M_1 \\ N_1 \end{pmatrix}, \quad \begin{pmatrix} X_1(t) \\ U_1(t) \end{pmatrix} = \begin{pmatrix} X_2(t) & \bar{X}_2(t) \\ U_2(t) & \bar{U}_2(t) \end{pmatrix} \begin{pmatrix} M_2 \\ N_2 \end{pmatrix}$$

on $[\alpha,\infty)_{\mathbb{T}}$, where $(\bar{X}_1, \bar{U}_1)$ and $(\bar{X}_2, \bar{U}_2)$ are the conjoined bases of ($\mathbb{S}$) satisfying the properties in Proposition 3.14 with respect to (X_1, U_1) and (X_2, U_2), respectively. If the equality $\operatorname{Im} X_1(\alpha) = \operatorname{Im} X_2(\alpha)$ is satisfied, then

(i) the matrices $M_1^T N_1$ and $M_2^T N_2$ are symmetric and $N_2 = -N_1^T$,

(ii) the matrices M_1 and M_2 are invertible and $M_2 = M_1^{-1}$,

(iii) the inclusions $\operatorname{Im} N_1 \subseteq \operatorname{Im} P_1$ and $\operatorname{Im} N_2 \subseteq P_2$ hold.

Moreover, the matrices M_1 and N_1 do not depend on the choice of $(\bar{X}_1, \bar{U}_1)$, and the matrices M_2 and N_2 do not depend on the choice of $(\bar{X}_2, \bar{U}_2)$.

The following properties are from [22, Remark 3.7] and they complement the results in Proposition 3.15 about the representation matrices M_i and N_i (for $i \in \{1,2\}$).

Remark 3.16. With the notation in Proposition 3.15, let us define the matrices

$$L_1 := X_1^\dagger(\alpha)\, X_2(\alpha), \quad L_2 := X_2^\dagger(\alpha)\, X_1(\alpha).$$

Then following properties hold for $i \in \{1,2\}$:

$$L_i L_{3-i} = P_i, \quad L_{3-i} = L_i^\dagger, \quad L_i = P_i M_i, \quad N_i = P_i N_i, \tag{3.28}$$

$$P_i \text{ is the projector onto } \operatorname{Im} L_i, \quad L_i^T N_i = M_i^T P_i N_i = M_i^T N_i \text{ is symmetric,} \tag{3.29}$$

$$X_{3-i}(t) = X_i(t)\,[L_i + S_i(t)\, N_i], \quad M_i + S_i(t)\, N_i \text{ is invertible,} \quad t \in [\alpha,\infty)_{\mathbb{T}}, \tag{3.30}$$

$$[L_i + S_i(t)\, N_i]^\dagger = L_{3-i} + S_{3-i}(t)\, N_{3-i}, \quad \operatorname{Im}[L_i + S_i(t)\, N_i] = \operatorname{Im} P_i, \quad t \in [\alpha,\infty)_{\mathbb{T}}, \tag{3.31}$$

$$S_{3-i}(t) = [L_i + S_i(t)\, N_i]^\dagger S_i(t)\, L_i^{\dagger T}, \quad t \in [\alpha,\infty)_{\mathbb{T}}, \tag{3.32}$$

where the matrix $S_i(t)$ is defined in (3.1) via the matrix $X_i(t)$.

3.3 Minimal conjoined bases and their properties

In this subsection we focus on minimal conjoined bases of (S). A conjoined basis (X, U) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞) is called *minimal* on $[\alpha, \infty)_{\mathbb{T}}$, if $\operatorname{rank} X(t) = n - d[\alpha, \infty)_{\mathbb{T}}$ holds for all $t \in [\alpha, \infty)_{\mathbb{T}}$. These special conjoined bases have the smallest possible rank according to estimate (3.23). They are used in order to derive many properties of other conjoined bases of (S). Note that if (X, U) is a minimal conjoined basis of (S) on the interval $[\alpha, \infty)_{\mathbb{T}}$, then necessarily the abnormality of (S) on $[\alpha, \infty)_{\mathbb{T}}$ is maximal, i.e., $d[\alpha, \infty)_{\mathbb{T}} = d_\infty$ holds. This follows from (3.14) and from estimate (3.23), since the rank of $X(t)$ is constant on $[\alpha, \infty)_{\mathbb{T}}$.

The following property will be used in the proof of Theorem 5.1 and it reveals a connection between orthogonal projectors P and $P_{S\infty}$ for a minimal conjoined basis (X, U) of (S). The stated equality $P = P_{S\infty}$ actually characterizes the property of (X, U) being a minimal conjoined basis of (S) on $[\alpha, \infty)_{\mathbb{T}}$, as mentioned (without the proof) in [22, Remark 5.3.]. In order to highlight its importance we state it separately and provide the details of its proof.

Lemma 3.17. *Let (X, U) be a conjoined basis of (S) on $[\alpha, \infty)_{\mathbb{T}}$ with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞), let the matrices P and $P_{S\infty}$ defined by (2.6) and (3.15), and assume that $d[\alpha, \infty)_{\mathbb{T}} = d_\infty$. Then (X, U) is a minimal conjoined basis of (S) on $[\alpha, \infty)_{\mathbb{T}}$ if and only if*

$$P = P_{S\infty}. \tag{3.33}$$

Proof. Let (X, U) be a minimal conjoined basis of (S) on $[\alpha, \infty)_{\mathbb{T}}$, so that (X, U) has constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞). From Lemma 3.2 it follows that

$$\operatorname{Im} S(t) \subseteq \operatorname{Im} P = \operatorname{Im} X^T(t), \quad t \in [\alpha, \infty)_{\mathbb{T}}, \tag{3.34}$$

and from Remark 3.5 we know that $\operatorname{Im} S(t)$ is nondecreasing on $[\alpha, \infty)_{\mathbb{T}}$. Moreover, from equation (3.21) in Lemma 3.7 we get $\operatorname{rank} P_{S\infty} = n - d_\infty = \lim_{t\to\infty} \operatorname{rank} S(t)$. Now from the fact that (X, U) is a minimal conjoined basis we get $\operatorname{rank} X(t) = n - d_\infty = \operatorname{rank} X^T(t)$ on $[\alpha, \infty)_{\mathbb{T}}$, which together with the inclusion (3.34) shows that

$$\operatorname{Im} S(t) = \operatorname{Im} X^T(t) \text{ for } t \in (\alpha, \infty)_{\mathbb{T}}. \tag{3.35}$$

This proves (3.33), since P is the orthogonal projector onto $\operatorname{Im} X^T(t)$ on $[\alpha, \infty)_{\mathbb{T}}$ and $P_{S\infty}$ is the orthogonal projector onto $\operatorname{Im} S(t)$ on $(\alpha, \infty)_{\mathbb{T}}$. Conversely, let (3.33) hold. Then by the definition of $P(t)$ in (2.4) and by Lemma 3.7 we have

$$\operatorname{rank} X(t) \overset{(2.4)}{=} \operatorname{rank} P \overset{(3.33)}{=} \operatorname{rank} P_{S\infty} \overset{(3.21)}{=} n - d_\infty, \quad t \in [\alpha, \infty)_{\mathbb{T}}.$$

This shows that (X, U) is a minimal conjoined basis on $[\alpha, \infty)_{\mathbb{T}}$. □

In the next result we present important properties of some special conjoined bases of (S) and their S-matrices, which are based on formulas (3.20) and (3.22) in Lemma 3.7 and on the properties of the Moore–Penrose pseudoinverse in Remark 2.1. These properties hold, in particular, for minimal conjoined bases of (S). We note that the formulation is slightly more general than in [22, Proposition 5.4], which we comment in the proof.

Proposition 3.18. *The following properties of conjoined bases of* (S) *hold.*

(i) Let (X, U) *be a conjoined basis of* (S) *with constant kernel on the interval* $[\alpha, \infty)_{\mathbb{T}}$ *and with* $\operatorname{rank} X(t) = n - d_\infty$ *on* $[\alpha, \infty)_{\mathbb{T}}$. *Let* P *be the associated projector defined in* (2.6). *Then* $d[\alpha, \infty)_{\mathbb{T}} = d_\infty$ *and*

$$\Lambda_0[\alpha, \infty)_{\mathbb{T}} = \operatorname{Im}[U(\alpha)\,(I - P)], \quad \operatorname{Im} X(\alpha) = \big(\Lambda_0[\alpha, \infty)_{\mathbb{T}}\big)^{\perp}. \tag{3.36}$$

Consequently, the initial subspace $\operatorname{Im} X(\alpha)$ *does not depend on the choice of the conjoined basis* (X, U) *of* (S) *with constant kernel and minimal rank on* $[\alpha, \infty)_{\mathbb{T}}$.

(ii) Let (X_1, U_1) *and* (X_2, U_2) *be two conjoined bases of* (S) *with constant kernel on the interval* $[\alpha, \infty)_{\mathbb{T}}$ *and with* $\operatorname{rank} X_1(t) = n - d_\infty = \operatorname{rank} X_2(t)$ *on* $[\alpha, \infty)_{\mathbb{T}}$. *Let* $S_1(t)$ *and* $S_2(t)$ *be the associated matrices defined in* (3.1). *If* $\beta \in [\alpha, \infty)_{\mathbb{T}}$ *is a point such that*

$$\operatorname{rank} S_1(t) = n - d[\alpha, \infty)_{\mathbb{T}} = \operatorname{rank} S_2(t), \quad t \in [\beta, \infty)_{\mathbb{T}},$$

then for $i \in \{1, 2\}$ *we have*

$$S^{\dagger}_{3-i}(t) = L_i^T S_i^{\dagger}(t)\, L_i + L_i^T N_i, \quad t \in [\beta, \infty)_{\mathbb{T}}, \tag{3.37}$$

where the matrices L_i *and* N_i *are from Proposition* 3.15 *and Remark* 3.16.

Proof. These results are proven in [22, Proposition 5.4], where it is in addition assumed that the conjoined basis (X, U) in part (i) has no focal points in the interval (α, ∞) (so that it is a minimal conjoined basis on $[\alpha, \infty)_{\mathbb{T}}$) and that the conjoined bases (X_1, U_1) and (X_2, U_2) in part (ii) have no focal points in the interval (α, ∞) (so that they are minimal conjoined bases on $[\alpha, \infty)_{\mathbb{T}}$). We emphasize that this additional assumption on no focal points of (X, U) or (X_1, U_1), (X_2, U_2) in the interval (α, ∞) is not needed for deriving the statements in (3.36) and (3.37), since the proofs actually follow only the continuous time case in [18, Theorems 5.15 and 5.17]. □

The last result of this subsection shows that for all minimal conjoined bases (X, U) of (S) on $[\alpha, \infty)_{\mathbb{T}}$ the first component of the associated conjoined basis $(\bar{X}, \bar{U})$ (that is, the matrix $\bar{X}$) is the same up to a right constant nonsingular multiple.

Lemma 3.19. *Let* (X_1, U_1) *and* (X_2, U_2) *be minimal conjoined bases of* (S) *on* $[\alpha, \infty)_{\mathbb{T}}$ *and let* $(\bar{X}_1, \bar{U}_1)$ *and* $(\bar{X}_2, \bar{U}_2)$ *be their associated conjoined bases from Proposition* 3.14. *Then there exists a constant invertible matrix* K *such that*

$$\bar{X}_2(t) = \bar{X}_1(t)\, K, \quad t \in [\alpha, \infty)_{\mathbb{T}}.$$

Proof. The proof is analogous to the proof of the continuous case in [23, Lemma 1] or to the proof of the discrete case in [24, Lemma 7.9] or [10, Lemma 6.100]. The details are therefore omitted. □

4 Antiprincipal solutions at infinity

In this section we introduce the main notion of this paper, i.e., an antiprincipal solution of (S) at infinity. This definition is based on the basic results about the matrices $S(t)$ and T in Subsection 3.1. We then derive several properties of antiprincipal solutions at infinity with the aid of Subsections 3.2 and 3.3. The results in this section are new in the time scales setting and they extend and unify their corresponding continuous and discrete time counterparts, as we emphasize by providing particular references with each statement.

Definition 4.1. A conjoined basis (X, U) of ($\mathbb{S}$) is said to be an *antiprincipal solution at infinity* with respect to the interval $[\alpha, \infty)_{\mathbb{T}}$ if

(i) the order of abnormality of ($\mathbb{S}$) on the interval $[\alpha, \infty)_{\mathbb{T}}$ is maximal, i.e.,

$$d[\alpha, \infty)_{\mathbb{T}} = d_\infty, \tag{4.1}$$

(ii) the conjoined basis (X, U) has constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞),

(iii) the matrix T defined in (3.6) corresponding to (X, U) satisfies $\operatorname{rank} T = n - d_\infty$.

Remark 4.2. By Theorem 3.4 we know that the limit of $S^\dagger(t)$ exists for all conjoined bases (X, U) of ($\mathbb{S}$) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞). Therefore the definition of an antiprincipal solution of ($\mathbb{S}$) at infinity using the corresponding T-matrix is possible. Note that so far we do not know anything about the existence of limit $S(t)$ itself. In addition, according to Remark 3.8 the rank of the matrix T of an antiprincipal solution of ($\mathbb{S}$) at infinity is maximal possible.

Let (X, U) be an antiprincipal solution of ($\mathbb{S}$) at infinity with respect to the interval $[\alpha, \infty)_{\mathbb{T}}$. By (3.21) and (3.23) together with property (4.1) from the above definition we obtain that $n - d_\infty \le \operatorname{rank} X(t) \le n$ for all $t \in [\alpha, \infty)_{\mathbb{T}}$. Denote by r the rank of (X, U) near infinity, i.e., $r := \operatorname{rank} X(t)$ for $t \in [\alpha, \infty)_{\mathbb{T}}$. If $r = n - d_\infty$, then (X, U) is called a *minimal antiprincipal solution* at infinity, which we denote by $(X_{\min}, U_{\min})$. If $r = n$, then (X, U) is called a *maximal antiprincipal solution* at infinity, which we denote by $(X_{\max}, U_{\max})$. In this case the matrix $X_{\max}(t)$ is invertible for all $t \in [\alpha, \infty)_{\mathbb{T}}$. Such minimal and maximal antiprincipal solutions of ($\mathbb{S}$) at infinity will be considered e.g. in Theorems 5.3, 6.4 and 6.5 or in Remark 6.7.

The next theorem shows that the definition of an antiprincipal solution does not depend on the choice of point $\alpha \in [a, \infty)_{\mathbb{T}}$ determining the interval $[\alpha, \infty)_{\mathbb{T}}$, on which we impose the conditions (i) and (ii) in Definition 4.1. For this reason the term "with respect to interval $[\alpha, \infty)_{\mathbb{T}}$" will be dropped in the terminology of antiprincipal solutions of ($\mathbb{S}$) at infinity in some situations. This statement is a unification of [20, Theorem 5.5] in the continuous case and of [24, Proposition 4.4] in the discrete case, see also [10, Proposition 6.125].

Theorem 4.3. *Every antiprincipal solution of* ($\mathbb{S}$) *at infinity with respect to the interval* $[\alpha, \infty)_{\mathbb{T}}$ *is also an antiprincipal solution of* ($\mathbb{S}$) *at infinity with respect to the interval* $[\beta, \infty)_{\mathbb{T}}$ *for all* $\beta \in (\alpha, \infty)_{\mathbb{T}}$.

Proof. Let (X, U) be an antiprincipal solution of ($\mathbb{S}$) at infinity with respect to the interval $[\alpha, \infty)_{\mathbb{T}}$ and let $\beta \in (\alpha, \infty)_{\mathbb{T}}$ be a given point. Since $d[t, \infty)_{\mathbb{T}}$ is a nondecreasing function in the argument t, we get $d[\beta, \infty)_{\mathbb{T}} \ge d[\alpha, \infty)_{\mathbb{T}} = d_\infty$. This implies $d[\beta, \infty)_{\mathbb{T}} = d_\infty$. The property

$$(X, U) \text{ has constant kernel on } [\beta, \infty)_{\mathbb{T}} \text{ and no focal point in } (\beta, \infty) \tag{4.2}$$

holds trivially since $\beta > \alpha$. In order to prove that (X, U) is an antiprincipal solution of ($\mathbb{S}$) at infinity with respect to the interval $[\beta, \infty)_{\mathbb{T}}$, we need to show that the associated matrices

$$S_\beta(t) := \int_\beta^t [X^\sigma(s)]^\dagger\, \mathcal{B}(s)\, [X^\dagger(s)]^T\, \Delta s, \quad t \in [\beta, \infty)_{\mathbb{T}}, \quad T_\beta := \lim_{t \to \infty} S_\beta^\dagger(t),$$

satisfy the relation $\operatorname{rank} T_\beta = n - d_\infty$. By (4.2) and Theorem 3.4 we know that matrix T_β, being defined as the above limit, exists. We will show that $\operatorname{Im} T_\beta = \operatorname{Im} T$, which will imply the desired equality for the rank of T_β. Note that, with the aid of $S(t)$ defined in (3.1), the matrix $S_\beta(t)$ can be easily expressed as $S_\beta(t) = S(t) - S(\beta)$ for all $t \in [\beta, \infty)_{\mathbb{T}}$. Using (3.17) and $S(\beta) = S(\beta)\, P_{S\infty} = S(\beta)\, S^\dagger(t)\, S(t)$ on $[\beta, \infty)_{\mathbb{T}}$ we then obtain

$$S_\beta(t) = S(t) - S(\beta) = P_{S\infty}\, S(t) - S(\beta)\, S^\dagger(t)\, S(t) = [P_{S\infty} - S(\beta)\, S^\dagger(t)]\, S(t), \quad t \in [\beta, \infty)_{\mathbb{T}}.$$

Then by Remark 2.1(vii) and using (3.25) we obtain

$$S_\beta^\dagger(t) \overset{(2.2)}{=} \big(P_{S\infty}\, S(t)\big)^\dagger \, \big([P_{S\infty} - S(\beta)\, S^\dagger(t)]\, P_{S\infty}\big)^\dagger = S^\dagger(t)\, [P_{S\infty} - S(\beta)\, S^\dagger(t)]^\dagger \tag{4.3}$$

for all $t \in [\beta, \infty)_{\mathbb{T}}$, see also the proof of [22, Proposition 6.4]. Moreover, by (3.25) and (3.26) in Proposition 3.10 together with (3.21) we know that

$$\operatorname{rank}[P_{S\infty} - S(\beta)\, S^\dagger(t)] = n - d_\infty = \operatorname{rank}[P_{S\infty} - S(\beta)\, T], \quad t \in [\beta, \infty)_{\mathbb{T}}.$$

Then by Remark 2.1(iv) the limit of the pseudoinverse $[P_{S\infty} - S(\beta)\, S^\dagger(t)]^\dagger$ for $t \to \infty$ exists and is equal to $[P_{S\infty} - S(\beta)\, T]^\dagger$. Therefore, we obtain that

$$T_\beta = \lim_{t\to\infty} S_\beta^\dagger(t) \overset{(4.3)}{=} \lim_{t\to\infty} S^\dagger(t)\, [P_{S\infty} - S(\beta)\, S^\dagger(t)]^\dagger = T\, [P_{S\infty} - S(\beta)\, T]^\dagger. \tag{4.4}$$

Equality (4.4) implies that $\operatorname{Im} T_\beta \subseteq \operatorname{Im} T$. On the other hand, by (3.26) in Proposition 3.10 and Remark 2.1(ii) we can express the matrix T as

$$T \overset{(3.18)}{=} T P_{S\infty} \overset{(3.26)}{=} T\, [P_{S\infty} - S(t)\, T]^\dagger\, [P_{S\infty} - S(t)\, T] \overset{(4.4)}{=} T_\beta\, [P_{S\infty} - S(t)\, T]$$

for all $t \in [\beta, \infty)_{\mathbb{T}}$. This yields that $\operatorname{Im} T \subseteq \operatorname{Im} T_\beta$, and hence we proved $\operatorname{Im} T = \operatorname{Im} T_\beta$. Consequently, $\operatorname{rank} T_\beta = \operatorname{rank} T = n - d_\infty$, which completes the proof. ☐

In the next theorem we show that an antiprincipal solution of (S) at infinity is characterized by the property of the existence of the limit of $S(t)$ for $t \to \infty$. It is a unification of [24, Theorem 4.3] or [10, Theorem 6.124] in the discrete case and of [20, Theorem 5.3 and Remark 5.4] in the continuous case. We will see important applications of this result in the proofs of Proposition 4.5 and of Theorems 6.4 and 6.5.

Theorem 4.4. *Let (X, U) be a conjoined basis of (S) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞), let the matrices $S(t)$ and T be given by (3.1) and (3.6), and assume that $d[\alpha, \infty)_{\mathbb{T}} = d_\infty$. Then the following statements are equivalent.*

(i) The conjoined basis (X, U) is an antiprincipal solution of (S) at ∞.

(ii) The limit of $S(t)$ for $t \to \infty$ exists.

(iii) The condition $\lim_{t\to\infty} S(t) = T^\dagger$ holds.

Proof. We divide the proof into three steps. First we show that (i) ⇒ (ii). Let (X, U) be an antiprincipal solution at infinity. By Theorem 3.4 we know that the limit of $S^\dagger(t)$ for $t \to \infty$ exists and is equal to T. In addition,

$$\operatorname{rank} S(t) \overset{(3.20)}{=} \operatorname{rank} \hat{X}^{[\alpha]}(t) \overset{(3.20)}{=} n - d[\alpha, \infty) = n - d_\infty = \operatorname{rank} T, \tag{4.5}$$

holds for all sufficiently large $t \in [\alpha, \infty)_{\mathbb{T}}$ by Lemma 3.7 and by the assumption that (X, U) is an antiprincipal solution of (S) at infinity. Therefore, by Remark 2.1(iv) with $M(t) := S^\dagger(t)$ and $M := T$ we conclude that the limit of $[S^\dagger(t)]^\dagger = S(t)$ for $t \to \infty$ exists.

Next we prove the implication (ii) ⇒ (iii). Suppose that the limit of $S(t)$ for $t \to \infty$ exists and let us denote this limit by S_∞. From Theorem 3.4 we know that limit T of $S^\dagger(t)$ for $t \to \infty$ also exists. Moreover, by Remark 2.1(i) and (4.5) we know that

$$\operatorname{rank} S^\dagger(t) = \operatorname{rank} S(t) \overset{(4.5)}{=} n - d_\infty = \operatorname{rank} T$$

for all sufficiently large $t \in [\alpha, \infty)_{\mathbb{T}}$. Now by using Remark 2.1(iv) in which we put $M(t) := S^\dagger(t)$ and $M := T$ we conclude that the limit of $S(t)$ for $t \to \infty$ exists with

$$S_\infty = \lim_{t\to\infty} S(t) = \lim_{t\to\infty} [S^\dagger(t)]^\dagger = T^\dagger.$$

Finally, we prove the implication (iii) ⇒ (i). Suppose that $\lim_{t\to\infty} S(t) = T^\dagger$. Since by Theorem 3.4 we also know that $\lim_{t\to\infty} S^\dagger(t) = T$, it follows from Remark 2.1(iv) that there exists $\beta \in [\alpha, \infty)_{\mathbb{T}}$ such that $\operatorname{rank} S(t) = \operatorname{rank} T^\dagger$ holds for all $t \in [\beta, \infty)_{\mathbb{T}}$. Then from (3.20) in Lemma 3.7 together with the assumptions of the theorem we get $\operatorname{rank} S(t) = \operatorname{rank} S^\dagger(t) = n - d_\infty$ for all $t \in [\beta, \infty)_{\mathbb{T}}$. Therefore, considering the symmetry of T, we get $\operatorname{rank} T = \operatorname{rank} T^\dagger = n - d_\infty$, which proves that (X, U) is an antiprincipal solution of (S) at infinity. □

Our next result shows that the property of being an antiprincipal solution of (S) at infinity is preserved under the multiplication by a constant nonsingular matrix.

Proposition 4.5. *Let (X, U) be an antiprincipal solution of (S) at infinity with respect to the interval $[\alpha, \infty)_{\mathbb{T}}$. Then for every invertible $n \times n$ matrix M the solution (XM, UM) of (S) is also an antiprincipal solution at infinity with respect to the interval $[\alpha, \infty)_{\mathbb{T}}$ and the rank of (XM, UM) is the same as the rank of (X, U).*

Proof. The solution $(\tilde{X}, \tilde{U}) := (XM, UM)$ is obviously a conjoined basis of (S) with the same rank as (X, U). Since $\operatorname{Ker} X(t)$ is constant on $[\alpha, \infty)_{\mathbb{T}}$, then also $\operatorname{Ker} \tilde{X}(t) = \operatorname{Ker} [X(t)\, M]$ is constant on $[\alpha, \infty)_{\mathbb{T}}$. Moreover, by (2.2) in Remark 2.1(vii) we have

$$\tilde{X}^\dagger(t) = [X(t)\, M]^\dagger = (PM)^\dagger X^\dagger(t), \quad t \in [\alpha, \infty)_{\mathbb{T}}. \tag{4.6}$$

This yields that for $t \in [\alpha, \infty)_{\mathbb{T}}$ we have

$$\begin{aligned} \tilde{X}(t)\, [\tilde{X}^\sigma(t)]^\dagger \mathcal{B}(t) &\overset{(4.6)}{=} X(t)\, M (PM)^\dagger [X^\sigma(t)]^\dagger \mathcal{B}(t) = X(t)\, PM (PM)^\dagger PM M^{-1} [X^\sigma(t)]^\dagger \mathcal{B}(t) \\ &= X(t)\, PM M^{-1} [X^\sigma(t)]^\dagger \mathcal{B}(t) = X(t)\, [X^\sigma(t)]^\dagger \mathcal{B}(t) \geq 0, \end{aligned}$$

showing that the conjoined basis $(\tilde{X}, \tilde{U})$ has no focal points in (α, ∞). For the matrix $\tilde{S}(t)$ in (3.1) associated with $(\tilde{X}, \tilde{U})$ we have

$$\tilde{S}(t) := \int_\alpha^t [\tilde{X}^\sigma(s)]^\dagger \mathcal{B}(s)\, [\tilde{X}^\dagger(s)]^T \Delta s \overset{(4.6)}{=} (PM)^\dagger S(t)\, [(PM)^\dagger]^T, \quad t \in [\alpha, \infty)_{\mathbb{T}}. \tag{4.7}$$

Since the limit of $S(t)$ as $t \to \infty$ exists and is equal to $T^\dagger$ by Theorem 4.4(iii), then the limit

$$\lim_{t\to\infty} \tilde{S}(t) \overset{(4.7)}{=} (PM)^\dagger T^\dagger [(PM)^\dagger]^T$$

also exists. Therefore, by Theorem 4.4(ii) again (now applied to $(X, U) := (\tilde{X}, \tilde{U})$) the conjoined basis $(\tilde{X}, \tilde{U})$ is an antiprincipal solution of (S) at infinity. □

The following two theorems show that the relation "to be contained in" or "to contain" (in Definition 3.11) preserves the property of being an antiprincipal solution of ($\mathbb{S}$) at infinity. It is a unification of [20, Theorem 5.7] in the continuous case and of [24, Proposition 4.6] in the discrete case, see also [10, Proposition 6.127].

Theorem 4.6. *Let (X, U) be an antiprincipal solution of ($\mathbb{S}$) at infinity with respect to the interval $[\alpha, \infty)_{\mathbb{T}}$. Then every conjoined basis, which is contained in (X, U) on $[\alpha, \infty)_{\mathbb{T}}$, is also an antiprincipal solution of ($\mathbb{S}$) at infinity with respect to the interval $[\alpha, \infty)_{\mathbb{T}}$.*

Proof. From the assumptions of the theorem we directly get $d[\alpha, \infty)_{\mathbb{T}} = d_\infty$. Let (X_*, U_*) be a conjoined basis of ($\mathbb{S}$), which is contained in (X, U) on $[\alpha, \infty)_{\mathbb{T}}$. The conjoined basis (X_*, U_*) has constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞) due to Proposition 3.12(i), because (X, U) possess the same properties. Finally according to Proposition 3.12(ii) we get that the S-matrices corresponding to (X, U) and (X_*, U_*) coincide, i.e., $S(t) = S_*(t)$ on $[\alpha, \infty)_{\mathbb{T}}$. Therefore the limit $T_* := \lim_{t\to\infty} S_*^{\dagger}(t)$ exists and equals to T. This yields that $\operatorname{rank} T_* = \operatorname{rank} T = n - d_\infty$, which proves that (X_*, U_*) is also an antiprincipal solution of ($\mathbb{S}$) at infinity with respect to the interval $[\alpha, \infty)_{\mathbb{T}}$. □

Theorem 4.7. *Let (X, U) be an antiprincipal solution of ($\mathbb{S}$) at infinity with respect to the interval $[\alpha, \infty)_{\mathbb{T}}$. Then every conjoined basis with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞), which contains (X, U) on $[\alpha, \infty)_{\mathbb{T}}$, is also an antiprincipal solution of ($\mathbb{S}$) at infinity with respect to the interval $[\alpha, \infty)_{\mathbb{T}}$.*

Proof. From the assumptions of the theorem we directly get $d[\alpha, \infty)_{\mathbb{T}} = d_\infty$. Denote by (X_*, U_*) the conjoined basis with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞), which contains (X, U) on the interval $[\alpha, \infty)_{\mathbb{T}}$. Then by Proposition 3.12(ii), applied to $(X, U) := (X_*, U_*)$ and $(X_*, U_*) := (X, U)$, we obtain the equality $S_*(t) = S(t)$ on $[\alpha, \infty)_{\mathbb{T}}$. This implies that for the T-matrices corresponding to (X_*, U_*) and (X, U) the equality $T_* = T$ also holds. Therefore, $\operatorname{rank} T_* = \operatorname{rank} T = n - d_\infty$ holds and (X_*, U_*) is an antiprincipal solution of ($\mathbb{S}$) at infinity with respect to $[\alpha, \infty)_{\mathbb{T}}$. □

5 Existence of antiprincipal solutions at infinity

In this section we prove the existence of antiprincipal solutions at infinity for a nonoscillatory system ($\mathbb{S}$). We show the existence of such solutions (Theorem 5.3) for any rank in the admissible range given by estimate (3.23) in Lemma 3.7. As a main tool for this construction we derive (Theorem 5.2, through Theorem 5.1) a characterization of the T-matrices associated with conjoined bases of a nonoscillatory system ($\mathbb{S}$).

Our first result describes all minimal conjoined bases of ($\mathbb{S}$) on some interval $[\alpha, \infty)_{\mathbb{T}}$. It is a generalization to arbitrary time scales of [20, Theorem 4.4 and Remark 4.5] for the continuous case and of [24, Theorem 3.4] for the discrete case, see also [10, Theorem 6.106].

Theorem 5.1. *Let (X, U) be a minimal conjoined basis of ($\mathbb{S}$) on $[\alpha, \infty)_{\mathbb{T}}$, let $P_{S\infty}$ and T defined by (3.15) and (3.6), and assume that $d[\alpha, \infty)_{\mathbb{T}} = d_\infty$. Then a solution $(\tilde{X}, \tilde{U})$ is a minimal conjoined basis on $[\alpha, \infty)_{\mathbb{T}}$ if and only if there exist matrices $M, N \in \mathbb{R}^{n\times n}$ such that*

$$\tilde{X}(\alpha) = X(\alpha)\, M, \quad \tilde{U}(\alpha) = U(\alpha)\, M + X^{\dagger T}(\alpha)\, N, \tag{5.1}$$

$$M \text{ is nonsingular}, \quad M^T N = N^T M, \quad \operatorname{Im} N \subseteq \operatorname{Im} P_{S\infty}, \tag{5.2}$$

$$N M^{-1} + T \geq 0. \tag{5.3}$$

In this case the matrix $\tilde{T}$ in (3.6) *corresponding to* $(\tilde{X}, \tilde{U})$ *satisfies*

$$\tilde{T} = M^T T M + M^T N, \quad \operatorname{rank} \tilde{T} = \operatorname{rank} (N M^{-1} + T). \tag{5.4}$$

Proof. Let (X, U) be the conjoined basis of (S) from the assumptions of the theorem, that is, (X, U) has constant kernel on $[\alpha, \infty)_{\mathbb{T}}$, no focal points in (α, ∞), and $\operatorname{rank} X(t) = n - d_\infty$ on $[\alpha, \infty)_{\mathbb{T}}$. Assume first that $(\tilde{X}, \tilde{U})$ is also a minimal conjoined basis of (S) on $[\alpha, \infty)_{\mathbb{T}}$. Then $\operatorname{rank} \tilde{X}(t) = n - d_\infty = \operatorname{rank} X(t)$ and from (3.36) in Proposition 3.18(i) we obtain

$$\operatorname{Im} \tilde{X}(\alpha) = \big(\Lambda_0[\alpha, \infty)_{\mathbb{T}}\big)^\perp = \operatorname{Im} X(\alpha). \tag{5.5}$$

Applying now Proposition 3.15, where we put $(X, U) = (X_1, U_1)$ and $(\tilde{X}, \tilde{U}) = (X_2, U_2)$ on $[\alpha, \infty)_{\mathbb{T}}$, we get that there exist matrices $M, N \in \mathbb{R}^{n \times n}$ such that

- M is nonsingular by Proposition 3.15(ii),
- $M^T N = N^T M$ by Proposition 3.15(i),
- $\operatorname{Im} N \subseteq \operatorname{Im} P = \operatorname{Im} P_{S\infty}$ by Proposition 3.15(iii) and Lemma 3.17.

This which proves the properties in (5.2). Moreover, the mutual representation between $(\tilde{X}, \tilde{U})$ and (X, U), which we use here, is provided by the relation

$$\begin{pmatrix} \tilde{X}(t) \\ \tilde{U}(t) \end{pmatrix} = \begin{pmatrix} X(t) & \bar{X}(t) \\ U(t) & \bar{U}(t) \end{pmatrix} \begin{pmatrix} M \\ N \end{pmatrix}, \quad t \in [\alpha, \infty)_{\mathbb{T}}, \tag{5.6}$$

where $(\bar{X}, \bar{U})$ is the conjoined basis chosen according to Proposition 3.14. In particular,

$$X^\dagger(\alpha)\, \bar{X}(\alpha) = 0 \tag{5.7}$$

holds. Let $R(t)$ and $\tilde{R}(t)$ be the orthogonal projectors defined in (2.4), which are associated respectively with (X, U) and $(\tilde{X}, \tilde{U})$. Then from (5.5) we get $R(\alpha) = \tilde{R}(\alpha)$. Now from (5.6) for $t = \alpha$ we get that $\tilde{X}(\alpha) = X(\alpha)\, M + \bar{X}(\alpha)\, N$. Multiplying this equality by $X^\dagger(\alpha)$ from the left, using (5.7), and $\tilde{R}(\alpha)\, \tilde{X}(\alpha) = \tilde{X}(\alpha)$ derived from the definition of the Moore–Penrose pseudoinverse, we get that $\tilde{X}(\alpha) = X(\alpha)\, M$. Similarly, condition (5.6) for $t = \alpha$ gives that $\tilde{U}(\alpha) = U(\alpha)\, M + \bar{U}(\alpha)\, N$. Now using the information that (X, U) and $(\bar{X}, \bar{U})$ are normalized we get $\bar{U}(\alpha)\, X^T(\alpha) - U(\alpha)\, \bar{X}^T(\alpha) = I$. Multiplying this equality by $X^{\dagger T}(t)\, N$ from the right, and using $PN = P_{S\infty} N = N$ derived from the property $\operatorname{Im} N \subseteq \operatorname{Im} P = \operatorname{Im} P_{S\infty}$, we get $\bar{U}(\alpha)\, N = X^{\dagger T}(\alpha)\, N$. This together with the previous part implies that (5.1) holds. Let T and $\tilde{T}$ be, respectively, the matrices defined in (3.6) corresponding to the minimal conjoined bases (X, U) and $(\tilde{X}, \tilde{U})$ on $[\alpha, \infty)_{\mathbb{T}}$. Then from Proposition 3.18(ii) and Remark 3.16 (where we put $(X_1, U_1) := (X, U)$, $T_1 := T$ and $(X_2, U_2) := (\tilde{X}, \tilde{U})$, $T_2 := \tilde{T}$ and consider $t \to \infty$) we obtain

$$\tilde{T} = L_1^T T L_1 + L_1^T N \overset{(3.28)}{=} M^T P T P M + M^T N \overset{(3.29)}{=} M^T T M + M^T N, \tag{5.8}$$

This together with the existence of M^{-1} implies that

$$N M^{-1} + T = M^{T^{-1}} \tilde{T} M^{-1} \geq 0, \tag{5.9}$$

since $\tilde{T} \geq 0$ by Theorem 3.4. Therefore, condition (5.3) holds. From inequality (5.9) we then conclude that $\operatorname{rank} \tilde{T} = \operatorname{rank} N M^{-1} + T$, which together with (5.8) shows (5.4).

Conversely, assume that $(\tilde{X}, \tilde{U})$ is a solution of ($\mathbb{S}$) and let $M, N \in \mathbb{R}^{n\times n}$ be such that (5.1), (5.2), and (5.3) hold. First we will show that $(\tilde{X}, \tilde{U})$ is a conjoined basis of ($\mathbb{S}$), i.e., we will show that the solution $(\tilde{X}, \tilde{U})$ satisfies the condition on the symmetry of $\tilde{X}^T(t)\,\tilde{U}(t)$ and the condition on $\operatorname{rank}(\tilde{X}^T(t), \tilde{U}^T(t))^T = n$ at some point $t \in [\alpha, \infty)_{\mathbb{T}}$. The symmetry of $\tilde{X}^T(t)\,\tilde{U}(t)$ can be seen by using (5.1), by the symmetry of $M^T N$ and $X^T(t)\,U(t)$ as a property of the conjoined basis (X, U), and by the relation

$$M^T X^\dagger(t)\,X(t)\,N = M^T P N \overset{(5.2)}{=} M^T N, \quad t \in [a, \infty)_{\mathbb{T}}. \tag{5.10}$$

More precisely, we have

$$\tilde{X}^T(\alpha)\,\tilde{U}(\alpha) \overset{(5.1)}{=} M^T X^T(\alpha)\,[U(\alpha)\,M + X^{\dagger T}(\alpha)\,N] \overset{(5.10)}{=} M^T X^T(\alpha)\,U(\alpha)\,M + M^T N,$$

where the last matrix is symmetric for all $t \in [a, \infty)_{\mathbb{T}}$. The condition on the rank of the matrix $(\tilde{X}^T(t), \tilde{U}^T(t))^T$ is also satisfied, since it follows again from (5.1) together with the fact that $\operatorname{rank}(X^T(t), U^T(t))^T = n$ on $[\alpha, \infty)_{\mathbb{T}}$, and from the fact that the subspaces $\operatorname{Im} X(\alpha)$ and $\operatorname{Im} X^{\dagger T}(\alpha)$ are equal. Thus, $(\tilde{X}, \tilde{U})$ is a conjoined basis of ($\mathbb{S}$). Next we will show that $(\tilde{X}, \tilde{U})$ is a minimal conjoined basis of ($\mathbb{S}$) on $[\alpha, \infty)_{\mathbb{T}}$. This means to prove, according to the definition, that $(\tilde{X}, \tilde{U})$ has constant kernel on $[\alpha, \infty)_{\mathbb{T}}$, the rank of $\tilde{X}(t)$ is equal to $n - d_\infty$ on $[\alpha, \infty)_{\mathbb{T}}$, and $(\tilde{X}, \tilde{U})$ has no focal points in (α, ∞). Let $S(t)$ be the matrix in (3.1) corresponding to (X, U) on $[\alpha, \infty)_{\mathbb{T}}$ and let $(\tilde{X}, \tilde{U})$ be expressed in terms of (X, U) as in Proposition 3.15 (where we put $(X_1, U_1) = (X, U)$ and $(X_2, U_2) = (\tilde{X}, \tilde{U})$). More precisely, $(\tilde{X}, \tilde{U})$ is represented as

$$\begin{pmatrix} \tilde{X}(t) \\ \tilde{U}(t) \end{pmatrix} = \Phi(t) \begin{pmatrix} \underline{M} \\ \underline{N} \end{pmatrix}, \quad \Phi(t) := \begin{pmatrix} X(t) & \bar{X}(t) \\ U(t) & \bar{U}(t) \end{pmatrix}, \quad t \in [\alpha, \infty)_{\mathbb{T}}, \tag{5.11}$$

that is, $(\bar{X}_1, \bar{U}_1) := (\bar{X}, \bar{U})$, $M_1 := \underline{M}$, and $N_1 := \underline{N}$ in Proposition 3.15. We will show that $\underline{M} = M$ and $\underline{N} = N$ by using the fact that the matrix $\Phi(t)$ is symplectic as a fundamental matrix of ($\mathbb{S}$). Thus, we can express its inverse as $\Phi^{-1}(t) = -\mathcal{J}\,\Phi^T(t)\,\mathcal{J}$ and evaluate it in (5.11) at $t = \alpha$ to get

$$\begin{pmatrix} \underline{M} \\ \underline{N} \end{pmatrix} = \Phi^{-1}(\alpha) \begin{pmatrix} \tilde{X}(\alpha) \\ \tilde{U}(\alpha) \end{pmatrix} = \begin{pmatrix} \bar{U}^T(\alpha) & -\bar{X}^T(\alpha) \\ -U^T(\alpha) & X^T(\alpha) \end{pmatrix} \begin{pmatrix} \tilde{X}(\alpha) \\ \tilde{U}(\alpha) \end{pmatrix}. \tag{5.12}$$

Using the fact that Wronskian of (X, U) and $(\bar{X}, \bar{U})$ equals to the identity matrix and using that $(\tilde{X}, \tilde{U})$ now satisfies condition (5.1), equality (5.12) implies that

$$\begin{aligned} \underline{M} &= \bar{U}^T(\alpha)\,X(\alpha)\,M - \bar{X}^T(\alpha)\,[U(\alpha)\,M + X^{\dagger T}(\alpha)\,N] \\ &= [\bar{U}^T(\alpha)\,X(\alpha) - \bar{X}^T(\alpha)\,U(\alpha)]\,M - \bar{X}^T(\alpha)\,X^{\dagger T}(\alpha)\,N = M - [X^\dagger(\alpha)\,\bar{X}(\alpha)]^T N \overset{(5.7)}{=} M. \end{aligned}$$

Considering now the symmetry of $X^T(t)\,U(t)$ and the third condition in assumption (5.2) in the form $P_{S\infty}\,N = N$ we get $PN = P_{S\infty}\,N = N$, since (X, U) is a minimal conjoined basis of ($\mathbb{S}$) on $[\alpha, \infty)_{\mathbb{T}}$. Therefore, from (5.12) we get

$$\begin{aligned} \underline{N} &= -U^T(\alpha)\,X(\alpha)\,M + X^T(\alpha)\,[U(\alpha)\,M + X^{\dagger T}(\alpha)\,N] \\ &= [X^T(\alpha)\,U(\alpha) - U^T(\alpha)\,X(\alpha)]\,M + X^T(\alpha)\,X^{\dagger T}(\alpha)\,N = X^\dagger(\alpha)\,X(\alpha)\,N = PN = N. \end{aligned}$$

From Remark 3.16 and equation (3.30) we then obtain

$$\tilde{X}(t) = X(t)\,[P_{S\infty}\,\underline{M} + S(t)\,\underline{N}] = X(t)\,[P_{S\infty}\,M + S(t)\,N], \quad t \in [\alpha, \infty)_{\mathbb{T}}. \tag{5.13}$$

Note that equation (5.13) is valid when the kernel of the first basis (X, U) is constant on $[\alpha, \infty)_{\mathbb{T}}$, which is now satisfied, and there is no requirement on the kernel of the second basis $(\tilde{X}, \tilde{U})$, analogically to discrete case, see [10, Remark 6.70(iii)]. Now we show that also $(\tilde{X}, \tilde{U})$ has constant kernel on $[\alpha, \infty)_{\mathbb{T}}$. More precisely, we show that $\operatorname{Ker} \tilde{X}(t) = \operatorname{Ker}(P_{S\infty} M)$ on $[\alpha, \infty)_{\mathbb{T}}$ in the following two steps.

(i) We show that $\operatorname{Ker}(P_{S\infty} M) \subseteq \operatorname{Ker} \tilde{X}(t)$ on $[\alpha, \infty)_{\mathbb{T}}$. Let $u \in \operatorname{Ker} P_{S\infty} M$ be given, i.e., $P_{S\infty} M u = 0$. Multiplying the last equality by $X(t)\, [I + S(t)\, N M^{-1}]$ from the left and using

$$NM^{-1} P_{S\infty} = M^{T-1} N^T P_{S\infty} = M^{T-1} N^T = NM^{-1}$$

derived from (5.2), we get

$$\tilde{X}(t)\, u \overset{(5.13)}{=} X(t)\, [P_{S\infty} + S(t)\, N]\, u = X(t)\, [I + S(t)\, NM^{-1}]\, P_{S\infty} M u = 0$$

for all $t \in [\alpha, \infty)_{\mathbb{T}}$. Thus, $u \in \operatorname{Ker} \tilde{X}(t)$ on $[\alpha, \infty)_{\mathbb{T}}$.

(ii) We show that $\operatorname{Ker} \tilde{X}(t) \subseteq \operatorname{Ker}(P_{S\infty} M)$ on $[\alpha, \infty)_{\mathbb{T}}$. Let $v \in \operatorname{Ker} \tilde{X}(t)$ on $[\alpha, \infty)_{\mathbb{T}}$ and set $w := P_{S\infty} M v$. Our aim now is to show that $w = 0$. By (5.13) we get

$$X(t)\, [w + S(t)\, NM^{-1} w] = 0, \quad t \in [\alpha, \infty)_{\mathbb{T}}. \tag{5.14}$$

Multiplying (5.14) by $X^{\dagger}(t)$ from the left, using $P_{S\infty} w = w$ derived from the properties of any orthogonal projector, and from (3.17) we get

$$w = -S(t)\, NM^{-1} w, \quad t \in [\alpha, \infty)_{\mathbb{T}}, \tag{5.15}$$

which implies that

$$w \in \operatorname{Im} S(t) \overset{(3.35)}{=} \operatorname{Im} X^{\dagger}(t) = \operatorname{Im} P \overset{(3.33)}{=} \operatorname{Im} P_{S\infty}, \quad t \in [\alpha, \infty)_{\mathbb{T}}, \tag{5.16}$$

where we used that (X, U) is a minimal conjoined basis of ($\mathbb{S}$) on $[\alpha, \infty)_{\mathbb{T}}$. Moreover, from $w = P_{S\infty} w$ and the above derived results we obtain for $t \in [\alpha, \infty)_{\mathbb{T}}$

$$w^T S^{\dagger}(t)\, w \overset{(5.15)}{=} -w^T S^{\dagger}(t)\, S(t)\, NM^{-1} w = -w^T P_{S\infty} NM^{-1} P_{S\infty} w.$$

Considering (5.3), which can be rewritten as $-NM^{-1} \leq T$, we get

$$w^T S^{\dagger}(t)\, w = -w^T P_{S\infty} NM^{-1} P_{S\infty} w \leq w^T P_{S\infty} T P_{S\infty} w \overset{(3.18)}{=} w^T T w$$

for $t \in [\alpha, \infty)_{\mathbb{T}}$, which implies that

$$w^T [S^{\dagger}(t) - T]\, w \leq 0, \quad t \in [\alpha, \infty)_{\mathbb{T}}.$$

But according to Proposition 3.9 the inequality $S^{\dagger}(t) \geq T \geq 0$ holds for large t, so that $w \in \operatorname{Ker}[S^{\dagger}(t) - T]$ for large t. But since $\operatorname{Im} S(t) = \operatorname{Im} P_{S\infty} = \operatorname{Im} P$ for $t \in (\alpha, \infty)_{\mathbb{T}}$ and (3.18) holds, we derive by using Proposition 3.10 the equality of kernels

$$\operatorname{Ker}[S^{\dagger}(t) - T] = \operatorname{Ker}[S(t)\, S^{\dagger}(t) - S(t)\, T] = \operatorname{Ker}[P_{\infty} - S(t)\, T] \overset{(3.26)}{=} \operatorname{Ker} P_{S\infty}$$

on $(\alpha, \infty)_{\mathbb{T}}$, which implies that $w \in \operatorname{Ker} P_{S\infty}$. This together with $w \in \operatorname{Im} P_{S\infty}$ from (5.16) implies that $w = 0$. Thus, $P_{S\infty} M v = 0$, which proves that $v \in \operatorname{Ker}(P_{S\infty} M)$.

The proof of the equality $\operatorname{Ker} \tilde{X}(t) = \operatorname{Ker}(P_{S\infty} M)$ on $[\alpha, \infty)_{\mathbb{T}}$ is now complete. It follows that

$$\operatorname{rank} \tilde{X}(t) = \operatorname{rank}(P_{S\infty}\, M) = \operatorname{rank} P_{S\infty} = n - d_\infty, \quad t \in [\alpha, \infty)_{\mathbb{T}}.$$

Thus, the conjoined basis $(\tilde{X}, \tilde{U})$ has constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and the lowest possible rank $n - d_\infty$ on $[\alpha, \infty)_{\mathbb{T}}$. It remains to prove that $(\tilde{X}, \tilde{U})$ has no focal points in the interval (α, ∞). Since $P = P_{S\infty}$ and $d[\alpha, \infty)_{\mathbb{T}} = d_\infty$, we have $\operatorname{Im} S(t) \equiv \operatorname{Im} P_{S\infty}$ on the interval $(\alpha, \infty)_{\mathbb{T}}$. Recall that $S(\alpha) = 0$. Since $\operatorname{Ker} \tilde{X}(t)$ is constant on $[\alpha, \infty)_{\mathbb{T}}$, the matrix

$$\tilde{S}(t) := \int_\alpha^t [\tilde{X}^\sigma(s)]^\dagger\, \mathcal{B}(s)\, [\tilde{X}^\dagger(s)]^T\, \Delta s, \quad t \in [\alpha, \infty)_{\mathbb{T}}, \tag{5.17}$$

is symmetric and, by (3.32) in Remark 3.16, the formula

$$\tilde{S}(t) = [PM + S(t)\, N]^\dagger S(t)\, M^{T-1} \tilde{P}, \quad t \in [\alpha, \infty)_{\mathbb{T}},$$

holds. Then, by (3.37) in Proposition 3.18(ii), the pseudoinverse of $\tilde{S}(t)$ has the form

$$\tilde{S}^\dagger(t) = M^T S^\dagger(t)\, M + M^T N, \quad t \in (\alpha, \infty)_{\mathbb{T}}. \tag{5.18}$$

Note that if the point α is right-scattered, then formula (5.18) holds for $t \in [\sigma(\alpha), \infty)_{\mathbb{T}}$ only. Since by Proposition 3.9 the matrix function $S^\dagger(t)$ is nonincreasing on $(\alpha, \infty)_{\mathbb{T}}$, it follows from (5.18) that the matrix function $\tilde{S}^\dagger(t)$ is nonincreasing on $(\alpha, \infty)_{\mathbb{T}}$ as well and hence, by Remark 2.1(v), the the matrix function $\tilde{S}(t)$ is nondecreasing on $(\alpha, \infty)_{\mathbb{T}}$. Moreover,

$$\tilde{S}^\dagger(t) \overset{(5.18)}{=} M^T S^\dagger(t)\, M + M^T N \overset{(3.24)}{\geq} M^T T M + M^T N \overset{(5.3)}{\geq} 0, \quad t \in (\alpha, \infty)_{\mathbb{T}}.$$

Therefore, in view of Remark 2.1(vi) we also have

$$\tilde{S}(t) \geq 0, \quad t \in (\alpha, \infty)_{\mathbb{T}}. \tag{5.19}$$

From the already established monotonicity of $\tilde{S}(t)$ on $(\alpha, \infty)_{\mathbb{T}}$ it now follows that $\tilde{S}^\Delta(t) \geq 0$ on $(\alpha, \infty)_{\mathbb{T}}$. Then with the aid of Proposition 3.6(ii) (applied to $(X, U) := (\tilde{X}, \tilde{U})$) we get

$$\begin{aligned} \tilde{X}(t)\, [\tilde{X}^\sigma(t)]^\dagger\, \mathcal{B}(t) &= \tilde{X}(t)\, [\tilde{X}^\sigma(t)]^\dagger\, \mathcal{B}(t)\, \tilde{R}(t) = \tilde{X}(t)\, [\tilde{X}^\sigma(t)]^\dagger\, \mathcal{B}(t)\, [\tilde{X}^\dagger(t)]^T \tilde{X}^T(t) \\ &\overset{(5.17)}{=} \tilde{X}(t)\, \tilde{S}^\Delta(t)\, \tilde{X}^T(t) \geq 0, \quad t \in (\alpha, \infty)_{\mathbb{T}}. \end{aligned} \tag{5.20}$$

This shows that the conjoined basis $(\tilde{X}, \tilde{U})$ has no focal points in the interval (α, ∞) if the point α is right-dense, and no focal points in the interval $(\sigma(\alpha), \infty)$ if the point α is right-scattered. However, in the latter situation (that is, for $\sigma(\alpha) > \alpha$) we know by property (5.19) at $t = \sigma(\alpha)$ that $\tilde{S}^\sigma(\alpha) \geq 0$, so that in this case

$$\tilde{S}^\Delta(\alpha) = [\tilde{S}^\sigma(\alpha) - \tilde{S}(\alpha)] / \mu(\alpha) \overset{(5.17)}{=} \tilde{S}^\sigma(\alpha) / \mu(\alpha) \geq 0.$$

As in (5.20) we then conclude that $(\tilde{X}, \tilde{U})$ has no focal points in the interval $(\alpha, \sigma(\alpha)]$ when α is right-scattered. This proves that $(\tilde{X}, \tilde{U})$ has no focal points in the interval (α, ∞) and hence, it is a minimal conjoined basis of (S) on $[\alpha, \infty)_{\mathbb{T}}$. The proof is complete. $\square$

The next theorem serves as a criterion for the classification of all T-matrices, which correspond to conjoined bases of (S) with constant kernel on some interval $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞). It is a unification of the continuous and discrete cases in [20, Theorem 4.9 and Corollary 4.11] and [24, Theorem 3.5], see also [10, Theorem 6.107 and Remark 6.108].

Theorem 5.2. *Assume that* ($\mathbb{S}$) *is nonoscillatory. Then* $D \in \mathbb{R}^{n\times n}$ *is a* T*-matrix of some conjoined basis* (X,U) *of* ($\mathbb{S}$) *with constant kernel on* $[\alpha,\infty)_{\mathbb{T}}$ *and no focal points in* (α,∞) *and with* $d[\alpha,\infty)_{\mathbb{T}} = d_\infty$ *if and only if*

$$\text{the matrix } D \text{ is symmetric, } D \geq 0, \text{ and } \operatorname{rank} D \leq n - d_\infty. \tag{5.21}$$

Moreover, (X,U) *can be chosen to be a minimal conjoined basis of* ($\mathbb{S}$) *on* $[\alpha,\infty)_{\mathbb{T}}$.

Proof. First we prove that the result holds for minimal conjoined bases of ($\mathbb{S}$) on $[\alpha,\infty)_{\mathbb{T}}$. Let D be a T-matrix of a minimal conjoined basis (X,U) on an interval $[\alpha,\infty)_{\mathbb{T}}$ with $d[\alpha,\infty)_{\mathbb{T}} = d_\infty$. Then according to Theorem 3.4 the matrix D is symmetric and $D \geq 0$. Note that by equation (3.21) in Lemma 3.7 we have $\operatorname{rank} P = \operatorname{rank} P_{S\infty} = n - d_\infty$. The inclusion in (3.18) implies that $\operatorname{rank} D \leq n - d_\infty$.

Conversely, assume that D is a symmetric matrix with $D \geq 0$ and $\operatorname{rank} D \leq n - d_\infty$. We will show through Theorem 5.1 that D is the T-matrix of some minimal conjoined basis of ($\mathbb{S}$) on some interval $[\alpha,\infty)_{\mathbb{T}}$ satisfying $d[\alpha,\infty)_{\mathbb{T}} = d_\infty$. First, we show that there exists a minimal conjoined basis $(X_{\min}, U_{\min})$ of ($\mathbb{S}$) on $[\alpha,\infty)_{\mathbb{T}}$. Then, using this conjoined basis, we will construct another one denoted by $(\tilde{X},\tilde{U})$ (via Theorem 5.1), such that its associated matrix $\tilde{T}$ is equal to D. Since we assume that system ($\mathbb{S}$) is nonoscillatory, then every conjoined basis of ($\mathbb{S}$) is nonoscillatory and by Proposition 3.13 (with $r := n - d_\infty$) there exists a minimal conjoined basis $(X_{\min}, U_{\min})$ of ($\mathbb{S}$) on $[\alpha,\infty)_{\mathbb{T}}$. Then by Proposition 3.18(i) condition $d[\alpha,\infty)_{\mathbb{T}} = d_\infty$ holds. The assumption $\operatorname{rank} D \leq n - d_\infty$ guarantees that there exists an orthogonal projector Q with $\operatorname{rank} Q = n - d_\infty$ such that

$$\operatorname{Im} D \subseteq \operatorname{Im} Q. \tag{5.22}$$

Then by Lemma 3.7 condition (3.21) holds, i.e., $\operatorname{rank} P_{S\infty} = n - d_\infty = \operatorname{rank} Q$. Moreover, by Proposition 2.2 (where we put $P_* := 0$) there exists an invertible matrix $E \in \mathbb{R}^{n\times n}$ satisfying $\operatorname{Im} EP_{S\infty} = \operatorname{Im} Q$, i.e., $\operatorname{Im} P_{S\infty} = \operatorname{Im} E^{-1}Q$. In particular, the equality $P_{S\infty} E^{-1}Q = E^{-1}Q$ holds. Define the matrices $M, N \in \mathbb{R}^{n\times n}$ by

$$M := E^T, \quad N := E^{-1}D - TE^T, \tag{5.23}$$

where T is the matrix in (3.6) corresponding to $(X_{\min}, U_{\min})$. We will show that the matrices M and N satisfy conditions (5.2) and (5.3) from Theorem 5.1, i.e., the following four properties of matrices M and N hold:

(i) The matrix M is invertible. This follows from the definition of M in (5.23).

(ii) The matrix M^TN is symmetric. This follows from the symmetry of D and T and from the calculation $M^TN = E(E^{-1}D - TE^T) = D - ETE^T$.

(iii) The inclusion $\operatorname{Im} N \subseteq \operatorname{Im} P_{S\infty}$ holds, since

$$\begin{aligned} N &= E^{-1}D - TE^T \overset{(5.22)}{=} E^{-1}QD - TE^T = P_{S\infty} E^{-1}QD - TE^T \\ &\overset{(3.18)}{=} P_{S\infty}E^{-1}QD - P_{S\infty} TE^T = P_{S\infty}(E^{-1}QD - TE^T) = P_{S\infty}N. \end{aligned}$$

(iv) The matrix $NM^{-1} + T$ is positive semidefinite, since $D \geq 0$ and

$$NM^{-1} + T = (E^{-1}D - TE^T)\, E^{T-1} + T = E^{-1}DE^{T-1} \geq 0.$$

Consider now the conjoined basis $(\tilde{X}, \tilde{U})$ of (S) on $[a, \infty)_{\mathbb{T}}$ with the initial conditions at the point α given by (5.1), where matrices M and N are given in (5.23) above. Then by Theorem 5.1 the solution $(\tilde{X}, \tilde{U})$ is a minimal conjoined basis of (S) on $[\alpha, \infty)_{\mathbb{T}}$ and, moreover, its associated $\tilde{T}$ satisfies (5.4). This yields that

$$\tilde{T} \overset{(5.4)}{=} M^T T M + M^T N \overset{(5.23)}{=} E T E^T + E(E^{-1} D - T E^T) = D.$$

Therefore, we showed that the matrix D is the T-matrix of the minimal conjoined basis $(\tilde{X}, \tilde{U})$ on $[\alpha, \infty)_{\mathbb{T}}$.

The general statement of the theorem now follows from Proposition 3.12(ii). Let (X, U) be a conjoined basis of (S) with constant constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞) and with $d[\alpha, \infty)_{\mathbb{T}} = d_\infty$. Let (X_*, U_*) be a minimal conjoined basis on $[\alpha, \infty)_{\mathbb{T}}$, which is contained in (X, U) on $[\alpha, \infty)_{\mathbb{T}}$. Note that such a minimal conjoined basis always exists by [22, Theorem 4.3] (with the choice $P_* := P_{S\infty}$). Then by the first part of the proof the matrix $D := T_*$ in (3.6) associated with (X_*, U_*) satisfies the conditions in (5.21). But by Proposition 3.12(ii) the matrices $S_*(t)$ and $S(t)$ coincide on the interval $[\alpha, \infty)_{\mathbb{T}}$, so that $T_* = T$ and hence, the matrix T satisfies (5.21) as well. □

Next we derive the existence of antiprincipal solutions at infinity with any admissible rank r for a nonoscillatory system (S), see the continuous case in [20, Theorem 5.8] and the discrete case in [24, Theorem 4.7] or in [10, Theorem 6.128]. It can also be viewed as a counterpart of [22, Theorem 6.8] regarding the principal solutions of (S) at infinity. The most important part consists of the existence of a minimal antiprincipal solution of (S) at infinity. This property will also be used later in Section 6 in the applications of antiprincipal solutions at infinity.

Theorem 5.3. *Assume that system (S) is nonoscillatory. Then there exists a minimal antiprincipal solution of (S) at infinity. Moreover, in this case for any integer r between $n - d_\infty$ and n there exists an antiprincipal solution (X, U) of (S) at infinity with the rank of $X(t)$ equal to r for large t.*

Proof. Assume that system (S) is nonoscillatory and let $D \in \mathbb{R}^{n \times n}$ be an arbitrary symmetric and positive semidefinite matrix with $\operatorname{rank} D = n - d_\infty$. Let $\alpha \in [a, \infty)_{\mathbb{T}}$ be large enough so that $d[\alpha, \infty)_{\mathbb{T}} = d_\infty$ holds. According to Theorem 5.2, there exists a minimal conjoined basis $(X_{\min}, U_{\min})$ on $[\alpha, \infty)_{\mathbb{T}}$ such that its corresponding matrix T is equal to D. By Definition 4.1, this conjoined basis is an antiprincipal solution of (S) at infinity with respect to the interval $[\alpha, \infty)_{\mathbb{T}}$, since $\operatorname{rank} T = \operatorname{rank} D = n - d_\infty$ due to above choice of D. In addition, since $\operatorname{rank} X_{\min}(t) = n - d_\infty$ on $[\alpha, \infty)_{\mathbb{T}}$, we get that $(X_{\min}, U_{\min})$ is a minimal antiprincipal solution of (S) at infinity, which proves the first part of the theorem. Furthermore, choose any integer r between $n - d_\infty$ and n. Then by Proposition 3.13, using the already established existence of $(X_{\min}, U_{\min})$, there exists a conjoined basis (X, U) of (S) with $\operatorname{rank} X(t) = r$ on $[\alpha, \infty)_{\mathbb{T}}$, which contains $(X_{\min}, U_{\min})$ on $[\alpha, \infty)_{\mathbb{T}}$. Then by Theorem 4.7 we know that the conjoined basis (X, U) is also an antiprincipal solution of (S), having also the desired rank r. □

Remark 5.4. On special time scales $\mathbb{T}$, which consist of disjoint closed intervals and/or isolated points, see [22, Section 7] and [28, Section 6], the converse statement in Theorem 5.3 also holds. That is, on such special time scales the existence of an antiprincipal solution at infinity implies the nonoscillation of system (S).

6 Applications of antiprincipal solutions at infinity

In this section we derive further properties of principal and antiprincipal solutions of system ($\mathbb{S}$) at infinity. First we recall the definition and basic properties of principal solutions of ($\mathbb{S}$) at infinity, which are a natural counterpart to antiprincipal solutions at infinity, when comparing the rank of their associated T-matrices.

According to [22, Definition 6.1], a conjoined basis $(\hat{X},\hat{U})$ of ($\mathbb{S}$) is a *principal solution at infinity*, if there exists $\alpha \in [a,\infty)_{\mathbb{T}}$ such that $(\hat{X},\hat{U})$ has constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and no focal points in (α,∞) and its associated matrix $\hat{T}$ defined in (3.6) through $\hat{X}(t)$ satisfies $\hat{T}=0$. If $\operatorname{rank}\hat{X}(t)=n-d_\infty$ or $\operatorname{rank}\hat{X}(t)=n$ on $[\alpha,\infty)_{\mathbb{T}}$, then $(\hat{X},\hat{U})$ is called a *minimal principal solution at infinity* or a *maximal principal solution at infinity*, respectively. According to [22, Theorem 6.6], if system ($\mathbb{S}$) is nonoscillatory, then the minimal principal solution exists and is unique up to a constant right invertible multiple. Complying with the previous notation, we will denote this (unique) minimal principal solution of ($\mathbb{S}$) at infinity by $(\hat{X}_{\min},\hat{U}_{\min})$. The result of [22, Theorem 6.9] then shows that the minimality property of the rank of $\hat{X}_{\min}(t)$ on $[\alpha,\infty)_{\mathbb{T}}$ and the uniqueness property of $(\hat{X}_{\min},\hat{U}_{\min})$ are in fact equivalent conditions.

The following result shows a construction of the minimal principal solution of ($\mathbb{S}$) at infinity from an arbitrary minimal conjoined basis of ($\mathbb{S}$). This construction is used in the proof of [22, Theorem 6.6] in order to establish the uniqueness of the minimal principal solution at infinity. For our future reference we present it as a separate statement.

Theorem 6.1. *Assume that system ($\mathbb{S}$) is nonoscillatory. Suppose that $\alpha \in [a,\infty)_{\mathbb{T}}$ is such that $d[\alpha,\infty)=d_\infty$ and there exists a conjoined basis of ($\mathbb{S}$) with constant kernel on $[\alpha,\infty)_{\mathbb{T}}$ and no focal points in (α,∞). Then a solution $(\hat{X},\hat{U})$ of ($\mathbb{S}$) is a minimal principal solution at infinity if and only if*

$$\begin{pmatrix}\hat{X}(t)\\ \hat{U}(t)\end{pmatrix}=\begin{pmatrix}X(t)\\ U(t)\end{pmatrix}-\begin{pmatrix}\bar{X}(t)\\ \bar{U}(t)\end{pmatrix}T, \quad t\in[\alpha,\infty)_{\mathbb{T}}, \tag{6.1}$$

for some minimal conjoined basis (X,U) of ($\mathbb{S}$) on $[\alpha,\infty)_{\mathbb{T}}$, where $(\bar{X},\bar{U})$ is the conjoined basis of ($\mathbb{S}$) from Proposition 3.14 associated with (X,U) and T is the matrix defined in (3.6).

Proof. Let $\alpha \in [a,\infty)_{\mathbb{T}}$ be as in the statement. If $(\hat{X},\hat{U})$ is a minimal principal solution of ($\mathbb{S}$) at infinity with respect to the interval $[\alpha,\infty)_{\mathbb{T}}$, then the corresponding matrix $\hat{T}$ in (3.6) satisfies $\hat{T}=0$ and $(\hat{X},\hat{U})$ is a minimal conjoined basis on $[\alpha,\infty)_{\mathbb{T}}$. Formula (6.1) then holds with $(X,U):=(\hat{X},\hat{U})$. Conversely, if (X,U) is a minimal conjoined basis of ($\mathbb{S}$) on $[\alpha,\infty)_{\mathbb{T}}$ and define the solution $(\hat{X},\hat{U})$ of ($\mathbb{S}$) by (6.1). Then in the proof of [22, Theorem 6.6] it is shown that $(\hat{X},\hat{U})$ is a minimal conjoined basis on $[\alpha,\infty)_{\mathbb{T}}$. Moreover, by (3.37) in Proposition 3.18(ii) (with $(X_1,U_1):=(X,U)$, $(X_2,U_2):=(\hat{X},\hat{U})$, $L_1:=X^\dagger(\alpha)\,\hat{X}(\alpha)=P$, and $N_1:=-T$) its associated matrix $\hat{S}(t)$ in (3.1) satisfies

$$\hat{S}^\dagger(t)=S^\dagger(t)-T, \quad t\in[\alpha,\infty)_{\mathbb{T}}. \tag{6.2}$$

Taking the limit for $t\to\infty$ in (6.2) and using that $S^\dagger(t)\to T$ for $t\to\infty$ we obtain that $\hat{S}^\dagger(t)\to\hat{T}=0$ for $t\to\infty$, i.e., $(\hat{X},\hat{U})$ is a minimal principal solution of ($\mathbb{S}$) at infinity with respect to the interval $[\alpha,\infty)_{\mathbb{T}}$. □

Remark 6.2. In [22, Theorem 6.7] it is shown that the minimal principal solution $(\hat{X}_{\min},\hat{U}_{\min})$ of ($\mathbb{S}$) at infinity can be determined from an arbitrary minimal conjoined basis (X,U) of ($\mathbb{S}$) on the interval $[\alpha,\infty)_{\mathbb{T}}$ by the initial conditions

$$\hat{X}_{\min}(\alpha)=X(\alpha), \quad \hat{U}_{\min}(\alpha)=U(\alpha)-[X^\dagger(\alpha)]^T\,T,$$

In the following considerations we will use an estimate for the maximal interval, on which the minimal principal solution $(\hat{X}_{\min}, \hat{U}_{\min})$ at infinity has constant kernel and no focal points. Thus, we define the point $\hat{\alpha}_{\min} \in [a, \infty)_{\mathbb{T}}$ as

$$\hat{\alpha}_{\min} := \left\{ \begin{array}{c} \inf \alpha \in [a, \infty)_{\mathbb{T}},\ (\hat{X}_{\min}, \hat{U}_{\min}) \text{ has constant kernel on } [\alpha, \infty)_{\mathbb{T}} \\ \text{and no focal points in } (\alpha, \infty) \end{array} \right\}. \tag{6.3}$$

Moreover, by estimate (3.23) and by $\operatorname{rank} \hat{X}_{\min}(t) = n - d_\infty$ on $[\alpha, \infty)_{\mathbb{T}}$ we obtain

$$d[\hat{\alpha}_{\min}, \infty)_{\mathbb{T}} = d_\infty = d[\alpha, \infty)_{\mathbb{T}} \quad \text{for every } \alpha \in [\hat{\alpha}_{\min}, \infty)_{\mathbb{T}}. \tag{6.4}$$

In the next theorem we use minimal antiprincipal solutions of (S) at infinity for a characterization of all antiprincipal solutions of (S) at infinity through the relation being contained. It is a unification of the continuous case in [20, Theorem 5.11] and the discrete case in [24, Theorem 4.11(ii)], see also [10, Theorem 6.131(ii)].

Theorem 6.3. *Assume that system* (S) *is nonoscillatory, let $\hat{\alpha}_{\min} \in [a, \infty)_{\mathbb{T}}$ be defined in* (6.3)*. Then a solution (X, U) of* (S) *is an antiprincipal solution at infinity if and only if (X, U) is a conjoined basis of* (S)*, which contains some minimal antiprincipal solution of* (S) *at infinity on $[\alpha, \infty)_{\mathbb{T}}$ for some $\alpha \in [\hat{\alpha}_{\min}, \infty)_{\mathbb{T}}$.*

Proof. Let $(\hat{X}_{\min}, \hat{U}_{\min})$ be the minimal principal solution of (S) at infinity. Then condition (6.4) holds. Let (X, U) be an antiprincipal solution of (S) at infinity with respect to the interval $[\alpha, \infty)_{\mathbb{T}}$. Due to Theorem 4.3 we may assume that $\alpha \in [\hat{\alpha}_{\min}, \infty)_{\mathbb{T}}$. By Proposition 3.13, there exists a conjoined basis $(X_{\min}, U_{\min})$ of (S) with constant kernel on $[\alpha, \infty)_{\mathbb{T}}$ and no focal points in (α, ∞) and, moreover, with $\operatorname{rank} X_{\min} = n - d_\infty$ on $[\alpha, \infty)_{\mathbb{T}}$ such that $(X_{\min}, U_{\min})$ is contained in (X, U). From Theorem 4.7 it then follows that $(X_{\min}, U_{\min})$ is also an antiprincipal solution of (S) at infinity. Conversely, let (X, U) be a conjoined basis of (S), which contains some minimal antiprincipal solution $(X_{\min}, U_{\min})$ of (S) at infinity on $[\alpha, \infty)_{\mathbb{T}}$ for some $\alpha \in [\hat{\alpha}_{\min}, \infty)_{\mathbb{T}}$. Then by Definition 4.1 (applied to $(X_{\min}, U_{\min})$) we know that $d[\alpha, \infty)_{\mathbb{T}} = d_\infty$. Therefore, by Theorem 4.6 we conclude that (X, U) is also an antiprincipal solution of (S) at infinity with respect to the interval $[\alpha, \infty)_{\mathbb{T}}$. □

The following result shows that principal solutions at finite points α for sufficiently large $\alpha \in [a, \infty)_{\mathbb{T}}$ are examples of minimal antiprincipal solutions of (S) at infinity. It is a unification of the continuous case in [18, Proposition 5.15] and the discrete case in [24, Theorem 5.10], see also [10, Theorem 6.143]. We recall from Lemma 3.7 that the principal solution of (S) at the point $\alpha \in [a, \infty)_{\mathbb{T}}$, denoted by $(\hat{X}^{[\alpha]}, \hat{U}^{[\alpha]})$, is the solution of (S) with the initial conditions $\hat{X}^{[\alpha]}(\alpha) = 0$ and $\hat{U}^{[\alpha]}(\alpha) = I$.

Theorem 6.4. *Assume that system* (S) *is nonoscillatory. Let the point $\hat{\alpha}_{\min}$ be defined in* (6.3)*. Then for every $\alpha \in [\hat{\alpha}_{\min}, \infty)_{\mathbb{T}}$ the principal solution $(\hat{X}^{[\alpha]}, \hat{U}^{[\alpha]})$ is a minimal antiprincipal solution of* (S) *at infinity.*

Proof. From [22, Theorem 6.6] we know that when system (S) is nonoscillatory, then there exists the minimal principal solution $(\hat{X}_{\min}, \hat{U}_{\min})$ of (S) at infinity, which we denote for simplicity by $(X, U) := (\hat{X}_{\min}, \hat{U}_{\min})$ in this proof. Consider its associated matrices P, $P_{S\infty}$, $S(t)$, and T defined in (2.6), (3.15), (3.1), and (3.6). Choose a point $\beta \in [\hat{\alpha}_{\min}, \infty)_{\mathbb{T}}$ such that $\operatorname{Im} S(t)$ is constant on $[\beta, \infty)_{\mathbb{T}}$. Then from (3.20) and (3.19) in Lemma 3.7 we get

$$\operatorname{rank} X^{[\alpha]}(t) = \operatorname{rank} S(t) = \operatorname{rank} P_{S\infty} \overset{(3.33)}{=} \operatorname{rank} P = \operatorname{rank} X(t) = n - d_\infty, \quad t \in [\beta, \infty)_{\mathbb{T}},$$

so that $(\hat{X}^{[\alpha]}, \hat{U}^{[\alpha]})$ is a minimal conjoined basis on $[\beta,\infty)_{\mathbb{T}}$, and

$$\hat{X}^{[\alpha]}(t) = X(t)\, S(t)\, X^T(\alpha) \quad t \in [\beta,\infty)_{\mathbb{T}}.$$

By checking the four properties in (2.1) of the Moore–Penrose pseudoinverse it follows that

$$[\hat{X}^{[\alpha]}(t)]^{\dagger} = [X^{\dagger}(\alpha)]^T S^{\dagger}(t)\, X^{\dagger}(t), \quad t \in [\beta,\infty)_{\mathbb{T}}. \tag{6.5}$$

Since the image of $S(t)$ is constant on $[\beta,\infty)_{\mathbb{T}}$, hence the kernel of $S(t)$ is constant on $[\beta,\infty)_{\mathbb{T}}$, it follows by formula (2.3) in Remark 2.1(viii) that

$$[S^{\dagger}(t)]^{\Delta} S(t) = -[S^{\sigma}(t)]^{\dagger} S^{\Delta}(t), \quad t \in [\beta,\infty)_{\mathbb{T}}. \tag{6.6}$$

Multiplying this equation by the matrix $S^{\dagger}(t)$ from the right and using the definition of the constant orthogonal projector $P_{S\infty}$ in (3.15) we obtain

$$\begin{aligned} [S^{\dagger}(t)]^{\Delta} &= [S^{\dagger}(t)\, P_{S\infty}]^{\Delta} = [S^{\dagger}(t)]^{\Delta}\, P_{S\infty} \overset{(3.15)}{=} [S^{\dagger}(t)]^{\Delta} S(t)\, S^{\dagger}(t) \\ &\overset{(6.6)}{=} -[S^{\sigma}(t)]^{\dagger}\, S^{\Delta}(t)\, S^{\dagger}(t), \quad t \in [\beta,\infty)_{\mathbb{T}}. \end{aligned} \tag{6.7}$$

Consider now the matrix $\hat{S}^{[\alpha]}_{\beta}(t)$ in (3.1) associated with the principal solution $(\hat{X}^{[\alpha]}, \hat{U}^{[\alpha]})$ on the interval $[\beta,\infty)_{\mathbb{T}}$, namely,

$$\hat{S}^{[\alpha]}_{\beta}(t) := \int_{\beta}^{t} [\hat{X}^{[\alpha]}(s)]^{\sigma\,\dagger} \mathcal{B}(s)\, [\hat{X}^{[\alpha]}(s)]^{\dagger\, T}\, \Delta s, \quad t \in [\beta,\infty)_{\mathbb{T}}. \tag{6.8}$$

Then by using (6.5), (6.7), and (6.8) we obtain for $t \in [\beta,\infty)_{\mathbb{T}}$

$$\begin{aligned} \hat{S}^{[\alpha]}_{\beta}(t) &\overset{(6.5)}{=} \int_{\beta}^{t} [X^{\dagger}(\alpha)]^T\, [S^{\dagger}(s)]^{\sigma}\, [X^{\dagger}(s)]^{\sigma}\, \mathcal{B}(s)\, [X^{\dagger}(s)]^T S^{\dagger}(s)\, X^{\dagger}(\alpha)\, \Delta s, \\ &= [X^{\dagger}(\alpha)]^T \left(\int_{\beta}^{t} [S^{\dagger}(s)]^{\sigma}\, S^{\Delta}(s)\, S^{\dagger}(s)\, \Delta s \right) X^{\dagger}(\alpha), \\ &\overset{(6.7)}{=} -[X^{\dagger}(\alpha)]^T \left(\int_{\beta}^{t} [S^{\dagger}(s)]^{\Delta}\, \Delta s \right) X^{\dagger}(\alpha) = [X^{\dagger}(\alpha)]^T\, [S^{\dagger}(\beta) - S^{\dagger}(t)]\, X^{\dagger}(\alpha). \end{aligned} \tag{6.9}$$

Now using the fact that (X,U) is the principal solution of (S) at infinity (i.e., $T = 0$), we get from (6.9) that the limit of $\hat{S}^{[\alpha]}_{\beta}(t)$ as $t \to \infty$ exists and

$$\lim_{t\to\infty} \hat{S}^{[\alpha]}_{\beta}(t) = [X^{\dagger}(\alpha)]^T\, [S^{\dagger}(\beta) - T]\, X^{\dagger}(\alpha) = [X^{\dagger}(\alpha)]^T S^{\dagger}(\beta)\, X^{\dagger}(\alpha).$$

This implies through Theorem 4.4(ii) that $(\hat{X}^{[\alpha]}, \hat{U}^{[\alpha]})$ is an antiprincipal solution of (S) at infinity. Since we have already proved that $(\hat{X}^{[\alpha]}, \hat{U}^{[\alpha]})$ is a minimal conjoined basis on $[\beta,\infty)_{\mathbb{T}}$, it follows that $(\hat{X}^{[\alpha]}, \hat{U}^{[\alpha]})$ is a minimal antiprincipal solution of (S) at infinity. □

In the following result we present another example of antiprincipal solutions of (S) at infinity. It is a unification of the continuous case in [23, Proposition 1] and the discrete case in [24, Proposition 7.5], see also [10, Proposition 6.155].

Theorem 6.5. *Assume that system* (S) *is nonoscillatory and let* (X,U) *be a minimal conjoined basis of* (S) *on an interval* $[\alpha,\infty)_{\mathbb{T}}$ *satisfying* $d[\alpha,\infty)_{\mathbb{T}} = d_{\infty}$. *Then the associated conjoined basis* $(\bar{X}, \bar{U})$ *from Proposition 3.14 is a maximal antiprincipal solution of* (S) *at infinity.*

Proof. Let the conjoined bases (X, U) and $(\bar{X}, \bar{U})$ be as in the assumptions of the theorem. Let P, $S(t)$, $P_{S\infty}$ be the matrices in (2.6), (3.1), (3.15) corresponding to (X, U). Since (X, U) is a minimal conjoined basis of ($\mathbb{S}$) on $[\alpha, \infty)_{\mathbb{T}}$, we have $P = P_{S\infty}$ by Lemma 3.17. Moreover, $d[\alpha, \infty)_{\mathbb{T}} = d_\infty$ it follows that $\operatorname{Im} S(t) \equiv \operatorname{Im} P_{S\infty}$ on $(\alpha, \infty)_{\mathbb{T}}$. Therefore, by Proposition 3.14(v) we then derive for all $t \in (\alpha, \infty)_{\mathbb{T}}$ that

$$\operatorname{Ker} \bar{X}(t) = \operatorname{Im} P \cap \operatorname{Ker} S(t) = \operatorname{Im} P_{S\infty} \cap [\operatorname{Im} S(t)]^\perp \equiv \operatorname{Im} P_{S\infty} \cap (\operatorname{Im} P_{S\infty})^\perp = \{0\}.$$

This shows that the matrix $\bar{X}(t)$ is invertible on $(\alpha, \infty)_{\mathbb{T}}$, in particular its kernel is constant on $(\alpha, \infty)_{\mathbb{T}}$. Fix any $\beta \in (\alpha, \infty)_{\mathbb{T}}$. We will show that $(\bar{X}, \bar{U})$ has no focal points in the interval (β, ∞). Recall that the matrix $S^\dagger(t)$ is nonincreasing on $[\beta, \infty)_{\mathbb{T}}$ and that, by Remark 2.1(viii) or by (6.7),

$$-[S^\sigma(t)]^\dagger S^\Delta(t)\, S^\dagger(t) = [S^\dagger(t)]^\Delta \le 0, \quad t \in [\beta, \infty)_{\mathbb{T}}. \tag{6.10}$$

Moreover, by Proposition 3.14(viii) we obtain for $t \in [\beta, \infty)_{\mathbb{T}}$ the equality

$$S^\dagger(t)\, X^\dagger(t) = \bar{X}^{-1}(t)\, X(t)\, P_{S\infty}\, X^\dagger(t) = \bar{X}^{-1}(t)\, X(t)\, P X^\dagger(t) = \bar{X}^{-1}(t)\, R(t). \tag{6.11}$$

Then by Proposition 3.6(ii) and by (6.11) with (6.10) we deduce that

$$\begin{aligned} [\bar{X}^\sigma(t)]^{-1} \mathcal{B}(t)\, \bar{X}^{T-1}(t) &= [\bar{X}^\sigma(t)]^{-1} R^\sigma(t)\, \mathcal{B}(t)\, R(t)\, \bar{X}^{T-1}(t) \\ &\overset{(6.11)}{=} [S^\dagger(t)]^\sigma\, [X^\dagger(t)]^\sigma\, \mathcal{B}(t)\, [X^\dagger(t)]^T S^\dagger(t) = [S^\dagger(t)]^\sigma S^\Delta(t)\, S^\dagger(t) \\ &\overset{(6.10)}{=} -[S^\dagger(t)]^\Delta \ge 0, \quad t \in [\beta, \infty)_{\mathbb{T}}, \end{aligned} \tag{6.12}$$

and consequently

$$\bar{X}(t)\, [\bar{X}^\sigma(t)]^{-1} \mathcal{B}(t) = \bar{X}(t)\, [\bar{X}^\sigma(t)]^{-1} \mathcal{B}(t)\, \bar{X}^{T-1}(t)\, \bar{X}^T(t) \overset{(6.12)}{\ge} 0, \quad t \in [\beta, \infty)_{\mathbb{T}}.$$

This proves that $(\bar{X}, \bar{U})$ has no focal points in the interval (β, ∞) and hence, it is a maximal conjoined basis on $[\beta, \infty)_{\mathbb{T}}$. It remains to show that $(\bar{X}, \bar{U})$ is an antiprincipal solution of ($\mathbb{S}$) at infinity. According to (3.1), we define the associated matrix $\bar{S}(t)$ by

$$\bar{S}(t) := \int_\beta^t [\bar{X}^\sigma(s)]^{-1} \mathcal{B}(s)\, \bar{X}^{T-1}(s)\, \Delta s, \quad t \in [\beta, \infty)_{\mathbb{T}}. \tag{6.13}$$

Then by using (6.12) in (6.13) we get

$$\bar{S}(t) \overset{(6.12)}{=} -\int_\beta^t [S^\dagger(t)]^\Delta\, \Delta s = S^\dagger(\beta) - S^\dagger(t), \quad t \in [\beta, \infty)_{\mathbb{T}}. \tag{6.14}$$

This implies that the limit

$$\lim_{t\to\infty} \bar{S}(t) \overset{(6.14)}{=} \lim_{t\to\infty} [S^\dagger(\beta) - S^\dagger(t)] = S^\dagger(\beta) - T$$

exists. By Theorem 4.4(ii) (applied to $(X, U) := (\bar{X}, \bar{U})$) it then follows that the conjoined basis $(\bar{X}, \bar{U})$ is an antiprincipal solution of ($\mathbb{S}$) at infinity. The proof is complete. □

In our next result we utilize antiprincipal solutions of ($\mathbb{S}$) at infinity in the Reid construction of the minimal principal solution $(\hat{X}_{\min}, \hat{U}_{\min})$ of ($\mathbb{S}$) at infinity. It is a unification of the continuous case in [23, Theorem 1] and the discrete case in [24, Theorem 7.3], see also [10, Theorem 6.153].

Theorem 6.6. *Assume that system* ($\mathbb{S}$) *is nonoscillatory. Let* (X, U) *be a minimal conjoined basis of* ($\mathbb{S}$) *on an interval* $[\alpha, \infty)_{\mathbb{T}}$ *satisfying* $d[\alpha, \infty)_{\mathbb{T}} = d_\infty$ *and let* $\beta \in [\alpha, \infty)_{\mathbb{T}}$ *be such that the associated conjoined basis* $(\bar{X}, \bar{U})$ *from Proposition 3.14 is a maximal antiprincipal solution of* ($\mathbb{S}$) *at infinity with respect to the interval* $[\beta, \infty)_{\mathbb{T}}$. *Then for all* $\tau \in [\beta, \infty)_{\mathbb{T}}$ *the solutions* (X_τ, U_τ) *of* ($\mathbb{S}$) *given by the initial conditions*

$$X_\tau(\tau) = 0 \quad \text{and} \quad U_\tau(\tau) = -[\bar{X}^{-1}(\tau)]^T \tag{6.15}$$

are conjoined bases of ($\mathbb{S}$) *satisfying*

$$\big(\hat{X}_{\min}(t), \hat{U}_{\min}(t)\big) = \lim_{\tau\to\infty} \big(X_\tau(t), U_\tau(t)\big), \quad t \in [a, \infty)_{\mathbb{T}}. \tag{6.16}$$

Proof. Let P, $P_{S\infty}$, $S(t)$, and T be the matrices in (2.6), (3.15), (3.1), and (3.6) associated with (X, U). Then $P = P_{S\infty}$ by Lemma 3.17 and by Proposition 3.14(viii) we get

$$S^\dagger(t) = \bar{X}^{-1}(t)\, X(t)\, P_{S\infty} = \bar{X}^{-1}(t)\, X(t)\, P = \bar{X}^{-1}(t)\, X(t), \quad t \in [\beta, \infty)_{\mathbb{T}}. \tag{6.17}$$

Fixed a point $\tau \in [\beta, \infty)_{\mathbb{T}}$. From (6.16) it follows that the solution (X_τ, U_τ) is a conjoined basis of ($\mathbb{S}$). Let us represent (X_τ, U_τ) in terms of (X, U) by using Proposition 3.15, i.e.,

$$\begin{pmatrix} X_\tau(t) \\ U_\tau(t) \end{pmatrix} = Z(t) \begin{pmatrix} M_\tau \\ N_\tau \end{pmatrix}, \quad Z(t) := \begin{pmatrix} X(t) & \bar{X}(t) \\ U(t) & \bar{U}(t) \end{pmatrix}, \quad t \in [a, \infty)_{\mathbb{T}}, \tag{6.18}$$

where the matrix $Z(t)$ is symplectic, i.e., $Z^{-1}(t) = -\mathcal{J} Z^T(t)\, \mathcal{J}$. Then the matrix $-M_\tau$ is the Wronskian of $(\bar{X}, \bar{U})$ and (X_τ, U_τ), and the matrix N_τ is the Wronskian of (X, U) and (X_τ, U_τ). Evaluating these Wronskians at the point τ we obtain

$$M_\tau = -[\bar{X}^T(\tau)\, U_\tau(\tau) - \bar{U}^T(\tau)\, X_\tau(\tau)] \overset{(6.15)}{=} I,$$
$$N_\tau = X^T(\tau)\, U_\tau(\tau) - U^T(\tau)\, X_\tau(\tau) \overset{(6.15)}{=} -X^T(\tau)\, [\bar{X}^{-1}(\tau)]^T \overset{(6.17)}{=} -[S^\dagger(\tau)]^T = -S^\dagger(\tau).$$

This shows that the limits of M_τ and N_τ for $\tau \to \infty$ exist and

$$\lim_{\tau\to\infty} M_\tau = I, \quad \lim_{\tau\to\infty} N_\tau = -\lim_{\tau\to\infty} S^\dagger(\tau) = -T. \tag{6.19}$$

Therefore, the limit of (X_τ, U_τ) for $\tau \to \infty$ also exists and by (6.18) it is equal to the solution

$$\begin{pmatrix} \hat{X}(t) \\ \hat{U}(t) \end{pmatrix} := \lim_{\tau\to\infty} \begin{pmatrix} X_\tau(t) \\ U_\tau(t) \end{pmatrix} \overset{(6.18)}{=} \lim_{\tau\to\infty} Z(t) \begin{pmatrix} M_\tau \\ N_\tau \end{pmatrix} \overset{(6.19)}{=} Z(t) \begin{pmatrix} I \\ -T \end{pmatrix}, \quad t \in [a, \infty)_{\mathbb{T}}. \tag{6.20}$$

In fact, since $\operatorname{rank}(I, -T)^T = n$ and the matrix T is symmetric, the solution $(\hat{X}, \hat{U})$ defined in (6.20) is a conjoined basis of ($\mathbb{S}$). By Theorem 6.1 we then conclude that $(\hat{X}, \hat{U})$ is the minimal principal solution of ($\mathbb{S}$) at infinity, i.e., $(\hat{X}, \hat{U}) = (\hat{X}_{\min}, \hat{U}_{\min})$. □

The following three comments complement the construction of $(\hat{X}_{\min}, \hat{U}_{\min})$ in Theorem 6.6.

Remark 6.7. The initial conditions in (6.15) show that the conjoined basis (X_τ, U_τ) is a constant nonsingular multiple of the principal solution $(\hat{X}^{[\tau]}, \hat{U}^{[\tau]})$ of ($\mathbb{S}$) at the point τ, namely

$$(X_\tau, U_\tau) = \big(\hat{X}^{[\tau]} M, \hat{U}^{[\tau]} M\big), \quad M := -\bar{X}^T(\tau).$$

Then, in view of Theorem 6.4 and Proposition 4.5, we may conclude that for all $\tau \in [a, \infty)_{\mathbb{T}}$ with $\tau \geq \max\{\hat{\alpha}_{\min}, \beta\}$ the conjoined bases (X_τ, U_τ) in Theorem 6.6 are minimal antiprincipal solutions of ($\mathbb{S}$) at infinity.

Remark 6.8. The limit formula for $(\hat{X}, \hat{U})$ in (6.20) shows, how this construction depend on the chosen initial conditions of (X_τ, U_τ) in (6.15). More precisely, let us consider instead of (6.15) the initial conditions $X_\tau(\tau) = 0$ and $U_\tau(\tau) = K_\tau$, where K_τ are invertible matrices for all $\tau \in [\beta, \infty)_{\mathbb{T}}$. Then the matrices M_τ and N_τ from the representation in (6.18) satisfy

$$M_\tau = -\bar{X}^T(\tau)\, K_\tau, \quad N_\tau = X^T(\tau)\, K_\tau \overset{(6.17)}{=} S^\dagger(\tau)\, \bar{X}^T(\tau)\, K_\tau,$$

where we used the fact that the matrix $\bar{X}(\tau)$ is invertible. This shows that for $t \in [a, \infty)_{\mathbb{T}}$ the limit of $(X_\tau(t), U_\tau(t))$ as $\tau \to \infty$ exists if and only the limit

$$M_\infty := \lim_{\tau \to \infty} \bar{X}^T(\tau)\, K_\tau$$

exists, and in this case the limiting solution $(\hat{X}, \hat{U})$ in (6.20) is equal to $(\hat{X}_{\min} M_\infty, \hat{U}_{\min} M_\infty)$.

Remark 6.9. The construction in Theorem 6.6 does not depend on the chosen minimal conjoined basis (X, U) on $[\alpha, \infty)_{\mathbb{T}}$. More precisely, suppose that we start with another minimal conjoined basis (X_*, U_*) of (S) on $[\alpha, \infty)_{\mathbb{T}}$ and denote by $(\bar{X}_*, \bar{U}_*)$ its associated conjoined basis from Proposition 3.14. Let us we represent (X_*, U_*) in terms of (X, U) and $(\bar{X}, \bar{U})$ as

$$\begin{pmatrix} X_*(t) \\ U_*(t) \end{pmatrix} = Z(t) \begin{pmatrix} M \\ N \end{pmatrix}, \quad t \in [a, \infty)_{\mathbb{T}},$$

where the fundamental matrix $Z(t)$ is given in (6.18) and the matrix M is invertible (see Proposition 3.15 with $(X_2, U_2) := (X_*, U_*)$, $(X_1, U_1) := (X, U)$, and $M_1 := M$). Then by using Lemma 3.19 we have $\bar{X}_*(t) = \bar{X}(t)\, M^{T-1}$ on $[\alpha, \infty)_{\mathbb{T}}$. Similarly to (6.15) we now consider for $\tau \in [\beta, \infty)_{\mathbb{T}}$ the conjoined bases $(X_{*\tau}, U_{*\tau})$ given by the initial conditions

$$X_{*\tau}(\tau) = 0 \quad \text{and} \quad U_{*\tau}(\tau) = -[\bar{X}_*^{-1}(\tau)]^T = -[\bar{X}^{-1}(\tau)]^T M.$$

Then $(X_{*\tau}, U_{*\tau}) = (X_\tau M, U_\tau M)$ on $[a, \infty)_{\mathbb{T}}$ and we derive that

$$\lim_{\tau \to \infty} \big(X_{*\tau}(t), U_{*\tau}(t)\big) = \lim_{\tau \to \infty} \big(X_\tau(t)\, M, U_\tau(t)\, M\big) = \big(\hat{X}_{\min}(t)\, M, \hat{U}_{\min}(t)\, M\big), \quad t \in [a, \infty)_{\mathbb{T}},$$

i.e., this modified construction leads to a constant nonsingular multiple of the minimal principal solution $(\hat{X}_{\min}, \hat{U}_{\min})$ of (S) at infinity given in (6.16).

7 Concluding remarks

In this paper we developed the theory of antiprincipal solutions at infinity for nonoscillatory symplectic dynamic systems on time scales. The motivation for this study comes from the theory of principal solutions at infinity for these systems, and from the corresponding theory of antiprincipal or dominant solutions at infinity, which exists in the continuous or discrete time setting. Our main results include in particular a characterization of antiprincipal solutions of (S) at infinity in terms of the limit of its associated S-matrix (Theorem 4.4), a characterization of minimal conjoined bases of (S) on a given interval $[\alpha, \infty)_{\mathbb{T}}$ in terms of the initial conditions at α (Theorem 5.1), the existence of antiprincipal solutions of (S) at infinity (Theorem 5.3), and several additional properties or applications of antiprincipal solutions of (S) at infinity (presented in Theorems 6.4, 6.5, and 6.6).

Note that, unlike in the continuous or discrete cases, the existence of a nonoscillatory conjoined basis of (S) does not (so far) imply the nonoscillation of system (S) on arbitrary time scale $\mathbb{T}$, see Remark 5.4. The reason is a nonexisting pointwise definition of the multiplicity of a focal point for general time scales. We believe that this problem might be solved by using the comparative index theory, see [12] or [10, Chapter 3], in combination with the theory of principal and antiprincipal solutions of symplectic systems on time scales. This is a work in progress.

Acknowledgements

This research was supported by the Czech Science Foundation under grant GA19-01246S. The authors wish to thank Peter Šepitka for several consultations regarding the topics of this paper, including the properties of minimal conjoined bases in Theorem 5.1 and its proof.

References

[1] C. D. Ahlbrandt, M. Bohner, J. Ridenhour, Hamiltonian systems on time scales, *J. Math. Anal. Appl.* **250**(2000), No. 2, 561–578. https://doi.org/10.1006/jmaa.2000.6992; MR1786081; Zbl 0966.39010

[2] C. D. Ahlbrandt, A. C. Peterson, *Discrete Hamiltonian systems. Difference equations, continued fractions, and Riccati equations*, Kluwer Texts in the Mathematical Sciences, Vol. 16, Kluwer Academic Publishers Group, Dordrecht, 1996. https://doi.org/10.1007/978-1-4757-2467-7; MR1423802; Zbl 0860.39001

[3] A. Ben-Israel, T. N. E. Greville, *Generalized inverses: theory and applications*, Second Edition, Springer-Verlag, New York, NY, 2003. https://doi.org/10.1007/b97366; MR1987382; Zbl 1026.15004

[4] D. S. Bernstein, *Matrix mathematics. Theory, facts, and formulas with application to linear systems theory*, Princeton University Press, Princeton, 2005. MR2123424; Zbl 1075.15001

[5] M. Bohner, O. Došlý, R. Hilscher, Linear Hamiltonian dynamic systems on time scales: Sturmian property of the principal solution, *Nonlinear Anal.* **47**(2001), No. 2, 849–860. https://doi.org/10.1016/S0362-546X(01)00228-0; MR1970703; Zbl 1042.34509

[6] M. Bohner, A. Peterson, *Dynamic equation on time scales. An introduction with applications*, Birkhäuser, Boston, 2001. https://doi.org/10.1007/978-1-4612-0201-1; MR1843232; Zbl 0978.39001

[7] M. Bohner, A. Peterson, editors, *Advances in dynamic equations on time scales*, Birkhäuser, Boston, 2003. https://doi.org/10.1007/978-0-8176-8230-9; MR1962542; Zbl 1025.34001

[8] S. L. Campbell, C. D. Meyer, *Generalized inverses of linear transformations*, Reprint of the 1991 corrected reprint of the 1979 original, Classics in Applied Mathematics, Vol. 56, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2009. https://doi.org/10.1137/1.9780898719048; MR3396208; Zbl 1158.15301

[9] O. Došlý, Principal and nonprincipal solutions of symplectic dynamic systems on time scales, in: *Proceedings of the Sixth Colloquium on the Qualitative Theory of Differential Equations (Szeged, Hungary, 1999)*, No. 5, 14 pp., *Electron. J. Qual. Theory Differ. Equ.*, Szeged, 2000. https://doi.org/10.14232/ejqtde.1999.5.5; MR1798655; Zbl 0971.34018

[10] O. Došlý, J. V. Elyseeva, R. Šimon Hilscher, *Symplectic difference systems: oscillation and spectral theory*, Pathways in Mathematics, Birkhäuser/Springer, Cham, 2019. https://doi.org/10.1007/978-3-030-19373-7; MR3970258; Zbl 1425.39001

[11] O. Došlý, R. Hilscher, Disconjugacy, transformations and quadratic functionals for symplectic dynamic systems on time scales, *J. Differ. Equations Appl.* **7**(2001), 265–295. https://doi.org/10.1080/10236190108808273; MR1923624; Zbl 0989.34027

[12] J. V. Elyseeva, Comparative index for solutions of symplectic difference systems, *Differential Equations* **45**(2009), No. 3, 445–459; translated from *Differencial'nyje Uravnenija* **45**(2009), No. 3, 431–444. https://doi.org/10.1134/S0012266109030148; MR2596770; Zbl 1182.39007

[13] R. Hilscher, Reid roundabout theorem for symplectic dynamic systems on time scales, *Appl. Math. Optim.* **43**(2001), No. 2, 129–146. https://doi.org/10.1007/s00245-001-0002-1; MR1814591; Zbl 0990.39017

[14] R. Hilscher, V. Zeidan, Time scale symplectic systems without normality, *J. Differential Equations* **230**(2006), No. 1, 140–173. https://doi.org/10.1016/j.jde.2006.06.010; MR2270550; Zbl 1104.37037

[15] R. Hilscher, V. Zeidan, Applications of time scale symplectic systems without normality, *J. Math. Anal. Appl.* **340**(2008), No. 1, 451–465. https://doi.org/10.1016/j.jmaa.2007.07.077; MR2376167; Zbl 1145.39006

[16] R. Hilscher, V. Zeidan, Riccati equations for abnormal time scale quadratic functionals, *J. Differential Equations* **244**(2008), No. 6, 1410–1447. https://doi.org/10.1016/j.jde.2007.10.012; MR2396504; Zbl 1355.49031

[17] W. Kratz, R. Šimon Hilscher, V. Zeidan, Eigenvalue and oscillation theorems for time scale symplectic systems, *Int. J. Dyn. Syst. Differ. Equ.* **3**(2011), No. 1–2, 84–131. https://doi.org/10.1504/IJDSDE.2011.038497; MR2797044; Zbl 1218.34039

[18] P. Šepitka, R. Šimon Hilscher, Minimal principal solution at infinity for nonoscillatory linear Hamiltonian systems, *J. Dynam. Differential Equations* **26**(2014), No. 1, 57–91. https://doi.org/10.1007/s10884-013-9342-1; MR3175260; Zbl 1305.34053

[19] P. Šepitka, R. Šimon Hilscher, Principal solutions at infinity of given ranks for nonoscillatory linear Hamiltonian systems, *J. Dynam. Differential Equations* **27**(2015), No. 1, 137–175. https://doi.org/10.1007/s10884-014-9389-7; MR3317394; Zbl 1348.37091

[20] P. Šepitka, R. Šimon Hilscher, Principal and antiprincipal solutions at infinity of linear Hamiltonian systems, *J. Differential Equations* **259**(2015), No. 9, 4651–4682. https://doi.org/10.1016/j.jde.2015.06.027; MR3373417; Zbl 1329.34024

[21] P. Šepitka, R. Šimon Hilscher, Recessive solutions for nonoscillatory discrete symplectic systems, *Linear Algebra Appl.* **469**(2015), 243–275. https://doi.org/10.1016/j.laa.2014.11.029; MR3299064; Zbl 1307.39007

[22] P. Šepitka, R. Šimon Hilscher, Principal solutions at infinity for time scale symplectic systems without controllability condition, *J. Math. Anal. Appl.* **444**(2016), No. 2, 852–880. https://doi.org/10.1016/j.jmaa.2016.06.057; MR3535739; Zbl 1343.93057

[23] P. Šepitka, R. Šimon Hilscher, Reid's construction of minimal principal solution at infinity for linear Hamiltonian systems, in: *Differential and Difference Equations with Applications (Proceedings of the International Conference on Differential & Difference Equations and Applications, Amadora, 2015)*, S. Pinelas, Z. Došlá, O. Došlý, and P. E. Kloeden, editors, Springer Proceedings in Mathematics & Statistics, Vol. 164, pp. 359–369, Springer, Berlin, 2016. https://doi.org/10.1007/978-3-319-32857-7_34; MR3571742; Zbl 1375.37058

[24] P. Šepitka, R. Šimon Hilscher, Dominant and recessive solutions at infinity and genera of conjoined bases for discrete symplectic systems, *J. Difference Equ. Appl.* **23**(2017), No. 4, 657–698. https://doi.org/10.1080/10236198.2016.1270274; MR3678531; Zbl 1376.39004

[25] P. Šepitka, R. Šimon Hilscher, Singular Sturmian separation theorems for nonoscillatory symplectic difference systems, *J. Difference Equ. Appl.* **24**(2018), No. 12, 1894–1934. https://doi.org/10.1080/10236198.2018.1544247; MR3890839; Zbl 1405.39006

[26] P. Šepitka, R. Šimon Hilscher, Singular Sturmian separation theorems on unbounded intervals for linear Hamiltonian systems, *J. Differential Equations* **266**(2019), No. 11, 7481–7524. https://doi.org/10.1016/j.jde.2018.12.007; MR3926108; Zbl 1408.37107

[27] P. Šepitka, R. Šimon Hilscher, Singular Sturmian comparison theorems for linear Hamiltonian systems, *J. Differential Equations* **269**(2020), No. 4, 2920–2955. https://doi.org/10.1016/j.jde.2020.02.016; MR4097237; Zbl 07200715

[28] R. Šimon Hilscher, V. Zeidan, Rayleigh principle for time scale symplectic systems and applications, *Electron. J. Qual. Theory Differ. Equ.* **2011**, No. 83, 1–26. https://doi.org/10.14232/ejqtde.2011.1.83; MR2854022; Zbl 1340.34355

[29] R. Šimon Hilscher, V. Zeidan, Hamilton–Jacobi theory over time scales and applications to linear-quadratic problems, *Nonlinear Anal.* **75**(2012), No. 2, 932–950. https://doi.org/10.1016/j.na.2011.09.027; MR2847468; Zbl 1230.49023

[30] R. Šimon Hilscher, V. Zeidan, Sufficiency and sensitivity for nonlinear optimal control problems on time scales via coercivity, *ESAIM Control Optim. Calc. Var.* **24**(2018), No. 4, 1705–1734. https://doi.org/10.1051/cocv/2017070; MR3922438; Zbl 1415.49015