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Admissibility and general dichotomies for evolution families

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Abstract. For an arbitrary noninvertible evolution family on the half-line and for $\rho\colon [0,\infty) \to [0,\infty)$ in a large class of rate functions, we consider the notion of a ρ-dichotomy with respect to a family of norms and characterize it in terms of two admissibility conditions. In particular, our results are applicable to exponential as well as polynomial dichotomies with respect to a family of norms. As a nontrivial application of our work, we establish the robustness of general nonuniform dichotomies.

Keywords: admissibility, dichotomies with growth rates, robustness.

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1 Introduction

Among many methods used to study the asymptotic behavior of nonautonomous systems, one of the most famous is the so-called admissibility method. This line of research in the context of differential equations has a long history that goes back to the pioneering work of Perron [26]. The main idea of Perron's work was to characterize the asymptotic properties of the linear differential equation

$$\dot{x}(t) = A(t)x(t), \qquad t \in \mathbb{J},$$

in terms of the (unique) solvability in $O(\mathbb{J}, X)$ of the equation

$$\dot{x}(t) = A(t)x(t) + f(t), \qquad t \in \mathbb{J},$$

for each test function $f \in I(\mathbb{J}, X)$, where $\mathbb{J} \in \{[0,\infty), \mathbb{R}\}$. Here X is a Banach space, while $I(\mathbb{J}, X)$ – the input-space and $O(\mathbb{J}, X)$ – the output space are suitably constructed function spaces. The milestones of this theory were grounded in the sixtieth in the remarkable works of Massera and Schäffer [18–20] and respectively in the seventies in the outstanding monographs of Coppel [10] and Daleckĭi and Kreĭn [11].

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Since then various authors obtained valuable contributions to this line of the research. For the results dealing with characterizations of *uniform* exponential behavior in terms of appropriate admissibility properties, we refer to the works of Huy [15], Latushkin, Randolph and Schnaubelt [16], Van Minh, Räbiger and Schnaubelt [22], Van Minh and Huy [23], Preda, Pogan and Preda [28, 29] as well as Sasu and Sasu [31–35]. For contributions dealing with various concepts of nonuniform exponential behavior, we refer to [4, 5, 17, 21, 27, 30, 36] and references therein. For a detailed description of this line of the research, we refer to [6].

We point out that all the above works deal with *exponential* behavior. Although this type of behavior has a somewhat privileged role due to its connections with the hyperbolic smooth dynamics, it is certainly not the only type of behavior that appears in the qualitative study of nonautonomous differential equations. To the best of our knowledge, the study of dichotomies with not necessarily exponential rates of expansion and contraction was initiated by Muldowney [24] and Naulin and Pinto [25]. More recently, in the context of nonuniform asymptotic behavior such dichotomies have been studied by Barreira and Valls [1,3] and Bento and Silva [8,9]. A special emphasis was devoted to the so-called polynomial dichotomies [2,7]. A complete characterization of polynomial dichotomies in terms of admissibility for evolution families was obtained by Dragičević [12] (see also [13] for related results in the case of discrete time) by building on the work of Hai [14], who considered polynomial stability.

The main objective of the present paper is to obtain similar results to that in [12] but for a much wider class of dichotomies. More precisely, for a large class of rate functions $\rho\colon [0,\infty) \to [0,\infty)$, we introduce the notion of a ρ-dichotomy with respect to a family of norms. We then obtain a complete characterization of this concept in terms of appropriate admissibility conditions. We point out that our results are new even in the particular case of *uniform* ρ-dichotomies. Indeed, although the proofs use the somewhat standard techniques, the major task accomplished in the present paper was to formulate appropriate admissibility conditions for the general dichotomies we study. In addition, the obtained results are new even for the class of polynomial dichotomies since in comparison to [12], we do not require that our evolution family exhibits polynomial bounded growth property. Consequently, we need to impose two admissibility conditions (rather than just one as in [12]) to characterize polynomial dichotomies. We stress that in the present paper we also use different admissibility spaces from those in [12].

The paper is organized as follows. In Section 2 we introduce the class of dichotomies we study as well as input and output spaces we are going to use. In Section 3, we show that the existence of ρ-dichotomies yields two types of admissibility properties. Then, in Section 4 we obtain a converse result by showing that those admissibility properties imply the existence of a ρ-dichotomy. Finally, in Section 5 we apply those results to establish the robustness of ρ-dichotomies.

2 Preliminaries

2.1 Generalized dichotomies

Let $X = (X, \|\cdot\|)$ be an arbitrary Banach space and let $\mathcal{B}(X)$ be the Banach algebra of all bounded linear operators on X. A family $\mathcal{T} = \{T(t,s)\}_{t\geq s\geq 0}$ of operators in $\mathcal{B}(X)$ is said to be an *evolution family* on X if the following properties hold:

- $T(t,t) = \mathrm{Id}$, for $t \geq 0$;

- $T(t,s)T(s,\tau) = T(t,\tau)$, for $t \geq s \geq \tau \geq 0$;
- for all $s \geq 0$ and $x \in X$ the mapping $t \mapsto T(t,s)x$ is continuous on $[s,\infty)$ and the mapping $t \mapsto T(s,t)x$ is continuous on $[0,s]$.

In this paper we always assume that $\mathcal{T} = \{T(t,s)\}_{t\geq s\geq 0}$ is an evolution family on X and let $\rho\colon [0,\infty) \to [0,\infty)$ be a strictly increasing function of class C^1 such that

$$\rho(0) = 0 \quad \text{and} \quad \lim_{t\to\infty} \rho(t) = \infty.$$

In particular, observe that ρ is bijective. Furthermore, assume that $\{\|\cdot\|_t\}_{t\geq 0}$ is a family of norms on X such that:

- there exist $C > 0$ and $\varepsilon \geq 0$ with
$$\|x\| \leq \|x\|_t \leq Ce^{\varepsilon\rho(t)}\|x\|, \quad \text{for } x \in X \text{ and } t \geq 0; \tag{2.1}$$
- the mapping $t \mapsto \|x\|_t$ is continuous for each $x \in X$.

We say that the evolution family $\mathcal{T}$ admits a *ρ-dichotomy* with respect to the family of norms $\|\cdot\|_t$, $t \geq 0$, if there exists a family $\{P(t)\}_{t\geq 0}$ of projections on X satisfying

$$T(t,s)P(s) = P(t)T(t,s), \quad \text{for } t \geq s \geq 0, \tag{2.2}$$

such that

$$T(t,s)|_{\operatorname{Ker} P(s)}\colon \operatorname{Ker} P(s) \to \operatorname{Ker} P(t) \text{ is invertible for all } t \geq s \geq 0, \tag{2.3}$$

and there exist constants $\lambda, D > 0$ such that:

- for $x \in X$ and $t \geq s \geq 0$,
$$\|T(t,s)P(s)x\|_t \leq De^{-\lambda(\rho(t)-\rho(s))}\|x\|_s; \tag{2.4}$$
- for $x \in X$ and $0 \leq t \leq s$,
$$\|T(t,s)(\mathrm{Id} - P(s))x\|_t \leq De^{-\lambda(\rho(s)-\rho(t))}\|x\|_s, \tag{2.5}$$
where
$$T(t,s) := \left(T(s,t)|_{\operatorname{Ker} P(t)}\right)^{-1}\colon \operatorname{Ker} P(s) \to \operatorname{Ker} P(t),$$
for $0 \leq t \leq s$.

In the following we recall the concept of *ρ-nonuniform exponential dichotomy* for evolution families (see [1,3]) and establish its connection with the notion of ρ-dichotomy with respect to a family of norms. An evolution family $\mathcal{T}$ is said to admit a *ρ-nonuniform exponential dichotomy* if there exists a family $\{P(t)\}_{t\geq 0}$ of projections on X satisfying (2.2) and (2.3), and there exist constants $\lambda, D > 0$ and $\varepsilon \geq 0$ such that

$$\|T(t,s)P(s)\| \leq De^{-\lambda(\rho(t)-\rho(s))+\varepsilon\rho(s)}, \quad \text{for } t \geq s \geq 0, \tag{2.6}$$

and

$$\|T(t,s)(\mathrm{Id} - P(s))\| \leq De^{-\lambda(\rho(s)-\rho(t))+\varepsilon\rho(s)}, \quad \text{for } 0 \leq t \leq s. \tag{2.7}$$

The concept of ρ-nonuniform exponential dichotomy includes as a special case the usual *exponential behavior* when $\rho(t) = t$. Also, for $\rho(t) = \ln(t+1)$ we obtain the concept of *nonuniform polynomial dichotomy* introduced independently by Barreira and Valls [2] and Bento and Silva [7], and more general for $\rho(t) = \int_0^t \mu(t)dt$, where $\mu : [0,\infty) \to (0,\infty)$ is a continuous function such that $\lim_{t\to\infty} \int_0^t \mu(t)dt = \infty$, we obtain the nonuniform version of the *generalized dichotomy* in the sense of Muldowney [24].

Proposition 2.1. *The following statements are equivalent:*

1. *$\mathcal{T}$ admits a ρ-nonuniform exponential dichotomy;*
2. *$\mathcal{T}$ admits a ρ-dichotomy with respect to a family of norms $\|\cdot\|_t$, $t \geq 0$ such that $t \mapsto \|x\|_t$ is continuous for each $x \in X$.*

Proof. Assume that $\mathcal{T}$ admits a ρ-nonuniform exponential dichotomy. For $t \geq 0$ and $x \in X$, set

$$\|x\|_t = \sup_{\tau\geq t} e^{\lambda(\rho(\tau)-\rho(t))}\|T(\tau,t)P(t)x\| + \sup_{\tau\in[0,t]} e^{\lambda(\rho(t)-\rho(\tau))}\|T(\tau,t)(\mathrm{Id}-P(t))x\|.$$

A simple computation shows that (2.1) holds for $C = 2D$. Moreover, by repeating the arguments in the proof of [6, Proposition 5.6], one can show that $t \mapsto \|x\|_t$ is continuous for each $x \in X$. Furthermore, for $t \geq s \geq 0$ and $x \in X$ we have

$$\begin{aligned}
\|T(t,s)P(s)x\|_t &= \sup_{\tau\geq t} e^{\lambda(\rho(\tau)-\rho(t))}\|T(\tau,s)P(s)x\| \\
&= \sup_{\tau\geq t} e^{-\lambda(\rho(t)-\rho(s))}e^{\lambda(\rho(\tau)-\rho(s))}\|T(\tau,s)P(s)x\| \\
&\leq e^{-\lambda(\rho(t)-\rho(s))} \sup_{\tau\geq s} e^{\lambda(\rho(\tau)-\rho(s))}\|T(\tau,s)P(s)x\| \\
&\leq e^{-\lambda(\rho(t)-\rho(s))}\|x\|_s,
\end{aligned}$$

and thus (2.4) holds. Similarly, one can prove (2.5). Hence, the evolution family $\mathcal{T}$ admits a ρ-dichotomy with respect to the family of norms $\|\cdot\|_t$, $t \geq 0$, defined above.

Conversely, assume that $\mathcal{T}$ admits a ρ-dichotomy with respect to a family of norms $\|\cdot\|_t$ on X satisfying (2.1) for some $C > 0$ and $\varepsilon \geq 0$. For $t \geq s \geq 0$ and $x \in X$ we have

$$\begin{aligned}
\|T(t,s)P(s)x\| &\leq \|T(t,s)P(s)x\|_t \\
&\leq De^{-\lambda(\rho(t)-\rho(s))}\|x\|_s \\
&\leq DCe^{\varepsilon\rho(s)}e^{-\lambda(\rho(t)-\rho(s))}\|x\|,
\end{aligned}$$

and thus (2.6) holds. Similarly, one can show (2.7). Therefore, the evolution family $\mathcal{T}$ admits a ρ-nonuniform exponential dichotomy. □

2.2 Admissible spaces

Let Y_1 be the space of all Bochner measurable functions $x\colon [0,\infty) \to X$ such that

$$\|x\|_1 := \int_0^\infty \|x(t)\|_t \, dt < \infty,$$

identifying functions that are equal Lebesque-almost everywhere. It is easy to show that $(Y_1, \|\cdot\|_1)$ is a Banach space (see [4, Theorem 1]). Moreover, consider the space Y_∞ of all continuous functions $x\colon [0,\infty) \to X$ such that

$$\|x\|_\infty := \sup_{t\geq 0} \|x(t)\|_t < \infty.$$

One can easily prove that $(Y_\infty, \|\cdot\|_\infty)$ is a Banach space. For a closed subspace $Z \subset X$, Y_∞^Z is the space of all $x \in Y_\infty$ such that $x(0) \in Z$. Obviously, Y_∞^Z is a closed subspace of Y_∞, therefore it is also a Banach space.

We consider another Banach function space $(Y'_\infty, \|\cdot\|'_\infty)$, which consists of all Bochner measurable functions $x\colon [0,\infty) \to X$ such that

$$\|x\|'_\infty := \operatorname*{ess\,sup}_{t\geq 0} \|x(t)\|_t < \infty,$$

where ess sup is taken with respect to the Lebesgue measure on $[0,\infty)$.

3 From dichotomy to admissibility

In this section we show that the existence of a ρ-dichotomy with respect to a family of norms for an evolution family $\mathcal{T} = \{T(t,s)\}_{t\geq s\geq 0}$ yields the admissibility of the pairs $\big(Y_\infty^Z, Y_1\big)$, $\big(Y_\infty^Z, Y'_\infty\big)$ for a certain closed subspace $Z \subset X$.

Proposition 3.1. *Assume that the evolution family $\mathcal{T}$ admits a ρ-dichotomy with respect to a family of norms $\|\cdot\|_t$, $t \geq 0$, and set $Z = \operatorname{Ker} P(0)$. Then, for each $y \in Y_1$ there exists a unique $x \in Y_\infty^Z$ such that*

$$x(t) = T(t,s)x(s) + \int_s^t T(t,\tau)y(\tau)\,d\tau, \quad \text{for } t \geq s \geq 0. \tag{3.1}$$

Proof. Take an arbitrary $y \in Y_1$. For $t \geq 0$, set

$$x(t) = \int_0^t T(t,s)P(s)y(s)\,ds - \int_t^\infty T(t,s)(\mathrm{Id} - P(s))y(s)\,ds.$$

It follows from (2.4) and (2.5) that

$$\begin{aligned}
\|x(t)\|_t &\leq \int_0^t \|T(t,s)P(s)y(s)\|_t\,ds + \int_t^\infty \|T(t,s)(\mathrm{Id} - P(s))y(s)\|_t\,ds \\
&\leq D\int_0^t e^{-\lambda(\rho(t)-\rho(s))}\|y(s)\|_s\,ds + D\int_t^\infty e^{-\lambda(\rho(s)-\rho(t))}\|y(s)\|_s\,ds \\
&\leq D\int_0^t \|y(s)\|_s\,ds + D\int_t^\infty \|y(s)\|_s\,ds = D\,\|y\|_1,
\end{aligned}$$

for every $t \geq 0$, and thus $x \in Y_\infty$. On the other hand, it is easy to check that $x(0) \in Z$. Therefore, $x \in Y_\infty^Z$. Moreover, for $t \geq s \geq 0$ we have

$$\begin{aligned}
x(t) - T(t,s)x(s) &= \int_0^t T(t,\tau)P(\tau)y(\tau)\,d\tau - T(t,s)\int_0^s T(s,\tau)P(\tau)y(\tau)\,d\tau \\
&\quad - \int_t^\infty T(t,\tau)(\mathrm{Id} - P(\tau))y(\tau)\,d\tau \\
&\quad + T(t,s)\int_s^\infty T(s,\tau)(\mathrm{Id} - P(\tau))y(\tau)\,d\tau \\
&= \int_s^t T(t,\tau)P(\tau)y(\tau)\,d\tau + \int_s^t T(t,\tau)(\mathrm{Id} - P(\tau))y(\tau)\,d\tau \\
&= \int_s^t T(t,\tau)y(\tau)\,d\tau,
\end{aligned}$$

and therefore we conclude that (3.1) holds. In order to establish the uniqueness, it is sufficient to consider the case when $y=0$. Let $x\in Y_\infty^Z$ such that

$$x(t)=T(t,s)x(s), \quad \text{for } t\ge s\ge 0.$$

Then, from (2.5) we have

$$\begin{aligned}\|x(0)\|_0 &= \|(\mathrm{Id}-P(0))x(0)\|_0=\|T(0,t)(\mathrm{Id}-P(t))x(t)\|_0\\ &\le De^{-\lambda\rho(t)}\|x(t)\|_t\\ &\le De^{-\lambda\rho(t)}\|x\|_\infty,\end{aligned}$$

for every $t\ge 0$. Passing to the limit when $t\to\infty$, we conclude that $x(0)=0$, which implies that $x=0$. □

Proposition 3.2. *Assume that the evolution family $\mathcal{T}$ admits a ρ-dichotomy with respect to a family of norms $\|\cdot\|_t$, $t\ge 0$, and set $Z=\operatorname{Ker}P(0)$. Then, for each $y\in Y'_\infty$ there exists a unique $x\in Y_\infty^Z$ such that*

$$x(t)=T(t,s)x(s)+\int_s^t \rho'(\tau)T(t,\tau)y(\tau)\,d\tau, \quad \text{for } t\ge s\ge 0. \tag{3.2}$$

Proof. Take $y\in Y'_\infty$. For $t\ge 0$, set

$$x(t)=\int_0^t \rho'(s)T(t,s)P(s)y(s)\,ds-\int_t^\infty \rho'(s)T(t,s)(\mathrm{Id}-P(s))y(s)\,ds.$$

It follows from (2.4) and (2.5) that

$$\begin{aligned}\|x(t)\|_t &\le \int_0^t \rho'(s)\|T(t,s)P(s)y(s)\|_t\,ds+\int_t^\infty \rho'(s)\|T(t,s)(\mathrm{Id}-P(s))y(s)\|_t\,ds\\ &\le D\int_0^t \rho'(s)e^{-\lambda(\rho(t)-\rho(s))}\|y(s)\|_s\,ds+D\int_t^\infty \rho'(s)e^{-\lambda(\rho(s)-\rho(t))}\|y(s)\|_s\,ds\\ &\le D\|y\|'_\infty\bigg(\int_0^t \rho'(s)e^{-\lambda(\rho(t)-\rho(s))}\,ds+\int_t^\infty \rho'(s)e^{-\lambda(\rho(s)-\rho(t))}\,ds\bigg)\\ &\le \frac{2D}{\lambda}\|y\|'_\infty, \quad \text{for every } t\ge 0.\end{aligned}$$

Since $x(0)\in Z$, we conclude that $x\in Y_\infty^Z$. A simple computation shows that (3.2) holds. The uniqueness part can be established as in the proof of Proposition 3.1. □

4 From admissibility to dichotomy

The aim of this section is to prove that the admissibility of the pairs (Y_∞^Z, Y_1), (Y_∞^Z, Y'_∞) for a closed subspace $Z\subset X$ yields the existence of a ρ-dichotomy with respect to the family of norms $\{\|\cdot\|_t\}_{t\ge 0}$. More precisely, our goal is to establish the following result.

Theorem 4.1. *Assume that there exists a closed subspace $Z\subset X$ such that:*

(i) for each $y\in Y_1$ there exists a unique $x\in Y_\infty^Z$ satisfying (3.1);

(ii) for each $y\in Y'_\infty$ there exists a unique $x\in Y_\infty^Z$ satisfying (3.2).

Then, the evolution family $\mathcal{T}$ admits a ρ-dichotomy with respect to the family of norms $\|\cdot\|_t$, $t\ge 0$.

Proof. Let

$$T_Z : \mathcal{D}(T_Z) \subset Y_\infty^Z \to Y_1, \quad T_Z x = y,$$

where

$$\mathcal{D}(T_Z) = \Big\{ x \in Y_\infty^Z : \text{ there exists } y \in Y_1 \text{ satisfying (3.1)} \Big\}.$$

Furthermore, let

$$T_Z' : \mathcal{D}(T_Z') \subset Y_\infty^Z \to Y_\infty', \quad T_Z' x = y,$$

where

$$\mathcal{D}(T_Z') = \Big\{ x \in Y_\infty^Z : \text{ there exists } y \in Y_\infty' \text{ satisfying (3.2)} \Big\}.$$

Lemma 4.2. *The operators $T_Z \colon \mathcal{D}(T_Z) \to Y_1$, $T_Z' \colon \mathcal{D}(T_Z') \to Y_\infty'$ are well-defined, linear and closed.*

Proof of the lemma. Assume that $x \in Y_\infty^Z$ and $y_1, y_2 \in Y_1$ such that

$$x(t) = T(t,\tau)x(\tau) + \int_\tau^t T(t,s)y_i(s)\, ds,$$

for $t \ge \tau \ge 0$ and $i \in \{1,2\}$. Hence,

$$\int_\tau^t T(t,s)(y_1(s) - y_2(s))\, ds = 0, \quad \text{for } t > \tau \ge 0.$$

Dividing by $t-\tau$ and letting $t - \tau \to 0$, it follows from the Lebesgue differentiation theorem that

$$y_1(t) = y_2(t) \quad \text{for almost every } t \ge 0.$$

We conclude that $y_1 = y_2$ in Y_1. Thus, T_Z is well-defined and, by definition it is linear.

We now show that T_Z is closed. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}(T_Z)$ converging to $x \in Y_\infty^Z$ such that $y_n = T_Z x_n$ converges to $y \in Y_1$. Then, for $t \ge \tau \ge 0$ we have that

$$x(t) - T(t,\tau)x(\tau) = \lim_{n\to\infty} (x_n(t) - T(t,\tau)x_n(\tau)) = \lim_{n\to\infty} \int_\tau^t T(t,s)y_n(s)\, ds.$$

On the other hand, we have

$$\begin{aligned}
\left\| \int_\tau^t T(t,s)y_n(s)\, ds - \int_\tau^t T(t,s)y(s)\, ds \right\| &\le M \int_\tau^t \|y_n(s) - y(s)\|\, ds \\
&\le M \int_\tau^t \|y_n(s) - y(s)\|_s\, ds \\
&\le M\|y_n - y\|_1,
\end{aligned}$$

where $M = M(t,\tau) = \sup\{\|T(t,s)\| : s \in [\tau,t]\}$ is finite by the Banach–Steinhaus theorem. Since $y_n \to y$ in Y_1, we conclude that

$$\lim_{n\to\infty} \int_\tau^t T(t,s)y_n(s)\, ds = \int_\tau^t T(t,s)y(s)\, ds,$$

and therefore (3.1) holds. We conclude that $x \in \mathcal{D}(T_Z)$ and $T_Z x = y$. Therefore, T_Z is a closed linear operator. Similarly, one can show that T_Z' is well-defined, linear and closed. □

By the assumption in Theorem 4.1, the linear operators T_Z, T'_Z are bijective, and by previous lemma and the Closed Graph Theorem they have bounded inverse $G_Z\colon Y_1 \to Y_\infty^Z$ and $G'_Z\colon Y'_\infty \to Y_\infty^Z$, respectively.

For $\tau \geq 0$, set

$$S(\tau) = \left\{ v \in X : \sup_{t \geq \tau} \|T(t,\tau)v\|_t < \infty \right\} \quad \text{and} \quad U(\tau) = T(\tau,0)Z.$$

Clearly, $S(\tau)$ and $U(\tau)$ are subspaces of X for each $\tau \geq 0$.

Lemma 4.3. *For $\tau \geq 0$, we have that*

$$X = S(\tau) \oplus U(\tau). \tag{4.1}$$

Proof of the lemma. Let $\tau \geq 0$ and take $v \in X$. Set

$$g(s) = \chi_{[\tau,\tau+1]}(s)T(s,\tau)v, \qquad s \geq 0.$$

Clearly, $g \in Y_1$. Since T_Z is invertible, there exists $h \in \mathcal{D}(T_Z) \subset Y_\infty^Z$ such that $T_Z h = g$. It follows from (3.1) that

$$h(t) = T(t,\tau)(h(\tau) + v) \quad \text{for } t \geq \tau + 1.$$

Since $h \in Y_\infty$, we conclude that $h(\tau) + v \in S(\tau)$. Similarly, it follows from (3.1) that

$$h(\tau) = T(\tau,0)h(0).$$

Since $h(0) \in Z$, we have that $h(\tau) \in U(\tau)$ and thus

$$v = (h(\tau) + v) + (-h(\tau)) \in S(\tau) + U(\tau).$$

We have proved that $X = S(\tau) + U(\tau)$.

Take now $v \in S(\tau) \cap U(\tau)$. Then, there exists $z \in Z$ such that $v = T(\tau,0)z$. We consider a function $h\colon [0,\infty) \to X$, defined by

$$h(t) = T(t,0)z \quad \text{for } t \geq 0.$$

Clearly, $h \in Y_\infty^Z$. Since $h(t) = T(t,s)h(s)$ for all $t \geq s \geq 0$, it follows that $T_Z h = 0$ and thus $h = 0$. We conclude that $v = h(\tau) = 0$, and hence $S(\tau) \cap U(\tau) = \{0\}$. This completes the proof of the lemma. □

Let $P(\tau)\colon X \to S(\tau)$ and $Q(\tau)\colon X \to U(\tau)$ be the projections associated with the decomposition (4.1), with $P(\tau) + Q(\tau) = \mathrm{Id}$. Observe that (2.2) holds. Indeed, observe that

$$T(t,\tau)S(\tau) \subset S(t) \quad \text{and} \quad T(t,\tau)U(\tau) \subset U(t), \quad \text{for } t \geq \tau \geq 0.$$

Hence, we have that for every $x \in X$ and $t \geq \tau \geq 0$,

$$P(t)T(t,\tau)x = P(t)T(t,\tau)P(\tau)x + P(t)T(t,\tau)Q(\tau)x = T(t,\tau)P(\tau)x.$$

We conclude that (2.2) holds.

Lemma 4.4. *For $t \geq \tau \geq 0$, the restriction $T(t,\tau)|_{U(\tau)}\colon U(\tau) \to U(t)$ is invertible.*

Proof of the lemma. Let $t \geq \tau \geq 0$ and take $x \in U(t)$. Then, there exists $z \in Z$ such that $x = T(t,0)z$. Since $T(\tau,0)z \in U(\tau)$ and $x = T(t,\tau)T(\tau,0)z$, we conclude that $T(t,\tau)|_{U(\tau)}$ is surjective.

Let now $x \in U(\tau)$ such that $T(t,\tau)x = 0$. Take $z \in Z$ such that $x = T(\tau,0)z$. We define $u\colon [0,\infty) \to X$ by $u(s) = T(s,0)z$, $s \geq 0$. Since $u(s) = 0$ for $s \geq t$, we have that $u \in Y_\infty^Z$ and $T_Z u = 0$. Consequently, $u = 0$ and $x = u(\tau) = 0$. This proves that $T(t,\tau)|_{U(\tau)}$ is also injective. The proof of the lemma is completed. □

Lemma 4.5. *There exists $M > 0$ such that*

$$\|P(\tau)v\|_\tau \leq M\|v\|_\tau, \quad \text{for all } v \in X \text{ and } \tau \geq 0. \tag{4.2}$$

Proof of the lemma. Take $v \in X$ and $\tau \geq 0$. Moreover, given $h > 0$, we define a function $g_h\colon [0,\infty) \to X$ by

$$g_h(t) = \frac{1}{h}\chi_{[\tau,\tau+h]}(t)T(t,\tau)v.$$

Clearly, $g_h \in Y_1$ and thus there exists $x_h \in \mathcal{D}(T_Z)$ such that $T_Z x_h = g_h$. We have

$$\|P(\tau)v\|_\tau = \|x_h(\tau) + v\|_\tau \leq \|x_h(\tau)\|_\tau + \|v\|_\tau \leq \|G_Z g_h\|_\infty + \|v\|_\tau.$$

Moreover,

$$\|G_Z g_h\|_\infty \leq \|G_Z\| \cdot \|g_h\|_1 = \|G_Z\| \frac{1}{h}\int_\tau^{\tau+h} \|T(t,\tau)v\|_t \, dt.$$

Letting $h \to 0$, we obtain

$$\|P(\tau)v\|_\tau \leq (1 + \|G_Z\|)\, \|v\|_\tau,$$

and we conclude that (4.2) holds for $M = 1 + \|G_Z\|$. □

Lemma 4.6. *There exist constants $\lambda, D > 0$ such that*

$$\|T(t,\tau)v\|_t \leq De^{-\lambda(\rho(t)-\rho(\tau))}\|v\|_\tau, \quad \text{for } t \geq \tau \geq 0 \text{ and } v \in S(\tau). \tag{4.3}$$

Proof of the lemma. Fix $\tau \geq 0$ and let $v \in S(\tau)$. We consider the function

$$u\colon [0,\infty) \to X, \quad u(t) = \chi_{[\tau,\infty)}(t)T(t,\tau)v.$$

Moreover, for any fixed $h > 0$, we define two functions $\varphi_h\colon [0,\infty) \to \mathbb{R}$ and $g_h\colon [0,\infty) \to X$ by

$$\varphi_h(t) = \begin{cases} 0, & 0 \leq t \leq \tau, \\ \frac{1}{h}(t-\tau), & \tau \leq t \leq \tau + h, \\ 1, & t \geq \tau + h, \end{cases}$$

and

$$g_h(t) = \frac{1}{h}\,\chi_{[\tau,\tau+h]}(t)\,T(t,\tau)v, \qquad t \geq 0.$$

It is easy to show that $g_h \in Y_1$, $\varphi_h u \in \mathcal{D}(T_Z)$ and $T_Z(\varphi_h u) = g_h$. We have

$$\begin{aligned}\sup_{t \geq \tau+h} \|u(t)\|_t = \sup_{t \geq \tau+h} \|\varphi_h(t)u(t)\|_t &\leq \|\varphi_h u\|_\infty = \|G_Z g_h\|_\infty \\ &\leq \|G_Z\| \cdot \|g_h\|_1 \\ &= \|G_Z\|\frac{1}{h}\int_\tau^{\tau+h} \|u(s)\|_s \, ds.\end{aligned}$$

Hence, letting $h \to 0$ we obtain the inequality

$$\|u(t)\|_t \leq \|G_Z\| \cdot \|v\|_\tau, \quad \text{for every } t \geq \tau.$$

Thus,

$$\|T(t,\tau)v\|_t \leq \|G_Z\| \cdot \|v\|_\tau, \quad \text{for every } t \geq \tau. \tag{4.4}$$

Let us take $t \geq \tau$ and $v \in S(\tau)$ such that $T(t,\tau)v \neq 0$, thus $T(s,\tau)v \neq 0$ for all $s \in [\tau, t]$. Let us consider $x, y\colon [0,\infty) \to X$ defined by

$$y(s) = \chi_{[\tau,t]}(s)\,\frac{T(s,\tau)v}{\|T(s,\tau)v\|_s}, \qquad s \geq 0,$$

and

$$x(s) = \begin{cases} 0, & 0 \leq s \leq \tau, \\ \int_\tau^s \rho'(r)\,\frac{T(s,\tau)v}{\|T(r,\tau)v\|_r}\,dr, & \tau < s \leq t, \\ \int_\tau^t \rho'(r)\,\frac{T(s,\tau)v}{\|T(r,\tau)v\|_r}\,dr, & s > t. \end{cases}$$

Note that $y \in Y'_\infty$ and $\|y\|'_\infty = 1$. Furthermore, since $v \in S(\tau)$ we get that

$$\|x(s)\|_s \leq \int_\tau^t \frac{\rho'(r)}{\|T(r,\tau)v\|_r}\,dr\,\|T(s,\tau)v\|_s \leq a_{t,\tau,v} \sup_{r\geq\tau} \|T(r,\tau)v\|_r < \infty,$$

for all $s \geq \tau$, where

$$a_{t,\tau,v} = \int_\tau^t \frac{\rho'(r)}{\|T(r,\tau)v\|_r}\,dr < \infty,$$

and thus $x \in Y_\infty^Z$. It is straightforward to show that $T'_Z x = y$. Consequently,

$$\|x\|_\infty = \|G'_Z y\|_\infty \leq \|G'_Z\| \cdot \|y\|'_\infty = \|G'_Z\|.$$

Therefore,

$$\|G'_Z\| \geq \|x\|_\infty \geq \|x(t)\|_t = \|T(t,\tau)v\|_t \int_\tau^t \frac{\rho'(r)}{\|T(r,\tau)v\|_r}\,dr. \tag{4.5}$$

From (4.4) it follows that

$$\frac{1}{\|T(r,\tau)v\|_r} \geq \frac{1}{\|G_Z\| \cdot \|v\|_\tau}, \quad \text{for all } r \in [\tau, t],$$

and thus, from (4.5) we get

$$\|G'_Z\| \cdot \|G_Z\| \cdot \|v\|_\tau \geq \|T(t,\tau)v\|_t\,(\rho(t) - \rho(\tau)), \text{ for } t \geq \tau \text{ and } v \in S(\tau).$$

Consequently,

$$(t-\tau)\left\|T\left(\rho^{-1}(t), \rho^{-1}(\tau)\right)v\right\|_{\rho^{-1}(t)} \leq \|G'_Z\| \cdot \|G_Z\| \cdot \|v\|_{\rho^{-1}(\tau)},$$

for $t \geq \tau$ and $v \in S\left(\rho^{-1}(\tau)\right)$. Let $N_0 \in \mathbb{N}^*$ such that $N_0 > e\|G'_Z\| \cdot \|G_Z\|$, and let $t \geq \tau + N_0$. Then,

$$\begin{aligned} N_0 \left\|T\left(\rho^{-1}(t), \rho^{-1}(\tau)\right)v\right\|_{\rho^{-1}(t)} &\leq (t-\tau)\left\|T\left(\rho^{-1}(t), \rho^{-1}(\tau)\right)v\right\|_{\rho^{-1}(t)} \\ &\leq \|G'_Z\| \cdot \|G_Z\| \cdot \|v\|_{\rho^{-1}(\tau)}, \end{aligned}$$

which implies that there exists $N_0 \in \mathbb{N}^*$ such that

$$\|T(\rho^{-1}(t), \rho^{-1}(\tau))v\|_{\rho^{-1}(t)} \le \frac{1}{e}\|v\|_{\rho^{-1}(\tau)}, \tag{4.6}$$

for $t \ge \tau$ with $t - \tau \ge N_0$ and $v \in S\left(\rho^{-1}(\tau)\right)$. Take an arbitrary $t \ge \tau$ with $t - \tau \ge N_0$ and write $t - \tau$ in the form

$$t - \tau = kN_0 + r,\ k = k(t,\tau) \in \mathbb{N}^* \quad \text{and} \quad r = r(t,\tau) \in [0, N_0).$$

Observing that

$$\begin{aligned} &T\left(\rho^{-1}(t), \rho^{-1}(\tau)\right) \\ &\quad = T\left(\rho^{-1}(t), \rho^{-1}(\tau + kN_0)\right) \prod_{j=0}^{k-1} T\left(\rho^{-1}(\tau + (k-j)N_0), \rho^{-1}(\tau + (k-j-1)N_0)\right), \end{aligned}$$

it follows from (4.4) and (4.6) that

$$\begin{aligned} \|T\left(\rho^{-1}(t), \rho^{-1}(\tau)\right) v\|_{\rho^{-1}(t)} &\le \|G_Z\|\, e^{-k}\, \|v\|_{\rho^{-1}(\tau)} \\ &\le e\,\|G_Z\|\, e^{-\frac{1}{N_0}(t-\tau)}\, \|v\|_{\rho^{-1}(\tau)}, \end{aligned}$$

and thus (4.3) holds with $\lambda = 1/N_0$ and $D = e\,\|G_Z\|$. The proof of the lemma is completed. □

Lemma 4.7. *There exist $\lambda, D > 0$ such that*

$$\|T(t,\tau)v\|_t \le De^{-\lambda(\rho(\tau)-\rho(t))}\|v\|_\tau, \text{ for } 0 \le t \le \tau \text{ and } v \in U(\tau). \tag{4.7}$$

Proof of the lemma. Take $\tau > 0$ and $z \in Z$. We define a function $u\colon [0,\infty) \to X$ by

$$u(t) = T(t,0)z, \quad \text{for } t \ge 0.$$

For sufficiently small $h > 0$, we define $\psi_h\colon [0,\infty) \to \mathbb{R}$,

$$\psi_h(t) = \begin{cases} 1, & 0 \le t \le \tau - h, \\ -\frac{t-\tau}{h}, & \tau - h \le t \le \tau, \\ 0, & t \ge \tau. \end{cases}$$

Finally, we consider

$$g_h\colon [0,\infty) \to X, \qquad g_h = -\frac{1}{h}\chi_{[\tau-h,\tau]}\, u.$$

It is easy to check that $g_h \in Y_1$, $\psi_h u \in \mathcal{D}(T_Z)$ and $T_Z(\psi_h u) = g_h$. Hence,

$$\begin{aligned} \sup_{t\in[0,\tau-h]} \|u(t)\|_t &= \sup_{t\in[0,\tau-h]} \|\psi_h(t)u(t)\|_t \le \|\psi_h u\|_\infty = \|G_Z g_h\|_\infty \\ &\le \|G_Z\| \cdot \|g_h\|_1 \\ &= \|G_Z\| \cdot \frac{1}{h}\int_{\tau-h}^{\tau} \|u(s)\|_s\, ds. \end{aligned}$$

Letting $h \to 0$, we get

$$\|u(t)\|_t \le \|G_Z\| \cdot \|u(\tau)\|_\tau, \quad \text{for } 0 \le t \le \tau,$$

which implies

$$\|T(t,0)z\|_t \le \|G_Z\| \cdot \|T(\tau,0)z\|_\tau, \quad \text{for } z \in Z \text{ and } 0 \le t \le \tau. \tag{4.8}$$

Take now $z \in Z \setminus \{0\}$ and $0 \le t \le \tau$. We define $x, y \colon [0,\infty) \to X$ by

$$y(s) = \begin{cases} -\frac{T(s,0)z}{\|T(s,0)z\|_s}, & 0 \le s \le \tau, \\ 0, & s > \tau, \end{cases}$$

and

$$x(s) = \begin{cases} \int_s^\tau \rho'(r) \frac{T(s,0)z}{\|T(r,0)z\|_r}\, dr, & 0 \le s \le \tau, \\ 0, & s > \tau. \end{cases}$$

Observe that $y \in Y'_\infty$ and $\|y\|'_\infty = 1$. Moreover, $x \in Y^Z_\infty$ and it is easy to check that $T'_Z x = y$. Hence,

$$\|x\|_\infty = \|G'_Z y\|_\infty \le \|G'_Z\|.$$

Consequently, for each $0 \le s \le \tau$ we have

$$\|G'_Z\| \ge \|T(s,0)z\|_s \int_s^\tau \rho'(r) \frac{1}{\|T(r,0)z\|_r}\, dr.$$

Letting $\tau \to \infty$, we conclude that

$$\|G'_Z\| \ge \|T(s,0)z\|_s \int_s^\infty \rho'(r) \frac{1}{\|T(r,0)z\|_r}\, dr \quad \text{for } s \ge 0 \text{ and } z \in Z \setminus \{0\}. \tag{4.9}$$

Take now $0 \le t \le \tau$ and $z \in Z \setminus \{0\}$. It follows from (4.8) and (4.9) that

$$\begin{aligned} \frac{1}{\|T(\rho^{-1}(t),0)z\|_{\rho^{-1}(t)}} &\ge \frac{1}{\|G'_Z\|} \int_{\rho^{-1}(t)}^\infty \rho'(r) \frac{1}{\|T(r,0)z\|_r}\, dr \\ &\ge \frac{1}{\|G'_Z\|} \int_{\rho^{-1}(t)}^{\rho^{-1}(\tau)} \rho'(r) \frac{1}{\|T(r,0)z\|_r}\, dr \\ &\ge \frac{1}{\|G'_Z\|} \int_{\rho^{-1}(t)}^{\rho^{-1}(\tau)} \rho'(r) \frac{1}{\|G_Z\| \cdot \|T(\rho^{-1}(\tau),0)z\|_{\rho^{-1}(\tau)}}\, dr \\ &= \frac{\tau - t}{\|G'_Z\| \cdot \|G_Z\|} \cdot \frac{1}{\|T(\rho^{-1}(\tau),0)z\|_{\rho^{-1}(\tau)}} \end{aligned}$$

and thus

$$(\tau - t)\|T(\rho^{-1}(t),0)z\|_{\rho^{-1}(t)} \le \|G_Z\| \cdot \|G'_Z\| \cdot \|T(\rho^{-1}(\tau),0)z\|_{\rho^{-1}(\tau)}.$$

We conclude that there exists $N_0 \in \mathbb{N}^*$ such that

$$\|T(\rho^{-1}(t),0)z\|_{\rho^{-1}(t)} \le \frac{1}{e}\|T(\rho^{-1}(\tau),0)z\|_{\rho^{-1}(\tau)},$$

for $z \in Z$ and $0 \le t \le \tau$ such that $\tau - t \ge N_0$. Hence,

$$\|T(\rho^{-1}(t),\rho^{-1}(\tau))v\|_{\rho^{-1}(t)} \le \frac{1}{e}\|v\|_{\rho^{-1}(\tau)},$$

for $v \in U(\rho^{-1}(\tau))$ and $0 \le t \le \tau$ such that $\tau - t \ge N_0$. By arguing as in the proof of Lemma 4.6, we find that there exist $\lambda, D > 0$ such that

$$\|T(\rho^{-1}(t),\rho^{-1}(\tau))v\|_{\rho^{-1}(t)} \le De^{-\lambda(\tau - t)}\|v\|_{\rho^{-1}(\tau)},$$

for $v \in U(\rho^{-1}(\tau))$ and $0 \le t \le \tau$, which readily implies the conclusion of the lemma. □

In order to complete the proof of the theorem, it is sufficient to observe that (4.2), (4.3) and (4.7) imply that (2.4) and (2.5) hold. □

Remark 4.8. It is worth observing that in order to deduce the existence of a ρ-dichotomy we imposed two admissibility conditions. In the following two examples we will illustrate that this was necessary.

Example 4.9. We consider an evolution family $\mathcal{T} = \{T(t,s)\}_{t \geq s \geq 0}$ given by

$$T(t,s) = \mathrm{Id}, \qquad t \geq s \geq 0.$$

Furthermore, take $Z = \{0\}$ and let $\|\cdot\|_t = \|\cdot\|$ for $t \geq 0$. Then for each $y \in Y_1$, the unique $x \in Y_Z$ satisfying (3.1) is given by

$$x(t) = \int_0^t T(t,s)y(s)\,ds = \int_0^t y(s)\,ds, \qquad t \geq 0.$$

Thus, the first assumption of Theorem 4.1 is fulfilled. On the other hand, $\mathcal{T}$ obviously doesn't admit a ρ-dichotomy with respect to the family of norms $\|\cdot\|_t$, $t \geq 0$.

The following example is a simple modification of [12, Example 1].

Example 4.10. Let $X = \mathbb{R}$ with the standard Euclidean norm $|\cdot|$. Furthermore, let $\|\cdot\|_t = |\cdot|$ for $t \geq 0$ and take $Z = \{0\}$. Furthermore, let $\rho(t) = \ln(1+t)$ for $t \geq 0$. We consider the sequence $(A_n)_{n \in \mathbb{N}}$ of operators on X (which can of course be identified with numbers) given by

$$A_n = \begin{cases} n & \text{if } n = 2^l \text{ for some } l \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, for $t \geq s \geq 0$ we define

$$T(t,s) = \begin{cases} A_{\lfloor t \rfloor - 1} \cdots A_{\lfloor s \rfloor}, & \lfloor t \rfloor \geq \lfloor s \rfloor + 1, \\ 1, & \lfloor t \rfloor = \lfloor s \rfloor. \end{cases}$$

Then, $\mathcal{T} = \{T(t,s)\}_{t \geq s \geq 0}$ is an evolution family. By arguing as in [12, Example 1], it is easy to check that the second assumption of Theorem 4.1 is satisfied and $\mathcal{T}$ doesn't admit a ρ-dichotomy with respect to the family of norms $\|\cdot\|_t$, $t \geq 0$.

5 Robustness of generalized dichotomies

In this section we apply our main results to prove that the concept of ρ-dichotomy with respect to a family $\{\|\cdot\|_t\}_{t \geq 0}$ of norms on X persist under sufficiently small linear perturbations. As a consequence, we establish the robustness property of ρ-nonuniform exponential dichotomy.

Theorem 5.1. *Assume that the evolution family $\{T(t,s)\}_{t \geq s \geq 0}$ admits a ρ-dichotomy with respect to a family $\{\|\cdot\|_t\}_{t \geq 0}$ of norms on X satisfying*

$$\|x\| \leq \|x\|_t \leq Ce^{\varepsilon\rho(t)}\|x\|, \quad \textit{for } x \in X \textit{ and } t \geq 0,$$

for some $C > 0$ and $\varepsilon \geq 0$, such that the mapping $t \mapsto \|x\|_t$ is continuous for each $x \in X$. If $B : [0, \infty) \to \mathcal{B}(X)$ is a strongly continuous operator-valued function such that

$$\|B(t)\| \leq \delta e^{-(\varepsilon+a)\rho(t)}\rho'(t), \qquad t \geq 0, \tag{5.1}$$

for some $a > 0$ and sufficiently small $\delta > 0$, then the perturbed evolution family $\{U(t,s)\}_{t\geq s\geq 0}$ satisfying

$$U(t,s) = T(t,s) + \int_s^t T(t,\tau)B(\tau)U(\tau,s)\,d\tau, \qquad t \geq s \geq 0, \tag{5.2}$$

admits a ρ-dichotomy with respect to the family of norms $\|\cdot\|_t$, $t \geq 0$.

Proof. Since $\{T(t,s)\}_{t\geq s\geq 0}$ admits a ρ-dichotomy with respect to the family of norms $\|\cdot\|_t$, $t \geq 0$, it follows from Proposition 3.1 and Proposition 3.2 that there exists a closed subspace $Z \subset X$ such that the operators

$$T_Z \colon \mathcal{D}(T_Z) \subset Y_\infty^Z \to Y_1 \quad \text{and} \quad T'_Z \colon \mathcal{D}(T'_Z) \subset Y_\infty^Z \to Y'_\infty,$$

defined in the proof of Theorem 4.1, are invertible and closed. We consider the graph norms:

$$\|x\|_{T_Z} := \|x\|_\infty + \|T_Z x\|_1, \qquad x \in \mathcal{D}(T_Z),$$

and

$$\|x\|_{T'_Z} := \|x\|_\infty + \|T'_Z x\|'_\infty, \qquad x \in \mathcal{D}(T'_Z).$$

Since T_Z, T'_Z are closed, it follows that $(\mathcal{D}(T_Z), \|\cdot\|_{T_Z})$, $\big(\mathcal{D}(T'_Z), \|\cdot\|_{T'_Z}\big)$ are Banach spaces. Furthermore,

$$T_Z : (\mathcal{D}(T_Z), \|\cdot\|_{T_Z}) \to (Y_1, \|\cdot\|_1)$$

and

$$T'_Z : \Big(\mathcal{D}(T'_Z), \|\cdot\|_{T'_Z}\Big) \to (Y'_\infty, \|\cdot\|'_\infty)$$

are bounded linear operators, denoted simply by T_Z and T'_Z, respectively.

We consider the linear operators $D : \mathcal{D}(T_Z) \to Y_1$, $D' : \mathcal{D}(T'_Z) \to Y'_\infty$ defined by

$$(Dx)(t) = B(t)x(t) \text{ and } (D'x)(t) = \frac{1}{\rho'(t)}B(t)x(t), \text{ for } t \geq 0.$$

One can easy check that these operators are well-defined. Furthermore, for each $x \in \mathcal{D}(T_Z)$ we have

$$\begin{aligned} \|Dx\|_1 &= \int_0^\infty \|B(t)x(t)\|_t\,dt \\ &\leq C\int_0^\infty e^{\varepsilon\rho(t)}\|B(t)x(t)\|\,dt \\ &\leq \delta C\int_0^\infty e^{-a\rho(t)}\,\rho'(t)\,\|x(t)\|\,dt \\ &\leq \frac{\delta C}{a}\,\|x\|_\infty, \end{aligned}$$

and thus

$$\|Dx\|_1 \leq \frac{\delta C}{a}\,\|x\|_{T_Z}, \quad x \in \mathcal{D}(T_Z). \tag{5.3}$$

On the other hand, for $x \in \mathcal{D}(T'_Z)$ we get

$$\begin{aligned} \|(D'x)(t)\|_t &= \frac{1}{\rho'(t)}\|B(t)x(t)\|_t \\ &\leq \frac{1}{\rho'(t)}Ce^{\varepsilon\rho(t)}\|B(t)x(t)\| \\ &\leq \delta\,Ce^{-a\rho(t)}\|x(t)\| \\ &\leq \delta\,C\,\|x\|_{T'_Z}, \end{aligned}$$

for all $t \geq 0$, hence

$$\|D'x\|'_\infty \leq \delta\, C\, \|x\|_{T'_Z}, \qquad x \in \mathcal{D}(T'_Z). \tag{5.4}$$

We define now the linear operators

$$U_Z : \mathcal{D}(U_Z) \to Y_1, \qquad U_Z x = y,$$

where $\mathcal{D}(U_Z)$ is the set of all functions $x \in Y_\infty^Z$ such that there exists $y \in Y_1$ satisfying

$$x(t) = U(t,s)x(s) + \int_s^t U(t,\tau)y(\tau)\, d\tau, \quad \text{for } t \geq s \geq 0,$$

and respectively,

$$U'_Z : \mathcal{D}(U'_Z) \to Y'_\infty, \qquad U'_Z x = y,$$

where $\mathcal{D}(U'_Z)$ is the set of all functions $x \in Y_\infty^Z$ such that there exists $y \in Y'_\infty$ satisfying

$$x(t) = U(t,s)x(s) + \int_s^t \rho'(\tau)U(t,\tau)y(\tau)\, d\tau, \quad \text{for } t \geq s \geq 0.$$

Lemma 5.2. *We have:*

$$\mathcal{D}(T_Z) = \mathcal{D}(U_Z) \quad \textit{and} \quad T_Z = U_Z + D, \tag{5.5}$$

and respectively,

$$\mathcal{D}(T'_Z) = \mathcal{D}(U'_Z) \quad \textit{and} \quad T'_Z = U'_Z + D'. \tag{5.6}$$

Proof of the lemma. Take $x \in \mathcal{D}(U_Z)$, that is $x \in Y_\infty^Z$ such that there exists $y \in Y_1$ with $U_Z x = y$. Then, for $t \geq s \geq 0$ we have

$$\begin{aligned} x(t) &= U(t,s)x(s) + \int_s^t U(t,\tau)y(\tau)\, d\tau \\ &= T(t,s)x(s) + \int_s^t T(t,\tau)B(\tau)U(\tau,s)x(s)\, d\tau + \int_s^t T(t,\tau)y(\tau)\, d\tau \\ &\quad + \int_s^t \int_\tau^t T(t,r)B(r)U(r,\tau)y(\tau)\, dr\, d\tau \\ &= T(t,s)x(s) + \int_s^t T(t,r)y(r)\, dr + \int_s^t T(t,r)B(r)U(r,s)x(s)\, dr \\ &\quad + \int_s^t \int_s^r T(t,r)B(r)U(r,\tau)y(\tau)\, d\tau\, dr \\ &= T(t,s)x(s) + \int_s^t T(t,r)\left(y(r) + B(r)x(r)\right)\, dr, \end{aligned}$$

thus $x \in \mathcal{D}(T_Z)$ and

$$T_Z x = y + Dx = (U_Z + D)x.$$

Reversing the arguments, we conclude that (5.5) holds. Similarly, one can prove (5.6). □

Now, we continue the proof of the theorem. From (5.5) and (5.3) we have

$$\|(U_Z - T_Z)x\|_1 = \|Dx\|_1 \leq \frac{\delta C}{a}\, \|x\|_{T_Z}, \quad \text{for all } x \in \mathcal{D}(T_Z) = \mathcal{D}(U_Z),$$

which implies that $U_Z : \mathcal{D}(U_Z) \to Y_1$ is bounded. Since T_Z is invertible, we obtain that U_Z is also invertible for sufficiently small $\delta > 0$. Similarly, one can show that U'_Z is invertible for sufficiently small $\delta > 0$. By Theorem 4.1 we conclude that the perturbed evolution family $\{U(t,s)\}_{t \geq s \geq 0}$ admits a ρ-dichotomy with respect to the family of norms $\|\cdot\|_t$, $t \geq 0$. □

From Proposition 2.1 and Theorem 5.1 we are able now to establish the robustness property of ρ-nonuniform exponential dichotomy.

Corollary 5.3. *Assume that $\mathcal{T} = \{T(t,s)\}_{t \geq s \geq 0}$ admits a ρ-nonuniform exponential dichotomy. If $B : [0,\infty) \to \mathcal{B}(X)$ is a strongly continuous operator-valued function satisfying (5.1) for some $a > 0$ and sufficiently small $\delta > 0$, then the perturbed evolution family satisfying (5.2) admits also a ρ-nonuniform exponential dichotomy.*

Remark 5.4. We stress that the robustness of ρ-nonuniform exponential dichotomies was established in [3, Theorem 1] using different techniques. However, we point out that we establish robustness under a wider class of perturbations than those considered in [3, Theorem 1]. On the other hand, we consider a smaller class of rate functions ρ.

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