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Fractional eigenvalue problems on $\mathbb{R}^N$

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Abstract. Let $N \geq 2$ be an integer. For each real number $s \in (0,1)$ we denote by $(-\Delta)^s$ the corresponding fractional Laplace operator. First, we investigate the eigenvalue problem $(-\Delta)^s u = \lambda V(x) u$ on $\mathbb{R}^N$, where $V : \mathbb{R}^N \to \mathbb{R}$ is a given function. Under suitable conditions imposed on V we show the existence of an unbounded, increasing sequence of positive eigenvalues. Next, we perturb the above eigenvalue problem with a fractional (t,p)-Laplace operator, when $t \in (0,1)$ and $p \in (1,\infty)$ are such that $t < s$ and $s - N/2 = t - N/p$. We show that when the function V is nonnegative on $\mathbb{R}^N$, the set of eigenvalues of the perturbed eigenvalue problem is exactly the unbounded interval (λ_1, ∞), where λ_1 stands for the first eigenvalue of the initial eigenvalue problem.

Keywords: fractional Laplacian, eigenvalue problem, weak solution, minimization problem, Nehari manifold.

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1 Introduction

Let $N \geq 2$ be an integer. For each real numbers $p \in (1,\infty)$ and $s \in (0,1)$ and each function $u : \mathbb{R}^N \to \mathbb{R}$ we define the nonlocal operator

$$(-\Delta_p)^s u(x) := 2 \lim_{\epsilon \searrow 0} \int_{|x-y| \geq \epsilon} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{N+sp}} \, dy, \; x \in \mathbb{R}^N . \tag{1.1}$$

For $p = 2$ the above definition reduces to the linear *fractional Laplacian* denoted by $(-\Delta)^s$. For that reason we will refer to $(-\Delta_p)^s$ as being a *fractional* (s,p)*-Laplacian operator* which is a nonlinear operator when $p \in (1,\infty) \setminus \{2\}$.

1.1 Statement of the problem and motivation

The main goal of this paper is to study an eigenvalue problem for the fractional Laplacian operator on $\mathbb{R}^N$ and a perturbed version of this problem when we perturb the fractional Laplacian by a nonlinear fractional (t,p)-Laplacian . More precisely, first we will study the eigenvalue problem

$$(-\Delta)^s \, u(x) = \mu V(x) u(x), \qquad \forall \, x \in \mathbb{R}^N, \tag{1.2}$$

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where $s \in (0,1)$ is a given real number, μ is a real parameter and $V : \mathbb{R}^N \to \mathbb{R}$ is a function that may change sign and which satisfies the hypothesis

($\widetilde{\mathrm{V}}$) $V \in L^1_{\text{loc}}\left(\mathbb{R}^N\right)$, $V^+ = V_1 + V_2 \neq 0$, $V_1 \in L^{\frac{N}{2s}}\left(\mathbb{R}^N\right)$ and $\lim_{x\to y} |x-y|^{2s} V_2(x) = 0$, for all $y \in \mathbb{R}^N$ and $\lim_{|x|\to\infty} |x|^{2s} V_2(x) = 0$.

Remark 1.1. Note that there exists functions $V : \mathbb{R}^N \to \mathbb{R}$ such that $V \notin L^{\frac{N}{2s}}\left(\mathbb{R}^N\right)$ but $\lim_{x\to y} |x-y|^{2s} V(x) = 0$, for all $y \in \mathbb{R}^N$ and $\lim_{|x|\to\infty} |x|^{2s} V(x) = 0$. Indeed, simple computations show that we can take $V(x) = |x|^{-2s}(1+|x|^{2s})^{-1}[\ln(2+|x|^{-2s})]^{-(2s)/N}$, if $x \neq 0$ and $V(0) = 1$.

Next, we will study a perturbation of problem (1.2), namely

$$(-\Delta)^s u(x) + \left(-\Delta_p\right)^t u(x) = \lambda V(x) u(x), \qquad \forall\, x \in \mathbb{R}^N, \tag{1.3}$$

under the assumption

$$0 < t < s < 1 \quad \text{and} \quad s - \frac{N}{2} = t - \frac{N}{p}, \tag{1.4}$$

where λ is a real parameter and $V : \mathbb{R}^N \to [0,\infty)$ is a function satisfying the hypothesis ($\widetilde{\mathrm{V}}$). Note that in the case of problem (1.3) we have $V = V^+$.

A first motivation in studying problems of type (1.2) comes from the paper by Szulkin & Willem [21] where a similar equation was investigated in the case when the fractional Laplacian $(-\Delta)^s$ is replaced by the classical Laplace operator Δ. In particular, we note that assumption ($\widetilde{\mathrm{V}}$) imposed here to the weight function V is suggested by condition (H) from [21]. At the same time we recall that some generalizations of the results from [21] to the case when the Laplace operator Δ is replaced by a more general class of degenerate elliptic operators of type $\operatorname{div}(|x|^\alpha \nabla)$, with $\alpha \in (0,2)$, was studied by Mihăilescu & Repovš in [18]. In the case of nonlocal operators, problems of type (1.2) were mainly investigated on bounded domains under the homogeneous Dirichlet boundary condition. Among the results obtained in this direction we recall the recent articles by Franzina & Palatucci [13], Lindgren & Lindqvist [15], Brasco, Parini & Squassina [3], Del Pezzo & Quass [5], Ferreira & Pérez-Llanos [11], Fărcășeanu [8], Del Pezzo, Ferreira & Rossi [4], Ercole, Pereira, & Sanchis [7]. Much less papers were devoted to the study of problem (1.2) on the whole Euclidian space $\mathbb{R}^N$. Here we just recall the study by Frank, Lenzmann, & Silvestre from [12] where the issue of the existence and uniqueness of bounded radial solutions which vanishes at infinity for problems of type (1.2) was considered. More precisely, in [12, Theorem 2.1] it is showed that if $u(x) = u(|x|)$ is a radial and bounded solution of (1.2) which vanishes at infinity then $u(0) = 0$ implies $u \equiv 0$, provided that the weight function V is radial and non-decreasing on $\mathbb{R}^N$ and $V \in C^{0,\gamma}(\mathbb{R}^N)$ for some real number $\gamma > \max\{0, 1-2s\}$.

Regarding the problem (1.3) we recall that it was studied on bonded domains form the Euclidian space $\mathbb{R}^N$ under the homogeneous Dirichlet boundary condition by Fărcășeanu, Mihăilescu, & Stancu-Dumitru in [10], in the case when $V \equiv 1$. In particular, we note that assumption (1.4) imposed here is suggested by condition (3) from [10]. We point out that in the case when the nonlocal operators from equation (1.3) are replaced by the corresponding differential operators (Laplacian and p-Laplacian) the resulting problem was analysed by Mihăilescu & Stancu-Dumitru in [19], while in the case of bounded domains similar results were obtained in [1, 9, 16, 17] under different boundary conditions. Thus, in particular, the results from this paper complement to the case of nonlocal operators some earlier results obtained in the case of differential operators.

The rest of the paper is organized as follows: in the next two subsections we introduce the natural function space setting where problems (1.2) and (1.3) will be studied and we point out the main results of the paper; in Section 2 we state and prove an auxiliary result that will be useful for the analysis of the main results; the last two sections are devoted to the proofs of the main results.

1.2 Fractional Sobolev spaces

In this subsection we introduce the natural function spaces where we will study equations (1.2) and (1.3) and we will recall some of their properties which will be useful in our analysis. For more details we refer the reader to the book by Grisvard [14] and to the papers [2, 3, 5, 6].

First, by [3, p. 1814] we recall that the natural setting for equations involving the operator $(-\Delta_p)^t$ is the fractional Sobolev space $\mathcal{D}_0^{t,p}(\mathbb{R}^N)$ defined as the closure of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|_{t,p} := \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+tp}}\,dxdy\right)^{1/p}.$$

The above function space is a reflexive Banach space. Moreover, in the particular case when $p=2$ the function space $\mathcal{D}_0^{t,2}(\mathbb{R}^N)$ is a Hilbert space.

From the above discussion it follows easily that the natural function space where we will study equation (1.2) will be the Hilbert space $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$. On the other hand, we note that in equation (1.3) are involved two nonlocal operators, $(-\Delta)^s$ and $(-\Delta_p)^t$, respectively. The natural function space where we analyse problems involving $(-\Delta)^s$ is the fractional Sobolev space $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$, while the function space where we study problems involving $\mathcal{D}_0^{t,p}(\mathbb{R}^N)$ is the fractional Sobolev space $\mathcal{D}_0^{t,p}(\mathbb{R}^N)$. Thus, in the case of equation (1.3) we should decide which of the spaces $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$ and $\mathcal{D}_0^{t,p}(\mathbb{R}^N)$ is the natural function space where we can seek solutions for the problem. A key condition in this case is assumption (1.4), which in view of [14, Theorem 1.4.4.1] assures that

$$\mathcal{D}_0^{s,2}(\mathbb{R}^N) \subset \mathcal{D}_0^{t,p}(\mathbb{R}^N). \tag{1.5}$$

Thus, the natural function space where we should study problem (1.3) is again the Hilbert space $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$.

Next, note that by [6, Theorem 6.5] there exists a positive constant $C = C(N,s)$ such that

$$\|u\|_{L^{2_s^*}(\mathbb{R}^N)} \le C\,\|u\|_{s,2}, \tag{1.6}$$

where $2_s^* := \frac{2N}{N-2s}$ is the so called *fractional critical exponent*. Consequently, the space $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$ is continuously embedded in $L^{2_s^*}(\mathbb{R}^N)$.

Further, we point out that a Hardy-type inequality can be established on the fractional Sobolev spaces. More precisely, by [2, Theorem 6.3] (see also [20]) we know that there exists a positive constant $C = C(N,s)$ such that

$$C\int_{\mathbb{R}^N} \frac{u(x)^2}{|x|^{2s}}\,dx \le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dxdy, \qquad \forall\, u \in C_0^\infty\left(\mathbb{R}^N\right). \tag{1.7}$$

1.3 The main results

In this subsection we make precise the concept of *eigenvalue* for the equations (1.2) and (1.3) and we present the main results of this paper.

Definition 1.2. We say that $\mu \in \mathbb{R}$ is an eigenvalue of problem (1.2), if there exists $u \in \mathcal{D}_0^{s,2}(\mathbb{R}^N) \setminus \{0\}$ such that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\, dxdy = \mu \int_{\mathbb{R}^N} V(x)u(x)\varphi(x)\, dx, \tag{1.8}$$

for all $\varphi \in \mathcal{D}_0^{s,2}(\mathbb{R}^N)$. Furthermore, u from the above relation will be called an eigenfunction corresponding to the eigenvalue μ.

The main result concerning problem (1.2) is given by the following theorem

Theorem 1.3. *Assume that condition $(\widetilde{V})$ is fulfilled. Then problem (1.2) has an unbounded, increasing sequence of positive eigenvalues.*

Definition 1.4. We say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (1.3), if there exists $u \in \mathcal{D}_0^{s,2}(\mathbb{R}^N) \setminus \{0\}$ such that

$$\begin{aligned} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} & \frac{(1+|u(x)-u(y)|^{p-2})(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+tp}}\, dxdy \\ &= \lambda \int_{\mathbb{R}^N} V(x)u(x)\varphi(x)\, dx, \end{aligned} \tag{1.9}$$

for all $\varphi \in \mathcal{D}_0^{s,2}(\mathbb{R}^N)$. Furthermore, u from the above relation will be called an eigenfunction corresponding to the eigenvalue λ.

Assume that $V : \mathbb{R}^N \to [0,\infty)$ is a function which satisfies condition $(\widetilde{\mathrm{V}})$ and define

$$\lambda_1 := \inf_{u \in C_0^\infty(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|_{s,2}^2}{\displaystyle\int_{\mathbb{R}^N} V(x)u^2\, dx}. \tag{1.10}$$

The main result regarding problem (1.3) is given by the following theorem.

Theorem 1.5. *Assume that $V : \mathbb{R}^N \to [0,\infty)$ is a function which satisfies condition $(\widetilde{V})$. Under assumption (1.4), the set of eigenvalues of problem (1.3) is the open interval (λ_1, ∞). Moreover, the corresponding eigenfunctions can be chosen to be non-negative.*

Remark. A simple analysis of the proof of Theorem 1.3 shows that in the case when function V satisfies $V(x) \geq 0$, for all $x \in \mathbb{R}^N$, then λ_1 defined in relation (1.10) is the smallest eigenvalue of problem (1.2).

2 An auxiliary result

In this section we prove an auxiliary result which will play an important role in our subsequent analysis. More precisely, we prove the following lemma.

Lemma 2.1. *Assume that condition* $(\widetilde{\mathrm{V}})$ *holds true. Then the functional* $T : \mathcal{D}_0^{s,2}(\mathbb{R}^N) \to \mathbb{R}$,

$$T(u) := \int_{\mathbb{R}^N} V^+(x) u^2 \, dx$$

is weakly continuous.

Proof. First, we show that the mapping $\mathcal{D}_0^{s,2}(\mathbb{R}^N) \ni u \to \int_{\mathbb{R}^N} V_1(x) u^2 \, dx$ is weakly continuous.

Let $\{u_n\} \subset \mathcal{D}_0^{s,2}(\mathbb{R}^N)$ be a sequence which converges weakly to $u \in \mathcal{D}_0^{s,2}(\mathbb{R}^N)$. Using the fact that $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$ is continuously embedded in $L^{2_s^*}(\mathbb{R}^N)$, we find that $\{u_n\}$ converges weakly to u in $L^{2_s^*}(\mathbb{R}^N) = L^{\frac{2N}{N-2s}}(\mathbb{R}^N)$. We infer that $\{u_n^2\}$ converges weakly to u^2 in $L^{\frac{N}{N-2s}}(\mathbb{R}^N)$.

Define $W : L^{\frac{N}{N-2s}}(\mathbb{R}^N) \to \mathbb{R}$ by

$$W(\xi) := \int_{\mathbb{R}^N} V_1(x) \xi \, dx, \qquad \forall \xi \in L^{\frac{N}{N-2s}}(\mathbb{R}^N).$$

Clearly, W is linear. Since $V_1 \in L^{\frac{N}{2s}}(\mathbb{R}^N)$ by Hölder's inequality we deduce that W is also continuous. Using the above pieces of information we find that

$$\lim_{n\to\infty} W(u_n) = W(u),$$

meaning that the mapping $\mathcal{D}_0^{s,2}(\mathbb{R}^N) \ni u \to \int_{\mathbb{R}^N} V_1(x) u^2 \, dx$ is weakly continuous.

In order to finish the proof, we shall prove that the mapping $\mathcal{D}_0^{s,2}(\mathbb{R}^N) \ni u \to \int_{\mathbb{R}^N} V_2(x) u^2 \, dx$ is also weakly continuous. Again, let $\{u_n\} \subset \mathcal{D}_0^{s,2}(\mathbb{R}^N)$ be a sequence which converges weakly to $u \in \mathcal{D}_0^{s,2}(\mathbb{R}^N)$. Let $\epsilon > 0$ arbitrary but fixed.

By hypothesis $(\widetilde{\mathrm{V}})$ we deduce that there exists $R > 0$ such that

$$|x|^{2s} V_2(x) \le \epsilon, \qquad \forall \, x \in \mathbb{R}^N \setminus B_R(0), \tag{2.1}$$

where $B_R(0)$ is the open ball centered at the origin of radius R.

Since $\{u_n\}$ converges weakly to u in $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$ we deduce that $\{u_n\}$ is bounded in $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$. Thus,

$$d := C \max \left\{ \sup_n \|u_n\|_{s,2}, \|u\|_{s,2} \right\} < +\infty,$$

where C is the constant given by relation (1.7).

Using relations (1.7) and (2.1) we find

$$\int_{\mathbb{R}^N \setminus B_R(0)} V_2(x) u_n^2 \, dx \le \epsilon \int_{\mathbb{R}^N \setminus B_R(0)} \frac{u_n^2}{|x|^{2s}} \, dx \le \frac{\epsilon}{C} \|u_n\|_{s,2}^2 \le \epsilon d^2 . \tag{2.2}$$

Analogously,

$$\int_{\mathbb{R}^N \setminus B_R(0)} V_2(x) u^2 \, dx \le \frac{\epsilon}{C} \|u\|_{s,2}^2 \le \epsilon d^2 . \tag{2.3}$$

Recalling again hypothesis $(\widetilde{\mathrm{V}})$ and using a compactness argument we find that $\overline{B}_R(0)$ is covered by a finite number of closed balls $\overline{B}_{r_1}(x_1), \overline{B}_{r_2}(x_2), \ldots, \overline{B}_{r_k}(x_k)$ such that for each $j \in \{1, \ldots, k\}$ we have

$$|x - x_j|^{2s} V_2(x) \le \epsilon, \qquad \forall \, x \in \overline{B}_{r_j}(x_j). \tag{2.4}$$

Next, we see that there exists $r > 0$ such that for each $j \in \{1, \ldots, k\}$ the following relation holds

$$|x - x_j|^{2s} V_2(x) \le \frac{\epsilon}{k}, \qquad \forall \, x \in \overline{B}_r(x_j).$$

Again, by relation (1.7) we get

$$\int_{\Omega} V_2(x)u_n^2\,dx \le \epsilon d^2 \quad \text{and} \quad \int_{\Omega} V_2(x)u^2\,dx \le \epsilon d^2\,, \tag{2.5}$$

where $\Omega := \cup_{i=1}^{k} B_r(x_j)$. Finally, by relation (2.4) we infer that $V_2 \in L^\infty(\overline{B}_R(0)\setminus\Omega)$. Since $\overline{B}_R(0)\setminus\Omega$ is bounded we deduce that $V_2 \in L^{\frac{N}{2s}}(\overline{B}_R(0)\setminus\Omega)$. Repeating the same arguments used in the first part of the proof we get

$$\lim_{n\to\infty}\int_{B_R(0)\setminus\Omega} V_2(x)u_n^2\,dx = \int_{B_R(0)\setminus\Omega} V_2(x)u^2\,dx. \tag{2.6}$$

By (2.2), (2.3), (2.5) and (2.6) we deduce that the mapping $\mathcal{D}_0^{s,2}(\mathbb{R}^N) \ni u \to \int_{\mathbb{R}^N} V_2(x)u^2\,dx$ is weakly continuous. Thus, the proof of the lemma is complete. □

3 Proof of Theorem 1.3

The conclusion of Theorem 1.3 will follow from the results of Propositions 3.1 and 3.2 below.

First, we consider the following minimization problem

$$(P_1) \qquad \underset{u\in\mathcal{D}_0^{s,2}(\mathbb{R}^N)}{\text{minimize}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dxdy, \text{ under restriction } \int_{\mathbb{R}^N} V(x)u^2\,dx = 1.$$

Proposition 3.1. *Under the hypothesis* $(\widetilde{\mathrm{V}})$, *problem* (P_1) *has a solution* $e_1 \ge 0$. *Moreover,* e_1 *is an eigenfunction of problem* (1.2) *having its corresponding eigenvalue*

$$\mu_1 := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|e_1(x)-e_1(y)|^2}{|x-y|^{N+2s}}\,dxdy. \tag{3.1}$$

Proof. Let $\{u_n\}_n \subset \mathcal{D}_0^{s,2}(\mathbb{R}^N)$ be a minimizing sequence of problem (P_1), i.e.,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dxdy = \inf_{w\in\mathcal{D}_0^{s,2}(\mathbb{R}^N)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|w(x)-w(y)|^2}{|x-y|^{N+2s}}\,dxdy$$

and

$$\int_{\mathbb{R}^N} V(x)u_n^2\,dx = 1, \qquad \forall\, n \ge 1.$$

It follows that $\{u_n\}$ is bounded in $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$ and consequently there exists $u \in \mathcal{D}_0^{s,2}(\mathbb{R}^N)$ such that u_n converges weakly to u in $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$. Since $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$ is a Hilbert space by the weakly lower semicontinuity of the norm $\|\cdot\|_{s,2}$ we get

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dxdy &\le \liminf_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dxdy \\ &= \inf_{w\in\mathcal{D}_0^{s,2}(\mathbb{R}^N)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|w(x)-w(y)|^2}{|x-y|^{N+2s}}\,dxdy.\end{aligned}$$

On the other hand, using the fact that $V(x) = V^+(x) - V^-(x)$ we deduce that

$$\int_{\mathbb{R}^N} V^-(x)u_n^2\,dx = \int_{\mathbb{R}^N} V^+(x)u_n^2\,dx - 1, \qquad \forall\, n \ge 1.$$

Fatou's lemma and Lemma 2.1 yield

$$\int_{\mathbb{R}^N} V^-(x)u^2\,dx \le \liminf_{n\to\infty}\int_{\mathbb{R}^N} V^-(x)u_n^2\,dx = \int_{\mathbb{R}^N} V^+(x)u^2\,dx - 1,$$

or

$$1 \le \int_{\mathbb{R}^N} V(x)u^2\,dx. \tag{3.2}$$

Define

$$e_1 := \frac{u}{\left(\int_{\mathbb{R}^N} V(x)u^2\,dx\right)^{1/2}}.$$

It is easy to check that

$$\int_{\mathbb{R}^N} V(x)e_1^2\,dx = 1.$$

Furthermore, using relation (3.2) we get

$$\begin{aligned}
\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|e_1(x)-e_1(y)|^2}{|x-y|^{N+2s}}\,dxdy &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|\frac{u(x)}{\left(\int_{\mathbb{R}^N}V(z)u^2\,dz\right)^{1/2}}-\frac{u(y)}{\left(\int_{\mathbb{R}^N}V(z)u^2\,dz\right)^{1/2}}\right|^2}{|x-y|^{N+2s}}\,dxdy\\
&= \frac{1}{\int_{\mathbb{R}^N}V(z)u^2\,dz}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dxdy\\
&\le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dxdy\\
&\le \inf_{w\in\mathcal{D}_0^{s,2}(\mathbb{R}^N)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|w(x)-w(y)|^2}{|x-y|^{N+2s}}\,dxdy.
\end{aligned}$$

This shows that e_1 is a solution of problem (P_1). Moreover, it is easy to see that $|e_1|$ is also a solution of problem (P_1) and consequently we can assume that $e_1 \ge 0$. Next, for each $\varphi \in \mathcal{D}_0^{s,2}(\mathbb{R}^N)$ we define $f:\mathbb{R}\to\mathbb{R}$ by

$$f(\epsilon) = \frac{\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|e_1(x)-e_1(y)+\epsilon(\varphi(x)-\varphi(y))|^2}{|x-y|^{N+2s}}\,dxdy}{\displaystyle\int_{\mathbb{R}^N}V(x)\left(e_1(x)+\epsilon\varphi(x)\right)^2\,dx}.$$

Clearly, f is of class C^1 and $f(0)\le f(\epsilon)$, for all $\epsilon\in\mathbb{R}$. Hence, 0 is a minimum point of f and thus,

$$f'(0) = 0,$$

or

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(e_1(x)-e_1(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dxdy\int_{\mathbb{R}^N}V(x)e_1(x)^2\,dx\\
&\qquad = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|e_1(x)-e_1(y)|^2}{|x-y|^{N+2s}}\,dxdy\int_{\mathbb{R}^N}V(x)e_1(x)\varphi(x)\,dx.
\end{aligned}$$

Since $\varphi\in\mathcal{D}_0^{s,2}(\mathbb{R}^N)$ has been chosen arbitrarily we deduce that the above relation holds true for each $\varphi\in\mathcal{D}_0^{s,2}(\mathbb{R}^N)$. Taking into account that $\int_{\mathbb{R}^N}V(x)e_1^2\,dx = 1$ it follows that μ_1 defined in (3.1) is an eigenvalue of problem (1.2) with the corresponding eigenfunction e_1. Thus, the proof is complete. □

Next, in order to find other eigenvalues of problem (1.2) we solve the following minimization problems

$$(P_n)\quad \begin{array}{l} \underset{u\in\mathcal{D}_0^{s,2}(\mathbb{R}^N)}{\text{minimize}} \displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dxdy, \text{ under restrictions } \int_{\mathbb{R}^N}V(x)u^2\,dx=1 \quad\text{and}\\ \displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(e_k(x)-e_k(y))(u(x)-u(y))}{|x-y|^{N+2s}}\,dxdy=0,\ \forall\, k\in\{1,\dots,n-1\},\end{array}$$

where e_k represents the solution of problem (P_k), for $k\in\{1,\dots,n-1\}$.

Proposition 3.2. *Assume that the hypothesis* $(\widetilde{\mathrm{V}})$ *is fulfilled. Then, for every* $n\geq 2$ *problem* (P_n) *has a solution* e_n*. Moreover,* e_n *is an eigenvector of problem* (1.2) *corresponding to the eigenvalue*

$$\mu_n:=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|e_n(x)-e_n(y)|^2}{|x-y|^{N+2s}}\,dxdy.$$

Furthermore, $\lim_{n\to\infty}\mu_n=\infty$.

Proof. The existence of e_n can be obtained in the same manner as in proof of Theorem 1.3, but replacing $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$ with its closed subspace

$$X_n:=\left\{u\in\mathcal{D}_0^{s,2}(\mathbb{R}^N):\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(e_k(x)-e_k(y))(u(x)-u(y))}{|x-y|^{N+2s}}\,dxdy=0,\ \text{for } k\in\{1,\dots,n-1\}\right\}.$$

Next, following the lines of the proof of Theorem 1.3 we find the existence of $e_n\in X_n$ which verifies

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(e_n(x)-e_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dxdy=\mu_n\int_{\mathbb{R}^N}V(x)e_n(x)\varphi(x)\,dx,\qquad\forall\,\varphi\in X_n,\quad(3.3)$$

where

$$\mu_n:=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|e_n(x)-e_n(y)|^2}{|x-y|^{N+2s}}\,dxdy$$

and

$$\int_{\mathbb{R}^N}V(x)e_n^2\,dx=1.$$

We note that for each $u\in X_n$ we have

$$\int_{\mathbb{R}^N}V(x)ue_k\,dx=0,\qquad\forall\,k\in\{1,\dots,n-1\}.$$

and

$$\int_{\mathbb{R}^N}V(x)e_je_k\,dx=\delta_{j,k},\qquad\forall\,j,k\in\{1,\dots,n-1\}.$$

Hence, for each $v\in\mathcal{D}_0^{s,2}(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N}V(x)\left[v-\sum_{j=1}^{n-1}\left(\int_{\mathbb{R}^N}V(x)ve_j\,dx\right)e_j\right]e_k\,dx=0,\qquad\forall\,k\in\{1,\dots,n-1\},$$

or

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(e_k(x)-e_k(y))(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,dxdy=0,\qquad\forall\,k\in\{1,\dots,n-1\},$$

where $\psi(x) := v(x) - \sum_{j=1}^{n-1} \left(\int_{\mathbb{R}^N} V(y) v e_j \, dy\right) e_j(x)$. This implies that $\psi \in X_n$.

Thus, for each $v \in \mathcal{D}_0^{s,2}(\mathbb{R}^N)$ relation (3.3) holds true for $\varphi = \psi$. On the other hand,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(e_n(x)-e_n(y))(e_k(x)-e_k(y))}{|x-y|^{N+2s}} \, dxdy = \mu_k \int_{\mathbb{R}^N} V(x) e_n e_k \, dx = \mu_n \int_{\mathbb{R}^N} V(x) e_n e_k \, dx = 0,$$

for all $k \in \{1, \ldots, n-1\}$. The above pieces of information yield

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(e_n(x)-e_n(y))(v(x)-v(y))}{|x-y|^{N+2s}} \, dxdy = \mu_n \int_{\mathbb{R}^N} V(x) e_n(x) v(x) \, dx, \qquad \forall\, v \in \mathcal{D}_0^{s,2}(\mathbb{R}^N),$$

which implies that

$$\mu_n := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|e_n(x)-e_n(y)|^2}{|x-y|^{N+2s}} \, dxdy$$

is an eigenvalue of problem (1.2) with the corresponding eigenfunction e_n.

Next, we point out that by construction $\{e_n\}_n$ is an orthonormal sequence in $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$ and $\{\mu_n\}_n$ is an increasing sequence of positive real numbers. We prove that $\lim_{n\to\infty} \mu_n = \infty$.

Indeed, let the sequence $f_n := \frac{e_n}{\sqrt{\mu_n}}$. Then $\{f_n\}_n$ is an orthonormal sequence in $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$ and

$$\|f_n\|_{s,2}^2 = \frac{1}{\mu_n} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|e_n(x)-e_n(y)|^2}{|x-y|^{N+2s}} \, dxdy = 1, \quad \forall\, n.$$

Consequently, $\{f_n\}_n$ is bounded in $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$ and, therefore, there exists $f \in \mathcal{D}_0^{s,2}(\mathbb{R}^N)$ such that $\{f_n\}_n$ converges weakly to f in $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$.

Let m be a positive integer. For each $n > m$ we have

$$\langle f_n, f_m \rangle_{s,2} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(f_n(x)-f_n(y))(f_m(x)-f_m(y))}{|x-y|^{N+2s}} \, dxdy = 0.$$

Passing to the limit as $n \to \infty$ we find that

$$\langle f, f_m \rangle_{s,2} = 0, \qquad \forall\, m.$$

Since the above relation holds for each positive integer m, we can pass to the limit as $m \to \infty$ and we find that $\|f\|_{s,2} = 0$. This means that $f = 0$ and thus, $\{f_n\}_n$ converges weakly to 0 in $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$. Lemma 2.1 assures us that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} V^+(x) f_n^2 \, dx = 0. \tag{3.4}$$

On the other hand, for each n we have

$$\frac{1}{\mu_n} = \frac{1}{\mu_n} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f_n(x)-f_n(y)|^2}{|x-y|^{N+2s}} \, dxdy = \int_{\mathbb{R}^N} V(x) f_n^2 \, dx \le \int_{\mathbb{R}^N} V^+(x) f_n^2 \, dx.$$

Combining the above estimate with relation (3.4) we find that $\lim_{n\to\infty} \mu_n = +\infty$.

The proof of Proposition 3.2 is complete. □

4 Proof of Theorem 1.5

The proof of Theorem 1.5 will be a simple consequence of Propositions 4.1, 4.2, 4.3 and 4.8 stated below in this section.

We recall that through this section we will assume that $V(x) \geq 0$, for all $x \in \mathbb{R}^N$, and conditions (1.4) and $(\widetilde{\mathrm{V}})$ hold true. Simple computations show that condition (1.4) implies $p > 2$. For each $0 < t < s < 1$ and $p > 2$ we define

$$\nu_1 := \inf_{u \in C_0^\infty(\mathbb{R}^N)\setminus\{0\}} \frac{\frac{1}{2}\|u\|_{s,2}^2 + \frac{1}{p}\|u\|_{t,p}^p}{\frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2\,dx}. \tag{4.1}$$

Proposition 4.1. $\lambda_1 = \nu_1$.

Proof. First, it is clear that $\lambda_1 \leq \nu_1$. Next, for each $u \in C_0^\infty(\mathbb{R}^N)$ and each $\theta > 0$ we have

$$\nu_1 \leq \frac{\frac{1}{2}\|\theta u\|_{s,2}^2 + \frac{1}{p}\|\theta u\|_{t,p}^p}{\frac{1}{2}\int_{\mathbb{R}^N} V(x)(\theta u)^2\,dx} = \frac{\frac{1}{2}\|u\|_{s,2}^2 + \frac{\theta^{p-2}}{p}\|u\|_{t,p}^p}{\frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2\,dx}. \tag{4.2}$$

Letting $\theta \to 0^+$ and passing to the infimum over $u \in C_0^\infty(\mathbb{R}^N)$ in the right hand-side of the above relation we deduce that $\nu_1 \leq \lambda_1$. The proof of this proposition is complete. □

Proposition 4.2. *For each $\lambda \in (-\infty, \lambda_1]$, problem (1.3) has no nontrivial solutions.*

Proof. First, note that if we assume that for some $\lambda \leq 0$ problem (1.3) has a nontrivial solution denoted by u, then testing in relation (1.9) with $\varphi = u$ we get a contradiction. Thus, for any $\lambda \in (-\infty, 0]$ problem (1.3) does not have nontrivial weak solutions.

Next, let $\lambda \in (0, \lambda_1)$. Assume by contradiction that there exists $u \in \mathcal{D}_0^{s,2}(\mathbb{R}^N) \setminus \{0\}$ a weak solution of problem (1.3). Taking $\varphi = u$ in (1.9) and by the definition of λ_1 we get

$$\lambda \int_{\mathbb{R}^N} V(x)u(x)^2\,dx = \|u\|_{s,2}^2 + \|u\|_{t,p}^p \geq \lambda_1 \int_{\mathbb{R}^N} V(x)u(x)^2\,dx,$$

a contradiction. It follows that problem (1.3) does not posses nontrivial weak solutions for any parameter $\lambda \in (0, \lambda_1)$.

In order to complete the proof of the proposition, we shall show that λ_1 cannot be an eigenvalue of problem (1.3). Again, if we assume by contradiction that there exists $u \in \mathcal{D}_0^{s,2}(\mathbb{R}^N) \setminus \{0\}$ such that (1.9) holds with $\lambda = \lambda_1$, then letting $\varphi = u$ in (1.9) and by the definition of λ_1 we get

$$\|u\|_{s,2}^2 + \|u\|_{t,p}^p = \lambda_1 \int_{\mathbb{R}^N} V(x)u(x)^2\,dx \leq \|u\|_{s,2}^2,$$

which is equivalent with $u \equiv 0$, a contradiction. Thus, for $\lambda = \lambda_1$ problem (1.3) does not have nontrivial solutions and thus, the proof of this proposition is now complete. □

Proposition 4.3. *For each $\lambda \in (\lambda_1, \infty)$ problem (1.3) has a nontrivial solution.*

In order to prove Proposition 4.3, for each $\lambda > \lambda_1$ we define the energy functional corresponding to problem (1.3) as $J_\lambda : \mathcal{D}_0^{s,2}(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}$ given by

$$J_\lambda(u) := \frac{1}{2} \|u\|_{s,2}^2 + \frac{1}{p} \|u\|_{t,p}^p - \frac{\lambda}{2} \int_{\mathbb{R}^N} V(x)u(x)^2 \, dx.$$

Using standard arguments one can deduce that $J_\lambda \in C^1(\mathcal{D}_0^{s,2}(\mathbb{R}^N), \mathbb{R})$ with the derivative given by

$$\begin{aligned} \langle J'_\lambda(u), w\rangle = & \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(w(x)-w(y))}{|x-y|^{N+2s}} \, dxdy - \lambda \int_{\mathbb{R}^N} V(x)u(x)w(x) \, dx. \\ & + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(w(x)-w(y))}{|x-y|^{N+tp}} \, dxdy. \end{aligned}$$

We note that problem (1.3) possesses a nontrivial weak solution for a certain λ if and only if J_λ possesses a non-trivial critical point. Since we cannot establish the coercivity of J_λ on $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$ we cannot apply the *Direct Method in the Calculus of Variations* in order to find critical points for this functional. For that reason we will study the functional J_λ on a subset of $\mathcal{D}^{s,2}(\mathbb{R}^N)$, the so-called *Nehari manifold* defined by

$$\begin{aligned} \mathcal{N}_\lambda &:= \left\{ u \in \mathcal{D}_0^{s,2}(\mathbb{R}^N) \setminus \{0\} : \langle J'_\lambda(u), u\rangle = 0 \right\} \\ &= \left\{ u \in \mathcal{D}_0^{s,2}(\mathbb{R}^N) \setminus \{0\} : \|u\|_{s,2}^2 + \|u\|_{t,p}^p = \lambda \int_{\mathbb{R}^N} V(x)u(x)^2 \, dx \right\}. \end{aligned}$$

Note that if $u \in \mathcal{N}_\lambda$ then

$$J_\lambda(u) = \left(\frac{1}{p} - \frac{1}{2}\right) \|u\|_{t,p}^p < 0 \tag{4.3}$$

and

$$\lambda \int_{\mathbb{R}^N} V(x)u(x)^2 \, dx > \|u\|_{s,2}^2 \, . \tag{4.4}$$

Lemma 4.4. $\mathcal{N}_\lambda \neq \emptyset$.

Proof. Since $\lambda > \lambda_1$, we infer that there exists $\varphi \in \mathcal{D}_0^{s,2}(\mathbb{R}^N) \setminus \{0\}$ for which

$$\|\varphi\|_{s,2}^2 < \lambda \int_{\mathbb{R}^N} V(x)\varphi(x)^2 \, dx.$$

Then there exists $\theta > 0$ such that $\theta\varphi \in \mathcal{N}_\lambda$, i.e.

$$\theta^2 \|\varphi\|_{s,2}^2 + \theta^p \|\varphi\|_{t,p}^p = \lambda\theta^2 \int_{\mathbb{R}^N} V(x)\varphi(x)^2 \, dx,$$

which holds true with

$$\theta = \left(\frac{\lambda \int_{\mathbb{R}^N} V(x)\varphi(x)^2 \, dx - \|\varphi\|_{s,2}^2}{\|\varphi\|_{t,p}^p} \right)^{\frac{1}{p-2}},$$

which completes the proof. □

Set

$$m_\lambda := \inf_{v\in\mathcal{N}_\lambda} J_\lambda(v).$$

Note that by (4.3) we know that $m_\lambda < 0$. We show that m_λ can be achieved on $\mathcal{N}_\lambda$.

Lemma 4.5. *Every minimizing sequence of functional J_λ on $\mathcal{N}_\lambda$ is bounded in $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$ and $\mathcal{D}_0^{t,p}(\mathbb{R}^N)$.*

Proof. Let $\{u_n\}_n \subset \mathcal{N}_\lambda$ be a minimizing sequence J_λ on $\mathcal{N}_\lambda$. We prove that $\{\, \|u_n\|_{s,2}^2 \,\}_n$ is a bounded sequence. Assume the contrary that $\|u_n\|_{s,2}^2 \to \infty$, as $n \to \infty$. Next, let $w_n := \frac{u_n}{\|u_n\|_{s,2}}$. Therefore $\|w_n\|_{s,2} = 1$ for each n, which means that $\{w_n\}_n$ is bounded in $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$. Thus, there exists $w \in \mathcal{D}_0^{s,2}(\mathbb{R}^N)$ such that w_n converges weakly to w in $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$.

Since $u_n \in \mathcal{N}_\lambda$, for each n, by (4.4) we deduce that $\lambda \int_{\mathbb{R}^N} V(x) w_n^2\, dx > 1$. Passing to the limit as $n \to \infty$ and taking into account Lemma 2.1, we obtain that

$$\lambda \int_{\mathbb{R}^N} V(x) w^2\, dx \geq 1. \tag{4.5}$$

On the other hand, since $u_n \in \mathcal{N}_\geq$ and $p > 2$, we get

$$\|w_n\|_{t,p}^p = \|u_n\|_{s,2}^{2-p} \left(\lambda \int_{\mathbb{R}^N} V(x) w_n^2\, dx - 1\right) \to 0, \quad \text{as } n \to \infty.$$

The above relation implies that w_n converges strongly to 0 in $\mathcal{D}_0^{t,p}(\mathbb{R}^N)$ and, consequently $w = 0$, which represents a contradiction with (4.5). It follows that $\{\, \|u_n\|_{s,2} \,\}_n$ is bounded. Since $u_n \in \mathcal{N}_\lambda$, by relation (4.3) we deduce that

$$J_\lambda(u_n) = \left(\frac{1}{p} - \frac{1}{2}\right) \|u_n\|_{t,p}^p = \left(\frac{1}{2} - \frac{1}{p}\right) \left(\|u_n\|_{s,2}^2 - \int_{\mathbb{R}^N} V(x) u_n^2\, dx\right).$$

Since $\{\, \|u_n\|_{s,2} \,\}_n$ is a bounded sequence and using the weak continuity of the mapping $\mathcal{D}_0^{s,2}(\mathbb{R}^N) \ni u \to \int_{\mathbb{R}^N} V(x) u^2\, dx$ given by Lemma 2.1, we deduce that $\{\, \|u_n\|_{t,p} \,\}_n$ is also a bounded sequence, and thus, the proof is complete. □

Lemma 4.6. *$m_\lambda \in (-\infty, 0)$.*

Proof. We already know that $m_\lambda < 0$. Let $\{u_n\}_n \subset \mathcal{N}_\lambda$ be a minimizing sequence J_λ on $\mathcal{N}_\lambda$ (in other words, $\{u_n\}_n$ is a minimizer of m_λ). Using the previous lemma we deduce the existence of a positive constant C such that $\|u_n\|_{s,2}^2 \leq C$ and $\|u_n\|_{t,p}^p \leq C$, for each positive integer n. Since $p > 2$ we have

$$J_\lambda(u_n) = \left(\frac{1}{p} - \frac{1}{2}\right) \|u_n\|_{t,p}^p \geq \left(\frac{1}{p} - \frac{1}{2}\right) C > -\infty.$$

Thus, m_λ is bounded from below by the constant $(\frac{1}{p} - \frac{1}{2})C$, which implies that $m_\lambda \in (-\infty, 0)$. This completes the proof of this lemma. □

Lemma 4.7. *There exists $u \in \mathcal{N}_\lambda$ such that $J_\lambda(u) = m_\lambda$.*

Proof. Let $\{u_n\}_n \subset \mathcal{N}_\lambda$ be a minimizing sequence for J_λ on $\mathcal{N}_\lambda$, i.e.

$$J_\lambda(u_n) = \left(\frac{1}{p} - \frac{1}{2}\right) \|u_n\|_{t,p}^p \to m_\lambda \quad \text{as} \quad n \to \infty.$$

By Lemma 4.5, we have that $\mathcal{N}_\lambda$ is bounded in $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$ and $\mathcal{D}_0^{t,p}(\mathbb{R}^N)$. We deduce that there exists a function $u \in \mathcal{D}_0^{s,2}(\mathbb{R}^N)$ such that u_n converges weakly to u in $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$ and also in $\mathcal{D}_0^{t,q}(\mathbb{R}^N)$. Then

$$\|u\|_{s,2}^2 \le \liminf_{n\to\infty} \|u_n\|_{s,2}^2 .$$

By Lemma 1 we deduce that

$$\lambda \int_{\mathbb{R}^N} V(x)u_n(x)^2\, dx \to \lambda \int_{\mathbb{R}^N} V(x)u(x)^2\, dx \quad \text{as } n \to \infty.$$

Using the above pieces of information we obtain

$$\begin{aligned} J_\lambda(u) &= \left(\frac{1}{2} - \frac{1}{p}\right)\left(\|u\|_{s,2}^2 - \lambda \int_{\mathbb{R}^N} V(x)u^2\, dx\right) \\ &\le \left(\frac{1}{2} - \frac{1}{p}\right) \liminf_{n\to\infty} \left(\|u_n\|_{s,2}^2 - \lambda \int_{\mathbb{R}^N} V(x)u_n^2\, dx\right) \\ &= \liminf_{n\to\infty} J_\lambda(u_n) = m_\lambda < 0. \end{aligned} \tag{4.6}$$

By the above calculus we deduce that

$$\|u\|_{s,2}^2 < \lambda \int_{\mathbb{R}^N} V(x)u^2\, dx,$$

which implies that certainly $u \not\equiv 0$. Since $u_n \in \mathcal{N}_\lambda$ for every n, we have

$$\|u_n\|_{s,2}^2 + \|u_n\|_{t,p}^p = \lambda \int_{\mathbb{R}^N} V(x)u_n^2\, dx.$$

Passing to the limit as $n \to \infty$ in the above relation and by weakly convergence of u_n to u in $\mathcal{D}_0^{s,2}(\mathbb{R}^N)$ and $\mathcal{D}_0^{t,p}(\mathbb{R}^N)$ and also by Lemma 2.1, we get

$$\|u\|_{s,2}^2 + \|u\|_{t,p}^p \le \lambda \int_{\mathbb{R}^N} V(x)u(x)^2\, dx. \tag{4.7}$$

In order to finish the proof, we show that the above relation is actually an equality. Assume by contradiction that the inequality in (4.7) is strict, i.e.

$$\|u\|_{s,2}^2 + \|u\|_{t,p}^p < \lambda \int_{\mathbb{R}^N} V(x)u^2\, dx. \tag{4.8}$$

Set

$$\theta := \left(\frac{\lambda \int_{\mathbb{R}^N} V(x)u^2\, dx - \|u\|_{s,2}^2}{\|u\|_{t,p}^p}\right)^{\frac{1}{p-2}}.$$

Since $u \in \mathcal{N}_\lambda$ we have that $\theta u \in \mathcal{N}_\lambda$. By (4.8) it is clear that $\theta > 1$. Since $p > 2$ we deduce that

$$\begin{aligned} J_\lambda(\theta u) &= \left(\frac{1}{2} - \frac{1}{p}\right) \theta^2 \left(\|u\|_{s,2}^2 - \lambda \int_{\mathbb{R}^N} V(x)u^2\, dx\right) \\ &< \left(\frac{1}{2} - \frac{1}{p}\right)\left(\|u\|_{s,2}^2 - \lambda \int_{\mathbb{R}^N} V(x)u^2\, dx\right) = J_\lambda(u) \\ &\le \liminf_{n\to\infty} J_\lambda(u_n) = m_\lambda, \end{aligned}$$

a contradiction. Thus, relation (4.8) cannot hold true. Therefore, relation (4.7) holds as an equality which implies that $u \in \mathcal{N}_\lambda$. By relation (4.6) we know that $J_\lambda(u) \leq m_\lambda$, and thus $J_\lambda(u) = m_\lambda$. Thus, the proof is complete. □

We are now ready to complete the proof of Proposition 4.1. Let u_λ be the minimizer of J_λ over $\mathcal{N}_\lambda$ given by Lemma 4.7, i.e.

$$J_\lambda(u_\lambda) = m_\lambda.$$

Since $u_\lambda \in \mathcal{N}_\lambda$, we have

$$\|u_\lambda\|_{s,2}^2 + \|u_\lambda\|_{t,p}^p = \lambda \int_{\mathbb{R}^N} V(x) u_\lambda^2 \, dx$$

and

$$\|u_\lambda\|_{s,2}^2 < \lambda \int_{\mathbb{R}^N} V(x) u_\lambda^2 \, dx.$$

We consider $\varphi \in \mathcal{D}_0^{s,2}(\mathbb{R}^N)$ is arbitrary but fixed, and $\delta > 0$ is sufficiently small such that for each $\epsilon \in (-\delta, \delta)$ the function $u_\lambda + \epsilon\varphi \not\equiv 0$ in $\mathbb{R}^N$ and

$$\|u_\lambda + \epsilon\varphi\|_{s,2}^2 < \lambda \int_{\mathbb{R}^N} V(x) |u_\lambda + \epsilon\varphi|^2 \, dx.$$

Define $\theta : (-\delta, \delta) \to (0, \infty)$ as

$$\theta(\epsilon) := \left(\frac{\lambda \int_{\mathbb{R}^N} V(x) |u_\lambda + \epsilon\varphi|^2 \, dx - \|u_\lambda + \epsilon\varphi\|_{s,2}^2}{\|u_\lambda + \epsilon\varphi\|_{t,p}^p} \right)^{\frac{1}{p-2}}.$$

We observe that $\theta(\epsilon)(u_\lambda + \epsilon\varphi) \in \mathcal{N}_\lambda$ and θ is a differentiable as a composition of some differentiable functions. Since $u_\lambda \in \mathcal{N}_\lambda$ we infer that $\theta(0) = 1$. Next, let $\gamma : (-\delta, \delta) \to \mathbb{R}$ be given by $\gamma(\epsilon) := J_\lambda(\theta(\epsilon)(u_\lambda + \epsilon\varphi))$. Clearly, $\gamma \in C^1(-\delta, \delta)$ and $m_\lambda = \gamma(0) \leq \gamma(\epsilon)$, for each $\epsilon \in (-\delta, \delta)$. Thus, we have

$$\begin{aligned} 0 = \gamma'(0) &= \langle J'(\theta(0)u_\lambda), \theta'(0)u_\lambda + \theta(0)\varphi \rangle \\ &= \theta'(0)\langle J'(u_\lambda), u_\lambda \rangle + \langle J'(u_\lambda), \varphi \rangle \\ &= \langle J'(u_\lambda), \varphi \rangle, \end{aligned}$$

where the last equality holds because $u_\lambda \in \mathcal{N}_\lambda$.

Since $\varphi \in \mathcal{D}_0^{s,2}(\mathbb{R}^N)$ was arbitrarily chosen we deduce that the last relation holds true for each $\varphi \in \mathcal{D}_0^{s,2}(\mathbb{R}^N)$ and thus, u_λ is a nontrivial critical point of J_λ, and consequently a nontrivial weak solution of equation (1.3). The proof of Proposition 4.1 is now complete.

Proposition 4.8. *If $u \in \mathcal{N}_\lambda$ is the minimizer of J_λ over $\mathcal{N}_\lambda$, given by Lemma 4.7, then $|u|$ is also a minimizer of I_λ over $\mathcal{N}_\lambda$.*

Proof. For each $\xi \in \mathcal{D}_0^{s,2}(\mathbb{R}^N)$ and for any $x, y \in \mathbb{R}^N$ we have

$$|\xi(y) - \xi(x)| \geq |\,|\xi(y)| - |\xi(y)|\,|,$$

and

$$|\xi(y) - \xi(x)| > |\,|\xi(y)| - |\xi(y)|\,|, \quad \text{if } \xi(x)\xi(y) < 0.$$

Using this, it follows that

$$\|\,|\xi|\,\|_{s,2}^2 \leq \|\xi\|_{s,2}^2 \quad \text{and} \quad \|\,|\xi|\,\|_{t,p}^p \leq \|\xi\|_{t,p}^p \qquad \forall \xi \in \mathcal{D}_0^{s,2}(\mathbb{R}^N).$$

By the above relation we deduce that

$$J_\lambda(|u|) \leq J_\lambda(u). \tag{4.9}$$

In what follows we will prove that $J_\lambda(|u|) \geq J_\lambda(u)$. We distinguish two cases. First, if $|u| \in \mathcal{N}_\lambda$ then taking into account that $p > 2$ we get

$$J_\lambda(|u|) = \left(\frac{1}{p} - \frac{1}{2}\right) \| |u| \|_{t,p}^p \geq \left(\frac{1}{p} - \frac{1}{2}\right) \|u\|_{t,p}^p = J_\lambda(u).$$

The above estimate and relation (4.9) yield $J_\lambda(|u|) = J_\lambda(u) = m_\lambda$ and everything is done.

Next, let us assume that $|u| \notin \mathcal{N}_\lambda$. Then

$$\| |u| \|_{s,2}^2 + \| |u| \|_{t,p}^p < \lambda \int_{\mathbb{R}^N} V(x) u^2 \, dx.$$

Set

$$\theta := \left(\frac{\lambda \int_{\mathbb{R}^N} V(x) u^2 \, dx - \| |u| \|_{s,2}^2}{\| |u| \|_{t,p}^p} \right)^{\frac{1}{p-2}}.$$

Since $p > 2$ we have that $\theta \in (1, \infty)$ and also $\theta |u| \in \mathcal{N}_\lambda$. We have that

$$\begin{aligned} m_\lambda \leq J_\lambda(\theta|u|) &= \left(\frac{1}{p} - \frac{1}{2}\right) \| |u| \|_{t,p}^p \, \theta^p \\ &= \left(\frac{1}{2} - \frac{1}{p}\right) \left(\| |u| \|_{s,2}^2 - \lambda \int_{\mathbb{R}^N} V(x) u^2 \, dx \right) \theta^2 \\ &< \left(\frac{1}{2} - \frac{1}{p}\right) \left(\| |u| \|_{s,2}^2 - \lambda \int_{\mathbb{R}^N} V(x) u^2 \, dx \right) \\ &\leq \left(\frac{1}{2} - \frac{1}{p}\right) \left(\|u\|_{s,2}^2 - \lambda \int_{\mathbb{R}^N} V(x) u^2 \, dx \right) = J_\lambda(u) = m_\lambda, \end{aligned}$$

which is a contradiction. Thus, $|u| \in \mathcal{N}_\lambda$. It follows that $|u|$ is also a minimizer of J_λ over $\mathcal{N}_\lambda$.

□

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References

[1] M. Bocea, M. Mihăilescu, Existence of nonnegative viscosity solutions for a class of problems involving the ∞-Laplacian, *NoDEA Nonlinear Differential Equations Appl.* **23**(2016), No. 2, Art. 11, 21 pp. https://doi.org/10.1007/s00030-016-0373-2; MR3478287

[2] L. Brasco, G. Franzina, Convexity properties of Dirichlet integrals and Picone-type inequalities, *Kodai Math. J.* **37**(2014), 769–799. https://doi.org/10.2996/kmj/1414674621; MR3273896

[3] L. Brasco, E. Parini, M. Squassina, Stability of variational eigenvalues for the fractional p-Laplacian, *Discrete Contin. Dyn. Syst.* **36**(2016), 1813–1845. https://doi.org/10.3934/dcds.2016.36.1813; MR3411543

[4] L. Del Pezzo, R. Ferreira, J. D. Rossi, Eigenvalues for a combination of local and nonlocal p-Laplacians, *Fract. Calc. Appl. Anal.* **22**(2019), 1414–1436. https://doi.org/10.1515/fca-2019-0074; MR4044581

[5] L. Del Pezzo, A. Quaas, Global bifurcation for fractional p-Laplacian and an application, *Z. Anal. Anwend.* **35**(2016), 411–447. https://doi.org/10.4171/ZAA/1572; MR3556755

[6] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math.* **136**(2012), 521–573. https://doi.org/10.1016/j.bulsci.2011.12.004; MR2944369

[7] G. Ercole, G. A. Pereira, R. Sanchis, Asymptotic behavior of extremals for fractional Sobolev inequalities associated with singular problems, *Ann. Mat. Pura Appl. (4)* **198**(2019), 2059–2079. https://doi.org/10.1007/s10231-019-00854-9; MR4031839

[8] M. Fărcăşeanu, On an eigenvalue problem involving the fractional (s,p)-Laplacian, *Fract. Calc. Appl. Anal.* **21**(2018), 94–103. https://doi.org/10.1515/fca-2018-0006; MR3776055

[9] M. Fărcăşeanu, M. Mihăilescu, D. Stancu-Dumitru, On the set of eigenvalues of some PDEs with homogeneous Neumann boundary condition, *Nonlinear Anal.* **116**(2015), 19–25. https://doi.org/10.1016/j.na.2014.12.019; MR3311349

[10] M. Fărcăşeanu, M. Mihăilescu, D. Stancu-Dumitru, Perturbated fractional eigenvalue problems, *Discrete Contin. Dyn. Syst.* **37**(2017), 6243–6255. https://doi.org/10.3934/dcds.2017270; MR3690302

[11] R. Ferreira, M. Pérez-Llanos, Limit problems for a fractional p-Laplacian as $p \to \infty$, *NoDEA Nonlinear Differential Equations Appl.* **23**(2016), No. 14, 28 pp. https://doi.org/10.1007/s00030-016-0368-z; MR3478965

[12] R. L. Frank, E. Lenzmann, L. Silvestre, Uniqueness of radial solutions for the fractional Laplacian, *Comm. Pure Appl. Math.* **69**(2016), 1671–1726. https://doi.org/10.1002/cpa.21591; MR3530361

[13] G. Franzina, G. Palatucci, Fractional p-eigenvalues, *Riv. Mat. Univ. Parma* **5**(2014), 315–328. MR3307955

[14] P. Grisvard, *Elliptic problems in nonsmooth domains*, Pitman, Boston, MA, 1985. MR0775683

[15] E. Lindgren, P. Lindqvist, Fractional eigenvalues, *Calc. Var. Partial Differential Equations* **49**(2014), 795–826. https://doi.org/10.1007/s00526-013-0600-1; MR3148135

[16] M. Mihăilescu, An eigenvalue problem possessing a continuous family of eigenvalues plus an isolated eigenvalue, *Commun. Pure Appl. Anal.* **10**(2011), 701–708. https://doi.org/10.3934/cpaa.2011.10.701; MR2754297

[17] M. Mihăilescu, G. Moroşanu, Eigenvalues of $-\Delta_p - \Delta_q$ under Neumann boundary condition, *Canad. Math. Bull.* **59**(2016), 606–616. https://doi.org/10.4153/CMB-2016-025-2; MR3563742

[18] M. Mihăilescu, D. Repovš, An eigenvalue problem involving a degenerate and singular elliptic operator, *Bull. Belg. Math. Soc. Simon Stevin* **18**(2011), No. 5, 839–847. https://doi.org/10.36045/bbms/1323787171; MR2918650

[19] M. Mihăilescu, D. Stancu-Dumitru, A perturbed eigenvalue problem on general domains, *Ann. Funct. Anal.* **7**(2016), 529–542. https://doi.org/10.1215/20088752-3660738; MR3543145

[20] H.-M. Nguyen, M. Squassina, Fractional Caffarelli–Kohn–Nirenberg inequalities, *J. Funct. Anal.* **274**(2018), 2661–2672. https://doi.org/10.1016/j.jfa.2017.07.007; MR3771839

[21] A. Szulkin, M. Willem, Eigenvalue problems with indefinite weights, *Studia Math.* **135**(1999), 191–201. MR1690753