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# Existence of weak solutions for quasilinear Schrödinger equations with a parameter

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**Abstract.** In this paper, we study the following quasilinear Schrödinger equation of the form

$$-\Delta_p u + V(x)|u|^{p-2}u - \left[\Delta_p (1+u^2)^{\alpha/2}\right] \frac{\alpha u}{2(1+u^2)^{(2-\alpha)/2}} = k(u), \qquad x \in \mathbb{R}^N,$$

where $p$-Laplace operator $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ $(1 < p \le N)$ and $\alpha \ge 1$ is a parameter. Under some appropriate assumptions on the potential $V$ and the nonlinear term $k$, using some special techniques, we establish the existence of a nontrivial solution in $C^{1,\beta}_{\text{loc}}(\mathbb{R}^N)$ $(0 < \beta < 1)$, we also show that the solution is in $L^\infty(\mathbb{R}^N)$ and decays to zero at infinity when $1 < p < N$.

**Keywords:** quasilinear Schrödinger equation, variational method, mountain-pass theorem, $p$-Laplace operator.

**2020 Mathematics Subject Classification:** 35J62, 35J20, 35Q55.

## 1 Introduction

In this work, we are interested in the existence of nontrivial solution to the following quasilinear Schrödinger equation

$$-\Delta_p u + V(x)|u|^{p-2}u - \left[\Delta_p (1+u^2)^{\alpha/2}\right] \frac{\alpha u}{2(1+u^2)^{(2-\alpha)/2}} = k(u), \qquad x \in \mathbb{R}^N, \tag{1.1}$$

where $p$-Laplace operator $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ $(1 < p \le N)$ and $\alpha \ge 1$ is a parameter. $V$ is a positive continuous potential and $k(u)$ is a nonlinear term of subcritical type.

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Such equations arise in various branches of mathematical physics. For instance, solutions of equation (1.1), in the case $p = 2$ and $\alpha = 1$ are closed related to the existence of solitary wave solutions for quasilinear Schrödinger equations

$$iz_t = -\Delta z + W(x)z - \tilde{k}(|z|^2)z - \Delta l(|z|^2)l'(|z|^2)z, \qquad x \in \mathbb{R}^N, \tag{1.2}$$

where $z : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential, $\tilde{k}$, $l : \mathbb{R}^+ \to \mathbb{R}$ are real functions. The form of (1.2) has been derived as models of several physical phenomena corresponding to various types of $l$. For instance, the case $l(s) = s$ models the time evolution of the condensate wave function in super-fluid film [15, 16], and is called the superfluid film equation in fluid mechanics by Kurihara [15]. In the case $l(s) = (1+s)^{1/2}$, problem (1.2) models the self-channeling of a high-power ultra short laser in matter, the propagation of a high-irradiance laser in a plasma creates an optical index depending nonlinearly on the light intensity and this leads to interesting new nonlinear wave equation (see [2, 4, 8, 28]). For more physical motivations and more references dealing with applications, we refer the reader to [1, 13, 17, 25–27] and references therein.

It is well known that, via the ansatz $z(t, x) = \exp(-iEt)u(x)$, where $E \in \mathbb{R}$ and $u$ is a real function, (1.2) can be reduced to the following elliptic equation

$$-\Delta u + V(x)u - \left[\Delta(l(u^2))\right] l'(u^2)u = k(u), \qquad x \in \mathbb{R}^N, \tag{1.3}$$

where $V(x) = W(x) - E$ and $k(u) = \tilde{k}(u^2)u$.

If we take $l(s) = s$ in (1.3), then we obtain the superfluid film equation in plasma physics

$$-\Delta u + V(x)u - \left[\Delta(u^2)\right] u = k(u), \qquad x \in \mathbb{R}^N. \tag{1.4}$$

Clearly, when $p = 2$ and $\alpha = 2$, equation (1.1) turns into equation (1.4). Equation (1.4) has been paid much attention in the past two decades. Many existence and multiplicity results of nontrivial solutions have been established by differential methods such as constrained minimization argument, changes of variables, Nehari method, a dual approach, perturbation method, see [7, 12, 14, 20–24, 26, 29, 31] and references therein.

If we take $l(s) = (1+s)^{1/2}$ in (1.3), then we get the equation

$$-\Delta u + V(x)u - \left[\Delta(1+u^2)^{1/2}\right] \frac{u}{2(1+u^2)^{1/2}} = k(u), \qquad x \in \mathbb{R}^N, \tag{1.5}$$

which models the self-channeling of a high-power ultrashort laser in matter. Obviously, equation (1.1) turns into (1.5) for the case $p = 2$ and $\alpha = 1$.

The existence of positive solutions for (1.5) has been studied recently. In [32], by a change of variables and the Ambrosetti–Rabinowitz mountain-pass theorem, the authors proved that (1.5) has a positive solution. They assume that the potential $V \in C(\mathbb{R}^N, \mathbb{R})$ and the nonlinearity $k : \mathbb{R} \to \mathbb{R}$ is Hölder continuous and satisfy the following conditions:

($V_1$) $V(x) \geq V_0 > 0$, for all $x \in \mathbb{R}^N$;

($V_2$) $\lim_{|x|\to\infty} V(x) = V(\infty) < \infty$ and $V(x) \leq V(\infty)$, for all $x \in \mathbb{R}^N$;

($H_1$) $k(s) = 0$ if s $\leq 0$;

($H_2$) $k(s) = o(s)$ as $s \to 0^+$;

($H_3$) There exists $2 < \theta < 2^*$ such that $|k(s)| \leq C(1 + |s|^{\theta-1})$;

($H_4$) There exists $\mu > \sqrt{6}$ such that $0 < \mu K(s) \leq sk(s)$ for all $s > 0$, where $K(s) = \int_0^s k(t)dt$.

In [5], by a dual approach, the authors studied the existence of positive solution for the following equation

$$-\Delta u+Ku-\left[\Delta(1+u^2)^{\alpha/2}\right]\frac{\alpha u}{2(1+u^2)^{(2-\alpha)/2}}=|u|^{q-1}u+|u|^{p-1}u,\qquad x\in\mathbb{R}^N, \tag{1.6}$$

where $K>0$, $N\geq 3,\alpha\geq 1$ and $2<q+1<p+1<\alpha 2^*$. Similar works can be found in [3,6,18,22] and reference therein.

However, to the best of our knowledge, in all works mentioned above, there are no existence results in the literature on the case $p\neq 2,\alpha\geq 1$ and the nonlinear term becomes general function. Motivated by the works mentioned above and [5,7,20,22,31,32], our purpose in this paper is to study the existence of nontrivial weak solutions of (1.1) under some assumptions on the potential $V(x)$ and nonlinear term $k(s)$.

**Definition 1.1.** We say that $u:\mathbb{R}^N\to\mathbb{R}$ is a weak solution of (1.1) if $u\in W^{1,p}(\mathbb{R}^N)\cap L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ and

$$\begin{aligned}\int_{\mathbb{R}^N}&\left[1+\frac{\alpha^p|u|^p}{2(1+u^2)^{(2-\alpha)p/2}}\right]|\nabla u|^{p-2}\nabla u\nabla\psi dx\\&+\frac{\alpha^p}{2}\int_{\mathbb{R}^N}\frac{[1+(\alpha-1)u^2]}{(1+u^2)^{1+(2-\alpha)p/2}}|\nabla u|^p|u|^{p-2}u\psi dx\\&=\int_{\mathbb{R}^N}\eta(x,u)\psi dx,\ \forall\psi\in C_0^\infty(\mathbb{R}^N),\end{aligned} \tag{1.7}$$

where $\eta(x,u)=k(u)-V(x)|u|^{p-2}u$.

In such a case, we can deduce formally that the Euler–Lagrange functional associated with the equation (1.1) is

$$J(u)=\frac{1}{p}\int_{\mathbb{R}^N}\left[1+\frac{\alpha^p|u|^p}{2(1+u^2)^{(2-\alpha)p/2}}\right]|\nabla u|^p dx+\frac{1}{p}\int_{\mathbb{R}^N}V(x)|u|^p dx-\int_{\mathbb{R}^N}K(u)dx,$$

where $K(s)=\int_0^s k(t)dt$.

For (1.1), due to the appearance of the nonlocal term $\int_{\mathbb{R}^N}\frac{\alpha^p|u|^p}{2(1+u^2)^{(2-\alpha)p/2}}|\nabla u|^p dx$, $J$ may be not well defined. To overcome this difficulty, enlightened by [7,20,32], we make a change of variables as

$$v=H(u)=\int_0^u h(t)dt, \tag{1.8}$$

where $h(t)=\left[1+\frac{\alpha^p|t|^p}{2(1+t^2)^{(2-\alpha)p/2}}\right]^{1/p}$, $t\in\mathbb{R}$. Since $H(t)$ is strictly increasing on $\mathbb{R}$, the inverse function $H^{-1}(t)$ of $H(t)$ exists. Then after the change of variables, $J(u)$ can be written by

$$\mathcal{F}(v)=J(H^{-1}(v))=\frac{1}{p}\int_{\mathbb{R}^N}|\nabla v|^p dx+\frac{1}{p}\int_{\mathbb{R}^N}V(x)|H^{-1}(v)|^p dx-\int_{\mathbb{R}^N}K(H^{-1}(v))dx. \tag{1.9}$$

According to Lemma 2.1 and our hypotheses on $V(x)$ and $k(s)$ below, it is clear that $\mathcal{F}$ is well defined in $W^{1,p}(\mathbb{R}^N)$ and $\mathcal{F}\in C^1$. The Euler–Lagrange equation associated to the functional $\mathcal{F}$ is

$$-\Delta_p v=\frac{\eta(x,H^{-1}(v))}{h(H^{-1}(v))},\qquad x\in\mathbb{R}^N. \tag{1.10}$$

In Proposition 2.2, we will show the relationship between the solutions of (1.10) and the solutions of (1.1).

Throughout this paper, let $1<p\leq N$, $\alpha\geq 1$. Besides, we assume that the potential $V(x)\in C(\mathbb{R}^N,\mathbb{R})$ and satisfies $(\mathrm{V}_1)-(\mathrm{V}_2)$, the nonlinearity $k(s)\in C(\mathbb{R},\mathbb{R})$ and satisfies the following conditions:

$(K_1)$ $k$ is odd and $k(s) = o(|s|^{p-2}s)$ as $s \to 0$;

$(K_2)$ There exists a constant $C > 0$ such that

$$|k(s)| \leq C(1 + |s|^{\theta-1}), \qquad \forall s \in \mathbb{R},$$

where $\alpha p < \theta < \alpha p^*$ if $1 < p < N$ and $\theta > \alpha p$ if $p = N$;

$(K_3)$ There exists $\mu \geq \widetilde{T}(p,\alpha)p$ such that $0 < \mu K(s) \leq sk(s)$ for all $s > 0$, where $K(s) = \int_0^s k(t)dt$, $\widetilde{T}(p,\alpha) = 1 + T(p,\alpha)$ and

$$T(p,\alpha) = \sup_{t\geq 0} \frac{th'(t)}{h(t)} = \sup_{t\geq 0} \frac{\alpha^p t^p \left[1 + (\alpha-1)t^2\right]}{(1+t^2)\left[2(1+t^2)^{(2-\alpha)p/2} + \alpha^p t^p\right]} > 0. \tag{1.11}$$

Our main result is the following.

**Theorem 1.2.** *Let $1 < p \leq N$, $\alpha \geq 1$. Suppose $(V_1)$–$(V_2)$ and $(K_1)$–$(K_2)$ hold. Then (1.1) admits a nontrivial weak solution $u \in C^{1,\beta}_{loc}(\mathbb{R}^N)$ $(0 < \beta < 1)$ provided that one of the following conditions is satisfied:*

*(a)* $(K_3)$ *holds with* $\mu > \widetilde{T}(p,\alpha)p$;

*(b)* $(K_3)$ *holds with* $\mu = \widetilde{T}(p,\alpha)p = 2p$ *and* $p < \theta < p^*$ *if* $1 < p < N$ *or* $\theta > p$ *if* $p = N$ *in* $(K_2)$.

*Furthermore, if $1 < p < N$, then $u \in L^\infty(\mathbb{R}^N)$ and $u(x) \to 0$ as $|x| \to \infty$.*

**Remark 1.3.** It is not difficult to verify that $T(p,\alpha) = \alpha - 1$ if $\alpha \geq 2$ and $\alpha - 1 \leq T(p,\alpha) < 1$ if $1 \leq \alpha < 2$. If $p = 2$, then $T(p,\alpha) = T(2,\alpha)$, which equals to the $T(\alpha)$ in [5]. If $p = 2$ and $\alpha = 1$, we obtain $T(2,1) = 5 - 2\sqrt{6}$. Thus, $\mu \geq \widetilde{T}(2,1)2 = (1 + T(2,1))2 \approx 2.202$ in $(K_3)$ is better than $\mu > 2\sqrt{6} \approx 2.449$ in $(H_4)$. If $p = 2$ and $\alpha = 2$, we have $\widetilde{T}(2,2)2 = 4$, which coincides with that in [7]. Therefore, our conclusion in Theorem 1.2 can be viewed as an extension result in [5,7,20,32].

The organization of this paper is as follows. In Section 2, we give some properties of $H(t)$ and some preliminary results. In Section 3, we present an auxiliary problem and some related results. In Section 4, we complete the proof of Theorem 1.2.

Throughout this paper, $C$ and $C_i$ stand for positive constants which may take different values at different places. $B_R$ denotes the open ball centered at the origin and radius $R > 0$, $C_0^\infty(\mathbb{R}^N)$ denotes functions infinitely differentiable with impact support in $\mathbb{R}^N$. For $1 \leq p \leq \infty$, $L^p(\mathbb{R}^N)$ denotes the usual Lebesgue space with the norms

$$\|u\|_p = \left(\int_{\mathbb{R}^N} |u|^p dx\right)^{1/p}, \qquad 1 \leq p < \infty;$$

$$\|u\|_\infty = \inf\left\{M > 0 : |u(x)| \leq M \text{ almost everywhere in } \mathbb{R}^N\right\}.$$

$W^{1,p}(\mathbb{R}^N)$ denotes the Sobolev spaces modelled in $L^p(\mathbb{R}^N)$ with its usual norm

$$\|u\| = \left(\int_{\mathbb{R}^N} (|\nabla u|^p + |u|^p) dx\right)^{1/p}.$$

$\langle \cdot, \cdot \rangle$ denotes the duality pairing between $X$ and its dual $X^*$. The weak (strong) convergence in $X$ is denoted by $\rightharpoonup$ $(\to)$, respectively.

## 2 Preliminaries

We first give some properties of the change of variables $H : \mathbb{R} \to \mathbb{R}$ defined by (1.8), which will be used frequently in the sequel of the paper.

**Lemma 2.1.** *For functions $h, H$ and $H^{-1}$, the following properties hold:*

*(1)* $H$ *is odd, strictly increasing, invertible and* $C^2$ *in* $\mathbb{R}$;

*(2)* $0 < (H^{-1})'(t) \le 1, \forall t \in \mathbb{R}$;

*(3)* $|H^{-1}(t)| \le |t|, \forall t \in \mathbb{R}$;

*(4)* $\lim_{t\to 0} \frac{H^{-1}(t)}{t} = 1$;

*(5)* $\lim_{t\to+\infty} \frac{\left(H^{-1}(t)\right)^{\alpha}}{t} = \begin{cases} \sqrt[p]{\frac{2}{3}}, & \alpha = 1, \\ \sqrt[p]{2}, & \alpha > 1; \end{cases}$

*(6)* $h(H^{-1}(t))H^{-1}(t) \le \widetilde{T}(p,\alpha)t \le \widetilde{T}(p,\alpha)h(H^{-1}(t))H^{-1}(t), \forall t \ge 0$;

*(7)* $h(H^{-1}(t))\left(H^{-1}(t)\right)^2 \le \widetilde{T}(p,\alpha)tH^{-1}(t) \le \widetilde{T}(p,\alpha)h(H^{-1}(t))\left(H^{-1}(t)\right)^2, \forall t \in \mathbb{R}$;

*(8)* $|H^{-1}(t)| \le C|t|^{1/\alpha}$ *for some* $C > 0$ *and* $\forall t \in \mathbb{R}$;

*(9) There exists* $C > 0$ *such that*

$$|H^{-1}(t)| \ge \begin{cases} C|t|, & |t| \le 1, \\ C|t|^{1/\alpha}, & |t| \ge 1. \end{cases}$$

*Proof.* By the definition of $H$, it is easy to verify that (1)–(4) hold.

(5) If $\alpha > 1$, since

$$h(t) = \left[1 + \frac{\alpha^p t^p}{2(1+t^2)^{(2-\alpha)p/2}}\right]^{1/p} = \left[1 + \frac{\alpha^p t^p}{2(1+t^2)^{p/2}}(1+t^2)^{(\alpha-1)p/2}\right]^{1/p}, \ t > 0,$$

one has $h(t) \sim (\frac{\alpha^p}{2} t^{p(\alpha-1)})^{1/p} = \frac{\alpha}{\sqrt[p]{2}} t^{\alpha-1}$ as $t \to +\infty$. Moreover, $H(t) = \int_0^t h(s)ds \sim \frac{1}{\sqrt[p]{2}} t^{\alpha}$ as $t \to +\infty$. Remember the fact $H^{-1}(t)$ is the inverse of $H(t)$, so we get $H^{-1}(t) \sim (\sqrt[p]{2}t)^{1/\alpha}$ as $t \to +\infty$, which implies $\lim_{t\to+\infty} \frac{(H^{-1}(t))^{\alpha}}{t} = \sqrt[p]{2}$. If $\alpha = 1$, the result is obvious since $h(t)$ is an increasing bounded function when $t > 0$.

(6) Denote $g_1(t) = h(H^{-1}(t))H^{-1}(t) - t, \ t \ge 0$. Obviously $g_1(0) = 0$. Since $\alpha \ge 1$, one has

$$\begin{aligned} g_1'(t) &= \frac{H^{-1}(t)h'(H^{-1}(t))}{h(H^{-1}(t))} \\ &= \frac{\alpha^p (H^{-1}(t))^p \left[1 + (\alpha-1)(H^{-1}(t))^2\right]}{(1+(H^{-1}(t))^2)\left[2\left(1+(H^{-1}(t))^2\right)^{(2-\alpha)p/2} + \alpha^p (H^{-1}(t))^p\right]} \ge 0, \qquad \forall t \ge 0, \end{aligned}$$

which implies

$$h(H^{-1}(t))H^{-1}(t) \ge t, \qquad \forall t \ge 0.$$

Consequently,

$$\widetilde{T}(p,\alpha)t \le \widetilde{T}(p,\alpha)h(H^{-1}(t))H^{-1}(t), \qquad \forall t \ge 0.$$

Set $g_2(t) = \widetilde{T}(p,\alpha)t - h(H^{-1}(t))H^{-1}(t),\ t \ge 0$. Clearly $g_2(0) = 0$. By virtue of $H^{-1}(t) \ge 0,\ t \ge 0$ and (1.11), we can deduce that

$$\begin{aligned} g_2'(t) &= T(p,\alpha) - \frac{H^{-1}(t)h'(H^{-1}(t))}{h(H^{-1}(t))} \\ &= T(p,\alpha) - \frac{sh'(s)}{h(s)}\Big|_{s=H^{-1}(t)} \\ &\ge 0, \qquad \forall t \ge 0, \end{aligned}$$

which implies

$$h(H^{-1}(t))H^{-1}(t) \le \widetilde{T}(p,\alpha)t, \qquad \forall t \ge 0.$$

(7) Since $tH^{-1}(t) \ge 0,\ \forall t \in \mathbb{R}$, utilizing (6), we have

$$h(H^{-1}(t))\big(H^{-1}(t)\big)^2 \le \widetilde{T}(p,\alpha)tH^{-1}(t) \le \widetilde{T}(p,\alpha)h(H^{-1}(t))\big(H^{-1}(t)\big)^2, \qquad \forall t \in \mathbb{R}.$$

It is not difficult to verify that (8) and (9) are right from (1), (4) and (5). □

Under the hypotheses $(\mathrm{V}_1)$–$(\mathrm{V}_2)$ and $(\mathrm{K}_1)$–$(\mathrm{K}_3)$, we readily derive that $\mathcal{F} \in C^1(W^{1,p}(\mathbb{R}^N))$ and

$$\langle \mathcal{F}'(v), \omega \rangle = \int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v \nabla \omega dx - \int_{\mathbb{R}^N} \frac{\eta(x, H^{-1}(v))}{h(H^{-1}(v))}\omega dx$$

for $v, \omega \in W^{1,p}(\mathbb{R}^N)$. Thus, the critical points of $\mathcal{F}$ correspond exactly to the weak solutions of (1.10). The following results characterize the relationship between the solutions of (1.10) and (1.1).

**Proposition 2.2.**

*(i) If $v \in W^{1,p}(\mathbb{R}^N) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ is a critical point of the functional $\mathcal{F}$, then $u = H^{-1}(v)$ is a weak solution of (1.1);*

*(ii) if $v$ is a classical solution of (1.10), then $u = H^{-1}(v)$ is a classical solution of (1.1).*

*Proof.* $(i)$ It is easy to see that $|u|^p = |H^{-1}(v)|^p \le |v|^p$ and $|\nabla u|^p = |(H^{-1})'(v)|^p |\nabla v|^p \le |\nabla v|^p$. Hence, $u \in W^{1,p}(\mathbb{R}^N) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$. Since $v$ is a critical point of $\mathcal{F}$, we get

$$\int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v \nabla \omega dx = \int_{\mathbb{R}^N} \frac{\eta(x, H^{-1}(v))}{h(H^{-1}(v))}\omega dx, \qquad \forall \omega \in W^{1,p}(\mathbb{R}^N). \tag{2.1}$$

Note that

$$\nabla v = H'(u)\nabla u = h(u)\nabla u = \left(1 + \frac{\alpha^p |u|^p}{2(1+u^2)^{(2-\alpha)p/2}}\right)^{1/p} \nabla u. \tag{2.2}$$

For all $\psi \in C_0^\infty(\mathbb{R}^N)$, one can achieve

$$h(H^{-1}(v))\psi = h(u)\psi = \left(1 + \frac{\alpha^p |u|^p}{2(1+u^2)^{(2-\alpha)p/2}}\right)^{1/p} \psi \in W^{1,p}(\mathbb{R}^N),$$

and

$$
\begin{aligned}
\nabla\left(h(H^{-1}(v))\psi\right) &= h'(u)\psi\nabla u + h(u)\nabla\psi \\
&= \frac{\alpha^p}{2}\left(1+\frac{\alpha^p|u|^p}{2(1+u^2)^{(2-\alpha)p/2}}\right)^{(1-p)/p}\frac{(1+(\alpha-1)u^2)}{(1+u^2)^{1+(2-\alpha)p/2}}|u|^{p-2}u\psi\nabla u \\
&\quad + \left(1+\frac{\alpha^p|u|^p}{2(1+u^2)^{(2-\alpha)p/2}}\right)^{1/p}\nabla\psi.
\end{aligned}
\tag{2.3}
$$

Letting $\omega = h(H^{-1}(v))\psi$ in (2.1) and combining (2.2)–(2.3) enable us to deduce (1.7), which means that $u = H^{-1}(v)$ is a weak solution of (1.1).

$(ii)$ From

$$
\Delta_p v = \sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(|\nabla v|^{p-2}\frac{\partial v}{\partial x_i}\right) = \sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(h^{p-1}(u)|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\right),
$$

we deduce that

$$
\begin{aligned}
\Delta_p v &= h^{p-1}(u)\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\right) + |\nabla u|^{p-2}\sum_{i=1}^{N}\frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_i}\left(h^{p-1}(u)\right) \\
&= h^{p-1}(u)\Delta_p u + (p-1)h^{p-2}(u)h'(u)|\nabla u|^p \\
&= \left(1+\frac{\alpha^p|u|^p}{2(1+u^2)^{(2-\alpha)p/2}}\right)^{(p-1)/p}\Delta_p u \\
&\quad + \left(1+\frac{\alpha^p|u|^p}{2(1+u^2)^{(2-\alpha)p/2}}\right)^{-1/p}\frac{(p-1)\alpha^p\left(1+(\alpha-1)u^2\right)}{2(1+u^2)^{1+(2-\alpha)p/2}}|u|^{p-2}u|\nabla u|^p.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\left(1+\frac{\alpha^p|u|^p}{2(1+u^2)^{(2-\alpha)p/2}}\right)^{(p-1)/p}\Delta_p u \\
&\quad + \left(1+\frac{\alpha^p|u|^p}{2(1+u^2)^{(2-\alpha)p/2}}\right)^{-1/p}\frac{(p-1)\alpha^p\left(1+(\alpha-1)u^2\right)}{2(1+u^2)^{1+(2-\alpha)p/2}}|u|^{p-2}u|\nabla u|^p \\
&= -\left(1+\frac{\alpha^p|u|^p}{2(1+u^2)^{(2-\alpha)p/2}}\right)^{-1/p}\eta(x,u),
\end{aligned}
$$

that is,

$$
\Delta_p u + \frac{\alpha^p|u|^p}{2(1+u^2)^{(2-\alpha)p/2}}\Delta_p u + \frac{(p-1)\alpha^p\left(1+(\alpha-1)u^2\right)}{2(1+u^2)^{1+(2-\alpha)p/2}}|u|^{p-2}u|\nabla u|^p = -\eta(x,u). \tag{2.4}
$$

Noticing that

$$
\begin{aligned}
&\frac{\alpha^p|u|^p}{2(1+u^2)^{(2-\alpha)p/2}}\Delta_p u + \frac{(p-1)\alpha^p(1+(\alpha-1)u^2)}{2(1+u^2)^{1+(2-\alpha)p/2}}|u|^{p-2}u|\nabla u|^p \\
&\quad = \left[\Delta_p(1+u^2)^{\alpha/2}\right]\frac{\alpha u}{2(1+u^2)^{(2-\alpha)/2}}.
\end{aligned}
$$

This together with the (2.4) derive

$$
-\Delta_p u - \left[\Delta_p(1+u^2)^{\alpha/2}\right]\frac{\alpha u}{2(1+u^2)^{(2-\alpha)/2}} = \eta(x,u).
$$

The proof is finished. □

## 3 Auxiliary problem

To prove the main result, we employ the results [9] for the equation

$$-\Delta_p v = g(v), \qquad x \in \mathbb{R}^N. \tag{3.1}$$

The energy functional associated to (3.1) is

$$I(v) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla v|^p dx - \int_{\mathbb{R}^N} G(v)dx,$$

where $G(s) = \int_0^s g(t)dt$. Obviously, $I \in C^1(W^{1,p}(\mathbb{R}^N))$ under the assumptions on $g(s)$ below:

($G_0$) $g$ is odd and $g \in C(\mathbb{R}, \mathbb{R})$;

($G_1$) $-\infty < \liminf_{s\to 0} \frac{g(s)}{|s|^{p-2}s} \le \limsup_{s\to 0} \frac{g(s)}{|s|^{p-2}s} = -\sigma < 0$ if $1 < p < N$,

$-\infty < \lim_{s\to 0} \frac{g(s)}{|s|^{N-2}s} = -\sigma < 0$ if $p = N$;

($G_2$) When $1 < p < N$, $\lim_{s\to\infty} \frac{|g(s)|}{|s|^{p^*-1}} = 0$, where $p^* = \frac{Np}{N-p}$; when $p = N$, for some positive constants $C$ and $\beta_0$, that

$$|g(s)| \le C\left[\exp\left(\beta_0 |s|^{N/(N-1)}\right) - S_{N-2}(\beta_0, s)\right]$$

for all $|s| \ge R > 0$, where

$$S_{N-2}(\beta_0, s) = \sum_{k=0}^{N-2} \frac{\beta_0^k}{k!} |s|^{kN/(N-1)};$$

($G_3$) There exists $\xi > 0$ such that $G(\xi) > 0$.

We recall that a solution $v(x)$ of (3.1) is said to be *a least energy solution (or ground state solution)* if and only if

$$I(v) = a, \quad \text{where } a = \inf\{I(w) : w \in W^{1,p}(\mathbb{R}^N) \setminus \{0\} \text{ is a solution of } (3.1)\}. \tag{3.2}$$

**Theorem 3.1** ([9, Theorem 1.4]). *Let $1 < p \le N$ and suppose ($G_0$)–($G_2$) hold. Then setting*

$$\Lambda = \left\{\gamma \in C\big([0,1], W^{1,p}(\mathbb{R}^N)\big) : \ \gamma(0) = 0,\ I(\gamma(1)) < 0\right\}, \qquad b = \inf_{\gamma\in\Lambda} \max_{0\le t\le 1} I(\gamma(t)),$$

*we have $\Lambda \neq \emptyset$ and $b = a$. Furthermore, for each least energy solution $w$ of (3.1), there exists a path $\gamma \in \Lambda$ such that $w \in \gamma([0,1])$ and*

$$\max_{t\in[0,1]} I(\gamma(t)) = I(w).$$

**Theorem 3.2** ([9, Theorem 1.6]). *Let $1 < p \le N$ and assume that ($G_0$)–($G_3$) are satisfied, then equation (3.1) has a least energy solution $v$ which is positive.*

**Theorem 3.3** ([9, Theorem 1.8]). *Assume that all conditions of Theorem 3.1 hold, then there exist $\lambda > 0$ and $\delta > 0$ such that $I(v) \ge \lambda\|v\|^p$ if $\|v\| \le \delta$.*

**Lemma 3.4.** *Assume that* $(V_1)$–$(V_2)$ *and* $(K_1)$–$(K_2)$ *are satisfied, then the functional* $\mathcal{F}$ *has a mountain-pass geometry.*

*Proof.* Let the energy functionals corresponding to the equations $-\Delta_p v = m_0(v)$ and $-\Delta_p v = m_\infty(v)$ be

$$\begin{aligned}\mathcal{F}_0(v) &= \frac{1}{p}\int_{\mathbb{R}^N} |\nabla v|^p dx + \frac{1}{p}\int_{\mathbb{R}^N} V_0|H^{-1}(v)|^p dx - \int_{\mathbb{R}^N} K(H^{-1}(v))dx,\\ \mathcal{F}_\infty(v) &= \frac{1}{p}\int_{\mathbb{R}^N} |\nabla v|^p dx + \frac{1}{p}\int_{\mathbb{R}^N} V(\infty)|H^{-1}(v)|^p dx - \int_{\mathbb{R}^N} K(H^{-1}(v))dx,\end{aligned}$$

respectively, where

$$\begin{aligned}m_0(v) &= \frac{1}{h(H^{-1}(v))}\left[k(H^{-1}(v)) - V_0|H^{-1}(v)|^{p-2}H^{-1}(v)\right],\\ m_\infty(v) &= \frac{1}{h(H^{-1}(v))}\left[k(H^{-1}(v)) - V(\infty)|H^{-1}(v)|^{p-2}H^{-1}(v)\right].\end{aligned}$$

Notice that $\mathcal{F}_0(v) \le \mathcal{F}(v) \le \mathcal{F}_\infty(v)$ for all $v \in W^{1,p}(\mathbb{R}^N)$.

Now, we claim that $m_0$ and $m_\infty$ satisfy $(G_0)$–$(G_2)$.

Obviously, $m_0$ and $m_\infty$ satisfy $(G_0)$.

By use of $k(s) = o(|s|^{p-2}s)$ as $s \to 0$ and Lemma 2.1 (4), we derive that

$$\lim_{s\to 0}\frac{m_0(s)}{|s|^{p-2}s} = -V_0 < 0, \qquad \lim_{s\to 0}\frac{m_\infty(s)}{|s|^{p-2}s} = -V(\infty) < 0, \quad \text{if } 1 < p < N,$$

and

$$\lim_{s\to 0}\frac{m_0(s)}{|s|^{N-2}s} = -V_0 < 0, \qquad \lim_{s\to 0}\frac{m_\infty(s)}{|s|^{N-2}s} = -V(\infty) < 0, \quad \text{if } p = N.$$

Hence, $m_0$ and $m_\infty$ satisfy $(G_1)$.

Similarly to the argument in the proof of Lemma 2.1 (5), we can show that

$$\lim_{s\to\infty}\frac{|H^{-1}(s)|^{\alpha-1}}{h(H^{-1}(s))} = \begin{cases}\sqrt[p]{\frac{2}{3}}, & \alpha = 1,\\ \frac{\sqrt[p]{2}}{\alpha}, & \alpha > 1.\end{cases} \tag{3.3}$$

When $1 < p < N$, it follows from $(K_2)$ and Lemma 2.1 (2), (3), (8) that

$$\begin{aligned}|m_0(s)| &\le \frac{1}{h(H^{-1}(s))}(C + C|H^{-1}(s)|^{\theta-1} + V_0|H^{-1}(s)|^{p-1})\\ &\le C + C\frac{|H^{-1}(s)|^{\theta-1}}{h(H^{-1}(s))} + V_0|s|^{p-1}\\ &= C + C|H^{-1}(s)|^{\theta-\alpha}\frac{|H^{-1}(s)|^{\alpha-1}}{h(H^{-1}(s))} + V_0|s|^{p-1}\\ &\le C + C|s|^{(\theta-\alpha)/\alpha}\frac{|H^{-1}(s)|^{\alpha-1}}{h(H^{-1}(s))} + V_0|s|^{p-1},\end{aligned} \tag{3.4}$$

where $\alpha p < \theta < \alpha p^*$. Combining (3.3) and (3.4), we can deduce

$$\lim_{s\to\infty}\frac{|m_0(s)|}{|s|^{p^*-1}} = 0.$$

On the other hand, when $p = N$, applying $(K_2)$ and Lemma 2.1 (2), (3), we conclude that

$$|m_0(s)| \le C_1 + C_2|s|^{\theta-1}.$$

Then there exist positive constants $C$ and $\beta_0$ such that

$$|m_0(s)| \le C\left[\exp\left(\beta_0|s|^{N/(N-1)}\right) - S_{N-2}(\beta_0, s)\right]$$

for all $|s| \ge R > 0$, where $S_{N-2}(\beta_0, s) = \sum_{k=0}^{N-2} \frac{\beta_0^k}{k!}|s|^{kN/(N-1)}$. Therefore, $m_0$ satisfies $(G_2)$. Analogously, $m_\infty$ also satisfies $(G_2)$.

Based upon Theorem 3.3, there exist $\lambda_1 > 0$ and $\delta_1 > 0$ such that

$$\mathcal{F}(v) \ge \mathcal{F}_0(v) \ge \lambda_1\|v\|^p \quad \text{if } \|v\| \le \delta_1.$$

Moreover, for the functional $\mathcal{F}_\infty$, by virtue of Theorem 3.1, we obtain that there exists $e \in W^{1,p}(\mathbb{R}^N)$ with $\|e\| > \delta_1$ such that $\mathcal{F}_\infty(e) < 0$, which implies $\mathcal{F}(e) < 0$. Thus $\Gamma \ne \emptyset$, where

$$\Gamma = \left\{\gamma \in C\big([0,1], W^{1,p}(\mathbb{R}^N)\big) : \ \gamma(0) = 0,\ \mathcal{F}(\gamma(1)) < 0\right\}.$$

The proof is complete. □

**Remark 3.5.** By $(K_3)$, for any given $s_0 > 0$, there exists $C > 0$ depending on $s_0$ such that $K(s) \ge Cs^\mu$ for all $s \ge s_0$. Particularly, we have $\lim_{s\to+\infty} K(s)/s^p = +\infty$. Thus, there exists $\xi > 0$ such that $M_0(\xi) > 0$ and $M_\infty(\xi) > 0$, where

$$M_0(s) = \int_0^s m_0(t)dt = K(H^{-1}(s)) - \frac{V_0}{p}|H^{-1}(s)|^p,$$
$$M_\infty(s) = \int_0^s m_\infty(t)dt = K(H^{-1}(s)) - \frac{V(\infty)}{p}|H^{-1}(s)|^p.$$

Hence, $m_0$ and $m_\infty$ also satisfy $(G_3)$. Taking advantage of Theorem 3.2, the equations

$$-\Delta_p v = m_0(v) \quad \text{and} \quad -\Delta_p v = m_\infty(v), \qquad x \in \mathbb{R}^N$$

have least energy solutions in $W^{1,p}(\mathbb{R}^N)$ which are positive.

## 4 Proof of Theorem 1.2

Since $\mathcal{F}$ has the mountain-pass geometry, we know (see [10]) that for the constant

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \mathcal{F}(\gamma(t)) > 0,$$

where

$$\Gamma = \left\{\gamma \in C\big([0,1], W^{1,p}(\mathbb{R}^N)\big) : \ \gamma(0) = 0,\ \mathcal{F}(\gamma(1)) < 0\right\},$$

there exists a Cerami sequence $\{v_n\}$ for $\mathcal{F}$ at the level $c$, that is,

$$\mathcal{F}(v_n) \to c \quad \text{and} \quad \|\mathcal{F}'(v_n)\|(1 + \|v_n\|) \to 0 \quad \text{as } n \to \infty. \tag{4.1}$$

**Lemma 4.1.** *Assume that $(V_1)$–$(V_2)$ and $(K_1)$–$(K_3)$ are satisfied. Let $\{v_n\} \subset W^{1,p}(\mathbb{R}^N)$ be a Cerami sequence for $\mathcal{F}$ at the level $c > 0$, then $\{v_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$.*

*Proof.* First, we will prove that if $\{v_n\}$ satisfies

$$\int_{\mathbb{R}^N} |\nabla v_n|^p dx + \int_{\mathbb{R}^N} V(x)|H^{-1}(v_n)|^p dx \le C \tag{4.2}$$

for some constant $C > 0$, then it is bounded in $W^{1,p}(\mathbb{R}^N)$. In fact, we only need to verify that $\int_{\mathbb{R}^N} |v_n|^p dx$ is bounded. We start splitting

$$\int_{\mathbb{R}^N} |v_n|^p dx = \int_{\{x:|v_n(x)|\le 1\}} |v_n|^p dx + \int_{\{x:|v_n(x)|>1\}} |v_n|^p dx.$$

Note that $\mu \ge \widetilde{T}(p,\alpha)p \ge \alpha p$, then it follows from Lemma 2.1 (9) and Remark 3.5 that there exists $C > 0$ such that $K(H^{-1}(s)) \ge C|s|^p$ for all $|s| > 1$. Consequently,

$$\int_{\{x:|v_n(x)|>1\}} |v_n|^p dx \le C^{-1} \int_{\{x:|v_n(x)|>1\}} K(H^{-1}(v_n))dx \le C^{-1} \int_{\mathbb{R}^N} K(H^{-1}(v_n))dx. \tag{4.3}$$

Using Lemma 2.1 (9) again, we derive that

$$\begin{aligned} \int_{\{x:|v_n(x)|\le 1\}} |v_n|^p dx &\le C^{-p} \int_{\{x:|v_n(x)|\le 1\}} |H^{-1}(v_n)|^p dx \\ &\le C^{-p} V_0^{-1} \int_{\mathbb{R}^N} V(x)|H^{-1}(v_n)|^p dx. \end{aligned} \tag{4.4}$$

Combining (4.1)–(4.4), we can achieve that $\{v_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$.

Next, we will show that (4.2) holds. By (4.1), we obtain

$$\frac{1}{p}\int_{\mathbb{R}^N} |\nabla v_n|^p dx + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|H^{-1}(v_n)|^p dx - \int_{\mathbb{R}^N} K(H^{-1}(v_n))dx = c + o_n(1), \tag{4.5}$$

and for all $\psi \in W^{1,p}(\mathbb{R}^N)$,

$$\begin{aligned} \langle \mathcal{F}'(v_n), \psi\rangle = &\int_{\mathbb{R}^N} |\nabla v_n|^{p-2}\nabla v_n \nabla\psi dx + \int_{\mathbb{R}^N} V(x)\frac{|H^{-1}(v_n)|^{p-2}H^{-1}(v_n)}{h(H^{-1}(v_n))}\psi dx \\ &- \int_{\mathbb{R}^N} \frac{k(H^{-1}(v_n))}{h(H^{-1}(v_n))}\psi dx. \end{aligned} \tag{4.6}$$

Denote $\psi_n = h(H^{-1}(v_n))H^{-1}(v_n)$, taking advantage of Lemma 2.1 (6), one can find $|\psi_n| \le \widetilde{T}(p,\alpha)|v_n|$ and

$$|\nabla\psi_n| = \left[1 + \frac{\alpha^p t^p[1+(\alpha-1)t^2]}{(1+t^2)\left(2(1+t^2)^{(2-\alpha)p/2} + \alpha^p t^p\right)}\Big|_{t=|H^{-1}(v_n)|}\right] |\nabla v_n| \le \widetilde{T}(p,\alpha)|\nabla v_n|.$$

Thus, $\|\psi_n\| \le \widetilde{T}(p,\alpha)\|v_n\|$. By choosing $\psi = \psi_n$ in (4.6), we deduce that

$$\begin{aligned} &\int_{\mathbb{R}^N} \left[1 + \frac{\alpha^p t^p[1+(\alpha-1)t^2]}{(1+t^2)\left(2(1+t^2)^{(2-\alpha)p/2} + \alpha^p t^p\right)}\Big|_{t=|H^{-1}(v_n)|}\right] |\nabla v_n|^p dx \\ &\quad + \int_{\mathbb{R}^N} V(x)|H^{-1}(v_n)|^p dx - \int_{\mathbb{R}^N} k(H^{-1}(v_n))H^{-1}(v_n)dx \\ &= \langle \mathcal{F}'(v_n), \psi_n\rangle = o_n(1). \end{aligned} \tag{4.7}$$

Combining (4.5), (4.7) and $(\mathrm{K}_3)$, one has

$$\begin{aligned}\int_{\mathbb{R}^N}&\left\{\frac{1}{p}-\frac{1}{\mu}\left[1+\frac{\alpha^p t^p[1+(\alpha-1)t^2]}{(1+t^2)\left(2(1+t^2)^{(2-\alpha)p/2}+\alpha^p t^p\right)}\Big|_{t=|H^{-1}(v_n)|}\right]\right\}|\nabla v_n|^p dx\\&+\left(\frac{1}{p}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N}V(x)|H^{-1}(v_n)|^p dx\\&\le c+o_n(1).\end{aligned}\tag{4.8}$$

If $\mu>\widetilde{T}(p,\alpha)p$ in $(\mathrm{K}_3)$, by virtue of (1.11), it follows that

$$\frac{\left[\mu-\widetilde{T}(p,\alpha)p\right]}{p\mu}\int_{\mathbb{R}^N}|\nabla v_n|^p dx+\frac{T(p,\alpha)}{\mu}\int_{\mathbb{R}^N}V(x)|H^{-1}(v_n)|^p dx\le c+o_n(1),$$

which implies that (4.2) holds and hence $\{v_n\}$ is bounded. If $\mu=\widetilde{T}(p,\alpha)p=2p$, applying Remark 1.3, we derive $\alpha=2$. In this case, we can apply the estimate (4.8) to derive

$$\frac{1}{2p}\int_{\mathbb{R}^N}\frac{|\nabla v_n|^p}{1+2^{p-1}|H^{-1}(v_n)|^p}dx+\frac{1}{2p}\int_{\mathbb{R}^N}V(x)|H^{-1}(v_n)|^p dx\le c+o_n(1).\tag{4.9}$$

Set $u_n=H^{-1}(v_n)$, we get that

$$|\nabla v_n|^p=\left(1+2^{p-1}|H^{-1}(v_n)|^p\right)|\nabla u_n|^p.\tag{4.10}$$

According to $(\mathrm{V}_1)$ and (4.9)–(4.10), it holds that

$$\frac{1}{2p}\min\{1,V_0\}\|u_n\|^p\le\frac{1}{2p}\int_{\mathbb{R}^N}|\nabla u_n|^p dx+\frac{1}{2p}\int_{\mathbb{R}^N}V(x)|u_n|^p dx\le c+o_n(1).$$

This implies $\{u_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$. The conditions $(\mathrm{K}_1)$–$(\mathrm{K}_2)$ yield that

$$K(s)\le|s|^p+C|s|^\theta.\tag{4.11}$$

Combining the condition (b) in Theorem 1.2 with (4.11), we can apply Sobolev embedding theorem to achieve that $\int_{\mathbb{R}^N}K(H^{-1}(v_n))dx=\int_{\mathbb{R}^N}K(u_n)dx$ is bounded. Thus, utilizing (4.5), we derive (4.2), which implies $\{v_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$. The proof is finished. □

### 4.1 Existence of nontrivial critical points for $\mathcal{F}$

According to Lemma 4.1, $\{v_n\}$ is a bounded Cerami sequence in $W^{1,p}(\mathbb{R}^N)$. Since $W^{1,p}(\mathbb{R}^N)$ is a reflexive Banach space, up to a subsequence, still denoted by $\{v_n\}$, such that $v_n\rightharpoonup v$. We assert that $\mathcal{F}'(v)=0$. In fact, since $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N)$, we only need to verify that $\langle\mathcal{F}'(v),\psi\rangle=0$ for all $\psi\in C_0^\infty(\mathbb{R}^N)$. Note that

$$\begin{aligned}&\langle\mathcal{F}'(v_n),\psi\rangle-\langle\mathcal{F}'(v),\psi\rangle\\&\quad=\int_{\mathbb{R}^N}(|\nabla v_n|^{p-2}\nabla v_n-|\nabla v|^{p-2}\nabla v)\nabla\psi dx\\&\qquad+\int_{\mathbb{R}^N}\Big[\frac{|H^{-1}(v_n)|^{p-2}H^{-1}(v_n)}{h(H^{-1}(v_n))}-\frac{|H^{-1}(v)|^{p-2}H^{-1}(v)}{h(H^{-1}(v))}\Big]V(x)\psi dx\\&\qquad-\int_{\mathbb{R}^N}\Big[\frac{k(H^{-1}(v_n))}{h(H^{-1}(v_n))}-\frac{k(H^{-1}(v))}{h(H^{-1}(v))}\Big]\psi dx.\end{aligned}$$

Remember the fact that $v_n \to v$ in $L^q_{\mathrm{loc}}(\mathbb{R}^N)$ for $q \in [1, p^*)$ if $1 < p < N$ and $q \geq 1$ if $p = N$, by virtue of the Lebesgue dominated convergence theorem and $(K_1)$–$(K_2)$, we derive that for all $\psi \in C_0^\infty(\mathbb{R}^N)$,

$$\langle \mathcal{F}'(v_n), \psi\rangle - \langle \mathcal{F}'(v), \psi\rangle \to 0 \quad \text{as } n \to \infty.$$

Since $\mathcal{F}'(v_n) \to 0$ as $n \to \infty$, the desired result is obtained immediately.

Now we will prove that $v \neq 0$. Assume on the contrary that $v = 0$. The argument will be divided into the following three steps.

*Step 1.* We claim that $\{v_n\}$ is also a Cerami sequence for the functional $\mathcal{F}_\infty$, which defined in Lemma 3.4, at the level $c$.

Indeed, since $V(x) \to V(\infty)$ as $|x| \to \infty$, $v_n \to 0$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ and Lemma 2.1 (3), one can get that

$$\begin{aligned}\mathcal{F}_\infty(v_n) - \mathcal{F}(v_n) &= \frac{1}{p}\int_{\mathbb{R}^N} \big(V(\infty) - V(x)\big)|H^{-1}(v_n)|^p dx\\ &\leq \frac{1}{p}\int_{\mathbb{R}^N} \big(V(\infty) - V(x)\big)|v_n|^p dx \to 0,\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{F}'_\infty(v_n) - \mathcal{F}'(v_n)\| &= \sup_{\|\psi\|\leq 1} |\langle \mathcal{F}'_\infty(v_n), \psi\rangle - \langle \mathcal{F}'(v_n), \psi\rangle|\\ &\leq \sup_{\|\psi\|\leq 1} \int_{\mathbb{R}^N} |v_n|^{p-1}|V(\infty) - V(x)||\psi| dx\\ &\leq \Big(\int_{\mathbb{R}^N} |v_n|^p |V(\infty) - V(x)|^{p/(p-1)} dx\Big)^{(p-1)/p} \to 0\end{aligned}$$

as $n \to \infty$, which implies

$$\|\mathcal{F}'_\infty(v_n)\|(1 + \|v_n\|) \leq \|\mathcal{F}'_\infty(v_n) - \mathcal{F}'(v_n)\|(1 + \|v_n\|) + \|\mathcal{F}'(v_n)\|(1 + \|v_n\|) \to 0$$

as $n \to \infty$.

*Step 2.* We claim that for all $R > 0$,

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |v_n|^p dx = 0 \tag{4.12}$$

cannot occur. Assume on the contrary that (4.12) occurs, that is, $\{v_n\}$ vanish, then by the Lions compactness lemma [19], we have $v_n \to 0$ in $L^q(\mathbb{R}^N)$ for any $q \in (p, p^*)$ if $1 < p < N$ and $q > p$ if $p = N$. It follows from $(K_1) - (K_2)$ that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$0 \leq k(H^{-1}(s))H^{-1}(s) \leq \varepsilon|H^{-1}(s)|^p + C_\varepsilon|H^{-1}(s)|^\theta, \qquad \forall s \in \mathbb{R}. \tag{4.13}$$

In view of (4.13) and Lemma 2.1 (3), (8), for any $v \in W^{1,p}(\mathbb{R}^N)$, one can get

$$\int_{\mathbb{R}^N} k(H^{-1}(v))H^{-1}(v) dx \leq \varepsilon\int_{\mathbb{R}^N} |v|^p dx + C_\varepsilon\int_{\mathbb{R}^N} |v|^\theta dx, \tag{4.14}$$

$$\int_{\mathbb{R}^N} k(H^{-1}(v))H^{-1}(v) dx \leq \varepsilon\int_{\mathbb{R}^N} |v|^p dx + C_\varepsilon\int_{\mathbb{R}^N} |v|^{\theta/\alpha} dx. \tag{4.15}$$

If $\mu = \widetilde{T}(p, \alpha)p = 2p$, we use inequality (4.14) , if $\mu > \widetilde{T}(p, \alpha)p$, we use inequality (4.15), we just think about the case $\mu > \widetilde{T}(p, \alpha)p$ because the other one is similar. Since $\theta/\alpha \in (p, p^*)$

if $1 < p < N$ and $\theta/\alpha > p$ if $p = N$. Combining Lemma 2.1 (6) and (4.15) enable us to deduce that for any $\varepsilon > 0$,

$$\begin{aligned}\lim_{n\to\infty}\int_{\mathbb{R}^N}\frac{k(H^{-1}(v_n))}{h(H^{-1}(v_n))}v_n dx &\le \lim_{n\to\infty}\int_{\mathbb{R}^N}k(H^{-1}(v_n))H^{-1}(v_n)dx\\ &\le \lim_{n\to\infty}\left(\varepsilon\int_{\mathbb{R}^N}|v_n|^p dx + C_\varepsilon\int_{\mathbb{R}^N}|v_n|^{\theta/\alpha}dx\right)\\ &\le \varepsilon\lim_{n\to\infty}\int_{\mathbb{R}^N}|v_n|^p dx,\end{aligned}$$

which implies

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\frac{k(H^{-1}(v_n))}{h(H^{-1}(v_n))}v_n dx = 0, \qquad \lim_{n\to\infty}\int_{\mathbb{R}^N}k(H^{-1}(v_n))H^{-1}(v_n)dx = 0. \tag{4.16}$$

Combining the first limit in (4.16) with the fact $\langle \mathcal{F}'(v_n), v_n\rangle \to 0$ as $n\to\infty$, we get

$$\int_{\mathbb{R}^N}|\nabla v_n|^p dx + \int_{\mathbb{R}^N}V(x)\frac{|H^{-1}(v_n)|^{p-2}H^{-1}(v_n)}{h(H^{-1}(v_n))}v_n dx \to 0 \tag{4.17}$$

as $n\to\infty$. Based upon (4.17) and Lemma 2.1 (7), we derive

$$\int_{\mathbb{R}^N}|\nabla v_n|^p dx + \int_{\mathbb{R}^N}V(x)|H^{-1}(v_n)|^p \to 0 \tag{4.18}$$

as $n\to\infty$. According to the second limit in (4.16) and $(\mathrm{K}_3)$, we deduce that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}K(H^{-1}(v_n))dx = 0. \tag{4.19}$$

$\lim_{n\to\infty}\mathcal{F}(v_n) = 0$ is obtained immediately from (4.18) and (4.19), we get a contradiction since $\lim_{n\to\infty}\mathcal{F}(v_n) = c > 0$. Thus, $\{v_n\}$ does not vanish and there exist $\tau, R > 0$ and $\{y_n\}\subset\mathbb{R}^N$ such that

$$\lim_{n\to\infty}\int_{B_R(y_n)}|v_n|^p dx \ge \tau > 0. \tag{4.20}$$

*Step 3.* Set $\widetilde{v}_n(x) = v_n(x+y_n)$. Since $\{v_n\}$ is a Cerami sequence for $\mathcal{F}_\infty$, it is easy to verify that $\{\widetilde{v}_n\}$ is also a Cerami sequence for $\mathcal{F}_\infty$. Arguing as in the case of $\{v_n\}$, up to a subsequence, still denoted by $\{\widetilde{v}_n\}$, we have $\widetilde{v}_n \rightharpoonup \widetilde{v}$ with $\mathcal{F}'_\infty(\widetilde{v}) = 0$. Since $\widetilde{v}_n \to \widetilde{v}$ in $L^p(B_R)$, by (4.20), we derive that

$$\int_{B_R}|\widetilde{v}|^p dx = \lim_{n\to\infty}\int_{B_R}|\widetilde{v}_n|^p dx = \lim_{n\to\infty}\int_{B_R(y_n)}|v_n|^p dx \ge \tau > 0,$$

which implies $\widetilde{v}\neq 0$.

Make use of Lemma 2.1 (7), we get

$$|H^{-1}(\widetilde{v}_n)|^p - \frac{|H^{-1}(\widetilde{v}_n)|^{p-2}H^{-1}(\widetilde{v}_n)}{h(H^{-1}(\widetilde{v}_n))}\widetilde{v}_n \ge 0,\ \forall n\in\mathbb{N}.$$

On the other hand, in view of Lemma 2.1 (6) and $(\mathrm{K}_3)$, it can be deduced that

$$\frac{k(H^{-1}(\widetilde{v}_n))}{h(H^{-1}(\widetilde{v}_n))}\widetilde{v}_n - pK(H^{-1}(\widetilde{v}_n)) \ge \frac{k(H^{-1}(\widetilde{v}_n))H^{-1}(\widetilde{v}_n)}{\widetilde{T}(p,\alpha)} - pK(H^{-1}(\widetilde{v}_n)) \ge 0, \qquad \forall n\in\mathbb{N}.$$

Note that $\widetilde{v}_n$ is a Cerami sequence for $\mathcal{F}_\infty$, by Fatou's lemma, straightforward computations generate that

$$\begin{aligned}
pc = & \liminf_{n\to\infty} \left[p\mathcal{F}_\infty(\widetilde{v}_n) - \langle \mathcal{F}'_\infty(\widetilde{v}_n), \widetilde{v}_n\rangle\right] \\
\geq & \liminf_{n\to\infty} \int_{\mathbb{R}^N} V(\infty)\left[|H^{-1}(\widetilde{v}_n)|^p - \frac{|H^{-1}(\widetilde{v}_n)|^{p-2}H^{-1}(\widetilde{v}_n)}{h(H^{-1}(\widetilde{v}_n))}\widetilde{v}_n\right]dx \\
& + \liminf_{n\to\infty} \int_{\mathbb{R}^N} \left[\frac{k(H^{-1}(\widetilde{v}_n))}{h(H^{-1}(\widetilde{v}_n))}\widetilde{v}_n - pK(H^{-1}(\widetilde{v}_n))\right]dx \\
\geq & \int_{\mathbb{R}^N} V(\infty)\left[|H^{-1}(\widetilde{v})|^p - \frac{|H^{-1}(\widetilde{v})|^{p-2}H^{-1}(\widetilde{v})}{h(H^{-1}(\widetilde{v}))}\widetilde{v}\right]dx \\
& + \int_{\mathbb{R}^N} \left[\frac{k(H^{-1}(\widetilde{v}))}{h(H^{-1}(\widetilde{v}))}\widetilde{v} - pK(H^{-1}(\widetilde{v}))\right]dx \\
= & \, p\mathcal{F}_\infty(\widetilde{v}) - \langle \mathcal{F}'_\infty(\widetilde{v}), \widetilde{v}\rangle \\
= & \, p\mathcal{F}_\infty(\widetilde{v}).
\end{aligned}$$

Thus, $\widetilde{v} \neq 0$ is a critical point of $\mathcal{F}_\infty$ satisfying $\mathcal{F}_\infty(\widetilde{v}) \leq c$.

In view of *Step 3*, we derive that the least energy level $a_\infty$ for $\mathcal{F}_\infty$ satisfies $a_\infty \leq c$. Denoting $\widehat{\omega}$ as a least energy solution of the equation $-\Delta_p v = m_\infty(v)$ (see Remark 3.5). Applying Theorem 3.1 to the functional $\mathcal{F}_\infty$, there exists a path $\gamma \in C\big([0,1], W^{1,p}(\mathbb{R}^N)\big)$ such that $\gamma(0) = 0$, $\mathcal{F}_\infty(\gamma(1)) < 0$, $\widehat{\omega} \in \gamma([0,1])$ and

$$\max_{t\in[0,1]} \mathcal{F}_\infty(\gamma(t)) = \mathcal{F}_\infty(\widehat{\omega}).$$

If $V(x) \equiv V(\infty)$, we prove the desired conclusion. So we assume that $V(x) \not\equiv V(\infty)$, we have

$$\mathcal{F}(\gamma(t)) < \mathcal{F}_\infty(\gamma(t)), \qquad \forall t \in (0,1]$$

and hence

$$c \leq \max_{t\in[0,1]} \mathcal{F}(\gamma(t)) < \max_{t\in[0,1]} \mathcal{F}_\infty(\gamma(t)) = \mathcal{F}_\infty(\widehat{\omega}) = a_\infty \leq c.$$

We get a contradiction. Therefore, $v$ is a nontrivial critical point of $\mathcal{F}$.

## 4.2 $L^\infty$-estimate and decay to zero at infinity

Let $v \in W^{1,p}(\mathbb{R}^N)$ be a nontrivial weak solution of (1.10), then for all $\omega \in W^{1,p}(\mathbb{R}^N)$, it holds that

$$\int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v \nabla\omega dx + \int_{\mathbb{R}^N} V(x)\frac{|H^{-1}(v)|^{p-2}H^{-1}(v)}{h(H^{-1}(v))}\omega dx = \int_{\mathbb{R}^N} \frac{k(H^{-1}(v))}{h(H^{-1}(v))}\omega dx. \tag{4.21}$$

Assume that $1 < p < N$. Without loss of generality, we suppose that $v \geq 0$. Otherwise, we work with the positive and negative parts of $v$. For each $m \geq 1$, define

$$v_m = \begin{cases} v, & \text{if } 0 \leq v \leq m, \\ m, & \text{if } v \geq m, \end{cases}$$

$$\zeta_m = v_m^{p(r-1)}v, \qquad \phi_m = v v_m^{r-1}$$

with $r > 1$ which will be given later. Choosing $\zeta_m$ as a test function in (4.21). Note that

$$k(H^{-1}(v)) \leq \frac{V_0}{2}(H^{-1}(v))^{p-1} + C(H^{-1}(v))^{\theta-1}$$

and $(\mathrm{V}_1)$, we can deduce

$$\begin{aligned}&\int_{\mathbb{R}^N} v_m^{p(r-1)}|\nabla v|^p dx + p(r-1)\int_{\mathbb{R}^N} v_m^{p(r-1)-1} v|\nabla v|^{p-2}\nabla v_m \nabla v dx \\ &\quad \leq C\int_{\mathbb{R}^N} \frac{(H^{-1}(v))^{\theta-1}}{h(H^{-1}(v))} v v_m^{p(r-1)} dx.\end{aligned}$$

Noticing $\nabla v_m \nabla v \geq 0$ in $\mathbb{R}^N$, using Lemma 2.1 (6) and (8), one has

$$\int_{\mathbb{R}^N} v_m^{p(r-1)}|\nabla v|^p dx \leq C\int_{\mathbb{R}^N} v^{\theta/\alpha} v_m^{p(r-1)} dx = C\int_{\mathbb{R}^N} v^{\hat{\theta}-p}\phi_m^p dx, \tag{4.22}$$

where $\hat{\theta} = \theta/\alpha$. It follows from the Gagliardo–Nirenberg inequality [11] and (4.22) that

$$\begin{aligned}\left(\int_{\mathbb{R}^N} \phi_m^{p^*} dx\right)^{p/p^*} &\leq C_1\int_{\mathbb{R}^N} |\nabla \phi_m|^p dx \\ &\leq C_1 2^{p-1}\left(\int_{\mathbb{R}^N} v_m^{p(r-1)}|\nabla v|^p dx + (r-1)^p\int_{\mathbb{R}^N} v^p v_m^{p(r-2)}|\nabla v_m|^p dx\right) \\ &\leq C_1 2^{p-1} r^p \int_{\mathbb{R}^N} v_m^{p(r-1)}|\nabla v|^p dx \\ &\leq C_2 r^p \int_{\mathbb{R}^N} v^{\hat{\theta}-p}\phi_m^p dx.\end{aligned}$$

According to the Hölder inequality, one sees that

$$\left(\int_{\mathbb{R}^N} \phi_m^{p^*} dx\right)^{p/p^*} \leq C_2 r^p \left(\int_{\mathbb{R}^N} v^{p^*} dx\right)^{(\hat{\theta}-p)/p^*}\left(\int_{\mathbb{R}^N} \phi_m^{pp^*/(p^*-\hat{\theta}+p)} dx\right)^{(p^*-\hat{\theta}+p)/p^*}.$$

As $0 \leq \phi_m \leq v^r$, the continuity of the embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$ leads to

$$\left(\int_{\mathbb{R}^N} (v v_m^{r-1})^{p^*} dx\right)^{p/p^*} \leq C_3 r^p \|v\|^{\hat{\theta}-p}\left(\int_{\mathbb{R}^N} v^{rpp^*/(p^*-\hat{\theta}+p)} dx\right)^{(p^*-\hat{\theta}+p)/p^*},$$

that is,

$$\left(\int_{\mathbb{R}^N} (v v_m^{r-1})^{p^*} dx\right)^{p/p^*} \leq C_3 r^p \|v\|^{\hat{\theta}-p}\|v\|_{r\lambda^*}^{pr} \tag{4.23}$$

with $\lambda^* = pp^*/(p^* - \hat{\theta} + p)$ and $r = p^*/\lambda^* = 1 + (p^* - \hat{\theta})/p > 1$. By virtue of Fatou's lemma, we conclude from (4.23) that

$$\|v\|_{rp^*} \leq \left(C_3 r^p \|v\|^{\hat{\theta}-p}\right)^{1/pr}\|v\|_{r\lambda^*}$$

or

$$\|v\|_{rp^*} \leq A^{1/r} r^{1/r}\|v\|_{r\lambda^*} \tag{4.24}$$

with $A > 0$ and $A^p = C_3\|v\|^{\hat{\theta}-p}$.

We now use the classical Moser's iteration scheme to prove $v \in L^\infty(\mathbb{R}^N)$. For each $k = 0,1,2,\ldots$, we define $r_{k+1}\lambda^* := p^* r_k$ with $r_0 = r$. Clearly, we have $r_k = r^{k+1} \uparrow +\infty$ as $k \to \infty$. Employing the previous argument for $r_1$, we get from (4.24) that

$$\begin{aligned}\|v\|_{r_1 p^*} &\le A^{1/r_1} r_1^{1/r_1} \|v\|_{r_1\lambda^*}\\ &= A^{1/r_1} r_1^{1/r_1} \|v\|_{rp^*}\\ &\le A^{1/r+1/r_1} r^{1/r} r_1^{1/r_1} \|v\|_{p^*}.\end{aligned}$$

By iteration scheme, we have

$$\|v\|_{r_k p^*} \le A^{S_k} e^{T_k} \|v\|_{p^*} \tag{4.25}$$

with $S_k = \sum_{i=0}^k \frac{1}{r_i} = \sum_{i=0}^k \frac{1}{r^{i+1}}$ and $T_k = \sum_{i=0}^k \frac{\ln r_i}{r_i} = \sum_{i=0}^k \frac{(i+1)\ln r}{r^{i+1}}$. Recall $r = p^*/\lambda^* > 1$, we get

$$\lim_{k\to\infty} S_k = p/(p^* - \hat{\theta}), \qquad \lim_{k\to\infty} T_k = r\ln r/(r-1)^2.$$

Letting $k \to \infty$ in (4.25) and by the Sobolev embedding theorem, we can deduce that $v \in L^\infty(\mathbb{R}^N)$ and

$$\begin{aligned}\|v\|_\infty &\le A^{p/(p^*-\hat{\theta})} r^{r/(r-1)^2} \|v\|_{p^*}\\ &\le C_3^{1/(p^*-\hat{\theta})} \|v\|^{(\hat{\theta}-p)/(p^*-\hat{\theta})} r^{r/(r-1)^2} \|v\|_{p^*}\\ &\le C_4 \|v\|^{(p^*-p)/(p^*-\hat{\theta})}.\end{aligned}$$

In the case $p = N$, [30, Theorem 1] enables us to derive that $v$ is locally bounded in $\mathbb{R}^N$. By a result in [33], we conclude that $v \in C^{1,\beta}_{\mathrm{loc}}(\mathbb{R}^N)(0 < \beta < 1)$ for $1 < p \le N$.

Next, when $1 < p < N$, we will show that $v(x) \to 0$ as $|x| \to \infty$. Since $v \in L^\infty(\mathbb{R}^N)$, it follows from $(V_1)$, $(K_2)$, Lemma 2.1 (8) and (4.21) that

$$\int_{\mathbb{R}^N} |\nabla v|^{p-2} \nabla v \nabla \psi dx \le C \int_{\mathbb{R}^N} (1 + |v|^{p-1}) \psi dx$$

for all $\psi \in C_0^\infty(\mathbb{R}^N), \psi \ge 0$. Applying [34, Theorem 1.3], one sees that for any $x \in \mathbb{R}^N$,

$$\sup_{y\in B_1(x)} v(y) \le C\|v\|_{L^p(B_2(x))}.$$

In particular, $v(x) \le C\|v\|_{L^p(B_2(x))}$. Since

$$\|v\|_{L^p(B_2(x))} \to 0 \quad \text{as } |x| \to \infty,$$

one has $v(x) \to 0$ as $|x| \to \infty$.

We conclude that $u = H^{-1}(v)$ is a nontrivial weak solution of (1.1) in $C^{1,\beta}_{\mathrm{loc}}(\mathbb{R}^N)(0 < \beta < 1)$ by Proposition 2.2. Since $|u| = |H^{-1}(v)| \le |v|$, we get that $u(x) \to 0$ as $|x| \to \infty$, which finalizes the proof of Theorem 1.2.

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