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On the principal eigenvalues of the degenerate elliptic systems

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Abstract. We study some qualitative properties for the set of principal eigenvalues of a degenerate elliptic system such as strict monotonicity with respect to the domain, local isolation and monotonicity and continuity of the principal eigenvalue with respect to the weight functions. Finally, explicit lower bounds for principal eigenvalues in terms of the measure of domain are also proved.

Keywords: principal eigenvalue, monotonicity, lower bound of eigenvalues.

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1 Introduction

In this paper we study the following system:

$$\begin{cases} -\Delta_p u = \lambda a(x)|v|^{\beta_1-1}v & \text{in } \Omega; \\ -\Delta_q v = \mu b(x)|u|^{\beta_2-1}u & \text{in } \Omega; \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\beta_1, \beta_2 > 0$ with $\beta_1\beta_2 = (p-1)(q-1)$, $(\lambda, \mu) \in \mathbb{R}^2$, $p, q \in (1, \infty)$, Ω is a bounded domain in $\mathbb{R}^n$ with a C^2-boundary and a and b are bounded functions on Ω satisfying

$$\operatorname*{ess\,inf}_{x\in\Omega} a(x) > 0 \quad \text{and} \quad \operatorname*{ess\,inf}_{x\in\Omega} b(x) > 0. \tag{1.2}$$

The p-Laplacian operator $\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,\frac{p}{p-1}}(\Omega)$ is defined by

$$\langle -\Delta_p u, v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx,$$

where $W^{-1,\frac{p}{p-1}}(\Omega)$ is the dual space of $W_0^{1,p}(\Omega)$.

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Consider the classical problem

$$\begin{cases} -\Delta_p u = f(x) & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Notice that, if $f \in L^\infty(\Omega)$, then problem (1.3) admits a unique weak solution $(-\Delta_p)^{-1}(f) := u \in W_0^{1,p}(\Omega)$. In this case, there exists $\alpha \in (0,1)$ such that $u \in C^{1,\alpha}(\overline{\Omega})$ (see [12,18,24,35]).

Thus, $(-\Delta_p)^{-1} : L^\infty(\Omega) \to C_0^{1,\alpha'}(\overline{\Omega})$ is continuous and bounded for $\alpha' = \alpha$ and compact whenever $0 < \alpha' < \alpha$. Moreover, the (weak and strong) comparison principles related to the p-Laplacian operator (see [6–10,15–19,31,34]) shows that $(-\Delta_p)^{-1}$ is order preserving, that is, for all $f, g \in L^\infty(\Omega)$, $f \leq g$ in Ω implies $(-\Delta_p)^{-1}f \leq (-\Delta_p)^{-1}g$ and it is also strictly order preserving, i.e., $f \leq (\not\equiv)\ g$ and $g\ (\not\equiv) \geq 0$ in Ω imply

$$(-\Delta_p)^{-1}f < (-\Delta_p)^{-1}g \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial(-\Delta_p)^{-1}g}{\partial\nu} < \frac{\partial(-\Delta_p)^{-1}f}{\partial\nu} \quad \text{on } \partial\Omega,$$

where $\nu \equiv \nu(y_0)$ denotes the exterior unit normal to $\partial\Omega$ at $y_0 \in \partial\Omega$. More generally, we have

$$(-\Delta_p)^{-1} : W^{-1,\frac{p}{p-1}}(\Omega) \to L^p(\Omega)$$

is well defined, compact and order preserving, when $p > 2$ (see [18, Corollary 8]).

By weak maximum principle in Ω means that for any weak solution $u \in W_0^{1,p}(\Omega)$ to

$$\begin{cases} -\Delta_p u = f(x) & \text{in } \Omega; \\ u \geq 0 & \text{on } \partial\Omega, \end{cases}$$

with $f \geq 0$ in Ω implies that $u \geq 0$ in Ω. Besides, the strong maximum principle is said to hold in Ω if, in addition, $u > 0$ in Ω whenever $f \not\equiv 0$ in Ω. The validity of the (weak and strong) maximum principles related to the p-Laplacian operator was established in [34,36]. Later, the paper [18] generalizes such results for operators involving the p-Laplacian. More generally, in [15] the authors showed an anti-maximum principle for a class of strictly cooperative elliptic systems.

In 1994, López-Gómez and Molina-Meyer [27] made a fairly complete characterization on maximum principles for linear second order elliptic operators and, more generally, in the context of cooperative systems. More recently, in [23] the authors established the connection between maximum principle for Lane–Emden systems and their principal spectral curves. We refer to [26] for a more detailed discussion of the maximum principle for elliptic problems and cooperative systems involving linear second order elliptic operators.

We shall introduce a bit of notation. Here X stands for the space $\left[C_0^1(\overline{\Omega})\right]^2$, X_+ is given by $\{(u,v) \in X : u \geq 0 \text{ and } v \geq 0 \text{ in } \Omega\}$, and $\mathring{X}_+$ is the topological interior of X_+ in X. Then, $\mathring{X}_+$ is nonempty and given by:

$$\left\{(u,v) \in X : u, v > 0 \text{ in } \Omega \text{ and } \frac{\partial u}{\partial \nu}, \frac{\partial v}{\partial \nu} < 0 \text{ on } \partial\Omega\right\}.$$

Let (u,v) in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$. The weak formulation of (1.1) is given by

$$\lambda \int_\Omega a(x)|v|^{\beta_1 - 1} v \Phi dx = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \Phi dx \tag{1.4}$$

and

$$\mu \int_\Omega b(x)|u|^{\beta_2 - 1} u \Psi dx = \int_\Omega |\nabla v|^{q-2} \nabla v \nabla \Psi dx$$

for any $(\Phi, \Psi) \in (C_0^1(\Omega))^2$.

We say that $(\lambda, \mu) \in \mathbb{R}_+^* \times \mathbb{R}_+^* = (0, \infty)^2$ is an eigenvalue of (1.1) if the system admits a nontrivial weak solution (φ, ψ) in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ which is called an eigenfunction corresponding to (λ, μ). We also say that (λ, μ) is a principal eigenvalue if admits a positive eigenfunction (φ, ψ). Finally, the couple (λ, μ) is said to be simple in $\mathring{X}_+$ if for any eigenfunctions $(\varphi, \psi), (\tilde{\varphi}, \tilde{\psi}) \in \mathring{X}_+$, there exists $\theta > 0$ such that $\tilde{\varphi} = \theta \varphi$ and $\tilde{\psi} = \theta \mu^{\frac{1}{\beta_2}} \psi$ in Ω.

During the past decades, the system (1.1) has been extensively studied in the case $p = q = 2$. For example, we can list the papers [4, 11, 14, 20, 28, 32], where several results on existence, nonexistence and uniqueness of nontrivial solutions have been developed when $\beta_1 \beta_2 \neq 1$. The case $\beta_1 \beta_2 = 1$ was treated in Montenegro [29]. Namely, the author proved that the set of principal eigenvalues (λ, μ) of the system (1.1) is nonempty and determines a curve in the cartesian plane which satisfies some properties as simplicity, continuity, monotonicity and local isolation. We also refer to [30] where a biparameter elliptic system was considered.

For the general case $p, q > 1$, we refer to [5] when $\beta_1 \beta_2 > (p-1)(q-1)$ and [7] when $\beta_1 \beta_2 = (p-1)(q-1)$. For instance, Cuesta and Takáč [7] showed that the set of principal eigenvalues of (1.1) is given by

$$\mathcal{C}_1(a, b, \Omega) := \left\{ (\lambda, \mu) \in (\mathbb{R}_+^*)^2 : \lambda^{\frac{1}{\sqrt{\beta_1 (p-1)}}} \mu^{\frac{1}{\sqrt{\beta_2 (q-1)}}} = \Lambda'(a, b, \Omega) \right\}$$

for some $\Lambda'(a, b, \Omega) > 0$, satisfying:

(a) (Uniqueness) $(\lambda, \mu) \in \mathcal{C}_1(a, b, \Omega)$ if and only if $(\lambda, \mu) \in \mathbb{R}_+ \times \mathbb{R}_+$ is a principal eigenvalue of the problem (1.1);

(b) (Simplicity in $\mathring{X}_+$) The principal curve $\mathcal{C}_1(a, b, \Omega)$ is simple in $\mathring{X}_+$, i.e., (λ, μ) is simple in $\mathring{X}_+$ for all $(\lambda, \mu) \in \mathcal{C}_1(a, b, \Omega)$;

(c) (Simplicity in X) Let $(\varphi, \psi) \in X$ be an eigenfunction associated to $(\lambda, \mu) \in \mathcal{C}_1(a, b, \Omega)$. So, either $(\varphi, \psi) \in \mathring{X}_+$ or $(-\varphi, -\psi) \in \mathring{X}_+$.

Let $\mathcal{R}_1(a, b, \Omega)$ be the open region in the first quadrant below $\mathcal{C}_1(a, b, \Omega)$, that is,

$$\mathcal{R}_1(a, b, \Omega) = \left\{ (\lambda, \mu) \in (\mathbb{R}_+^*)^2 : \lambda^{\frac{1}{\sqrt{\beta_1 (p-1)}}} \mu^{\frac{1}{\sqrt{\beta_2 (q-1)}}} < \Lambda'(a, b, \Omega) \right\}.$$

We say that the principal curve $\mathcal{C}_1(a, b, \Omega)$ is locally isolated above (or below) if for each $(\lambda_1, \mu_1) \in \mathcal{C}_1(a, b, \Omega)$, there is $\varepsilon = \varepsilon(\lambda_1, \mu_1) > 0$ such that the system (1.1) does not have any eigenvalue in $B_\varepsilon(\lambda_1, \mu_1) \cap \overline{\mathcal{R}_1(a, b, \Omega)}^c$ (or $B_\varepsilon(\lambda_1, \mu_1) \cap \mathcal{R}_1(a, b, \Omega)$).

Theorem 1.1. *Let $p, q \in (1, \infty)$, Ω be a bounded domain in $\mathbb{R}^n$ with a C^2-boundary, $\beta_1, \beta_2 > 0$ be such that $\beta_1 \beta_2 = (p-1)(q-1)$ and a, b, $\tilde{a}$ and $\tilde{b}$ be functions in $L^\infty(\Omega)$ satisfying (1.2) in Ω. Then, the curve $\mathcal{C}_1(a, b, \Omega)$ to the system (1.1) satisfies:*

(i) (Strict monotonicity with respect to the domain) Let D be a bounded domain in $\mathbb{R}^n$ with a C^2-boundary, such that $\overline{D} \subset \Omega$. Then, $\Lambda'(a, b, \Omega) < \Lambda'(a, b, D)$;

(ii) *(Monotonicity with respect to the weights) Suppose that* $a \le \tilde{a}$ *and* $b \le \tilde{b}$ *in* Ω. *Then,* $\Lambda'(a,b,\Omega) \ge \Lambda'(\tilde{a},\tilde{b},\Omega)$. *Moreover, if* $(a,b) \not\equiv (\tilde{a},\tilde{b})$ *then* $\Lambda'(a,b,\Omega) > \Lambda'(\tilde{a},\tilde{b},\Omega)$;

(iii) *(Local isolation above) The curve* $\mathcal{C}_1(a,b,\Omega)$ *is locally isolated above;*

(iv) *(Local isolation below) The system* (1.1) *does not admit any eigenvalues in* $\mathcal{R}_1(a,b,\Omega)$. *In particular, the curve* $\mathcal{C}_1(a,b,\Omega)$ *is locally isolated below;*

(v) *(Continuity of the principal eigenvalue with respect to the weight functions a and b) Let* $(a_k)_{k\ge1}$ *and* $(b_k)_{k\ge1}$ *be sequences of weight functions in* $L^\infty(\Omega)$ *which are positive in* Ω. *Assume that* $a_k \to a$ *and* $b_k \to b$ *uniformly in* Ω. *If* $a,b>0$ *in* $\overline{\Omega}$, *then* $\Lambda'(a_k,b_k,\Omega) \to \Lambda'(a,b,\Omega)$.

Note that the part (i) of Theorem 1.1 is essential for establish the Harnack inequality associated to the system (1.1). A very important application of Harnack inequality is the obtention of principal eigenvalues associated to the problems in general domains. Parts (ii) and (v) of Theorem 1.1 are important tools to furnish a min–max type characterization for principal curves associated to the problems whose solutions are not usually classical.

Now, we show an explicit lower estimate for principal eigenvalues of system (1.1) in terms of the Lebesque measure of Ω, more specifically, a counterpart of [2, Theorem 10.1] to degenerate elliptic systems. More recently, it was proved in [23] for Lane–Emden systems involving second order uniformly elliptic operators. Their proof use in a crucial way the celebrated Faber–Krahn inequality due to Faber [13] and Krahn [22]. We present now some essential ingredients:

For $1 \le p < n$, we use the sharp Sobolev inequality for any $u \in W_0^{1,p}(\Omega)$,

$$\|u\|_{L^{p^*}(\Omega)} \le c_{n,p}\|\nabla u\|_{L^p(\Omega)}, \tag{1.5}$$

where $p^* = \frac{np}{n-p}$ and an explicit formula of $c_{n,p}$ depending only on n and p was proved in [1,33].

For $p = n$ and $u \in W_0^{1,p}(\Omega)$, we have

$$\|u\|_{L^\eta(\Omega)} \le C(n)|\Omega|^{\frac{1}{\eta}}\|\nabla u\|_{L^p(\Omega)}, \tag{1.6}$$

where $C(n)>0$, $1 \le \eta < \infty$ and $|\cdot|$ stands for the Lebesgue measure of $\mathbb{R}^n$.

For $p > n$, there is a constant $C(n,p) > 0$ such that

$$\|u\|_{L^\infty(\Omega)} \le C(n,p)|\Omega|^{-\frac{1}{p^*}}\|\nabla u\|_{L^p(\Omega)}, \tag{1.7}$$

for all $u \in W_0^{1,p}(\Omega)$.

Consider now the nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u - \lambda |u|^{p-2}u = 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In [25], the author proved that the first eigenvalue $\lambda_{1,p}(\Omega)$ has the following properties, it is strictly positive, simple in any bounded connected Ω and characterized by

$$\lambda_{1,p}(\Omega) = \min_{\varphi \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla\varphi(x)|^p dx}{\int_\Omega |\varphi(x)|^p dx}.$$

By the Cheeger's constant (see [3,21]), we have

$$\lambda_{1,p}(B_1) \geq \left(\frac{n}{p}\right)^p, \tag{1.8}$$

where B_1 is the unit ball of $\mathbb{R}^n$.

Faber–Krahn inequality for the first eigenvalue of $-\Delta_p$. Let $1 < p < \infty$ and Ω be a open set in $\mathbb{R}^n$ with finite Lebesgue measure. Then,

$$\lambda_{1,p}(\Omega) \geq \lambda_{1,p}(B_1)|B_1|^{\frac{p}{n}}|\Omega|^{-\frac{p}{n}}.$$

Our main result gives an explicit lower estimate for principal eigenvalues of system (1.1) in terms of the measure of Ω and the weighted functions a and b.

Precisely, we have:

Theorem 1.2. *Let (λ, μ) be a principal eigenvalue of (1.1). Suppose $\beta_1 \geq \beta_2$, $p \leq q$ and $|\Omega| \leq 1$.*

(i) For $1 < p < n$, $q < p^*$ and

$$q - 1 \leq \beta_1 < \frac{np - n + p}{n - p},$$

there exists an explicit constant $C = C(p, q, \beta_1, \beta_2, n, a, b) > 0$ such that

$$\lambda + \mu^{\frac{p(q-1)}{q\beta_2}} \geq C\left(\frac{n}{q}\right)^{p\theta_p}|B_1|^{\theta_p\frac{p}{n}}|\Omega|^{-\theta_p\frac{p}{n}}, \tag{1.9}$$

where

$$\frac{1}{\beta_1 + 1} = \frac{\theta_p}{p} + \frac{1 - \theta_p}{p^*};$$

(ii) For $p = q = n$ and $q - 1 \leq \beta_1 < \infty$, the estimate (1.9) holds with

$$\frac{1}{\beta_1 + 1} = \frac{\theta_p}{p} + \frac{1 - \theta_p}{2(\beta_1 + 1)};$$

(iii) For $n < p$ and $q - 1 \leq \beta_1 < \infty$, there is an explicit constant $C = C(p, q, \beta_1, \beta_2, n, a, b) > 0$ such that

$$\lambda + \mu^{\frac{p(q-1)}{q\beta_2}} \geq C\left(\frac{n}{q}\right)^{\frac{\theta_q\beta_1 p}{p-1}}|B_1|^{\theta_p\frac{p}{n}}|\Omega|^{-\theta_p\frac{p}{n}}, \tag{1.10}$$

where $\theta_p = \frac{p}{\beta_1+1}$ and $\theta_q = \frac{q}{\beta_1+1}$;

(iv) For $n = p < q$ and $q - 1 \leq \beta_1 < \infty$, we have (1.10) holds, with

$$\frac{1}{\beta_1 + 1} = \frac{\theta_p}{p} + \frac{1 - \theta_p}{2(\beta_1 + 1)} \quad \textit{and} \quad \theta_q = \frac{q}{\beta_1 + 1}.$$

In particular,

$$\lim_{|\Omega| \downarrow 0} \Lambda'(a, b, \Omega) = \infty.$$

Using the ideas of the proof of Theorem 1.2, we obtain the following result:

Theorem 1.3. *Let (λ,μ) be a principal eigenvalue of (1.1). Suppose $\beta_1 \leq \beta_2$, $p \leq q$ and $|\Omega| \leq 1$.*

(i) For $1 < p < n$, $q < p^*$ and

$$\frac{(p-1)(q-1)(n-p)}{np-n+p} < \beta_1 \leq p-1,$$

there exists an explicit constant $C = C(p,q,\beta_1,\beta_2,n,a,b) > 0$ such that

$$\lambda^{\frac{q(p-1)}{p\beta_1}} + \mu \geq C\left(\frac{n}{q}\right)^{rq} |B_1|^{r\frac{q}{n}} |\Omega|^{-r\frac{q}{n}}, \tag{1.11}$$

where $r := \min\left\{\theta_p, \theta_q \frac{\beta_2}{q-1}\right\}$,

$$\frac{1}{\beta_2+1} = \frac{\theta_p}{p} + \frac{1-\theta_p}{p^*}$$

and

$$\begin{cases} \frac{1}{\beta_2+1} = \frac{\theta_q}{q} + \frac{1-\theta_q}{q^*} & \text{if } 1 < q < n; \\ \frac{1}{\beta_2+1} = \frac{\theta_q}{q} + \frac{1-\theta_q}{p^*} & \text{if } q = n; \\ \theta_q = \frac{q}{\beta_2+1} & \text{if } q > n; \end{cases}$$

(ii) For $p = q = n$ and $0 < \beta_1 \leq p-1$, the estimate (1.11) holds with $r = \theta_q \frac{\beta_2}{q-1}$ and

$$\frac{1}{\beta_2+1} = \frac{\theta_q}{q} + \frac{1-\theta_q}{2(\beta_2+1)};$$

(iii) For $n < p$ and $0 < \beta_1 \leq p-1$, there is an explicit constant $C = C(p,q,\beta_1,\beta_2,n,a,b) > 0$ such that

$$\lambda^{\frac{q(p-1)}{p\beta_1}} + \mu \geq C\left(\frac{n}{q}\right)^{sq} |B_1|^{r\frac{q}{n}} |\Omega|^{-r\frac{q}{n}}, \tag{1.12}$$

where $s := \max\left\{\theta_p, \theta_q \frac{\beta_2}{q-1}\right\}$, $\theta_p = \frac{p}{\beta_2+1}$ and $\theta_q = \frac{q}{\beta_2+1}$;

(iv) For $n = p < q$ and $0 < \beta_1 \leq p-1$, we have (1.12) holds, with

$$\frac{1}{\beta_2+1} = \frac{\theta_p}{p} + \frac{1-\theta_p}{2(\beta_2+1)} \quad \text{and} \quad \theta_q = \frac{q}{\beta_2+1}.$$

In particular,

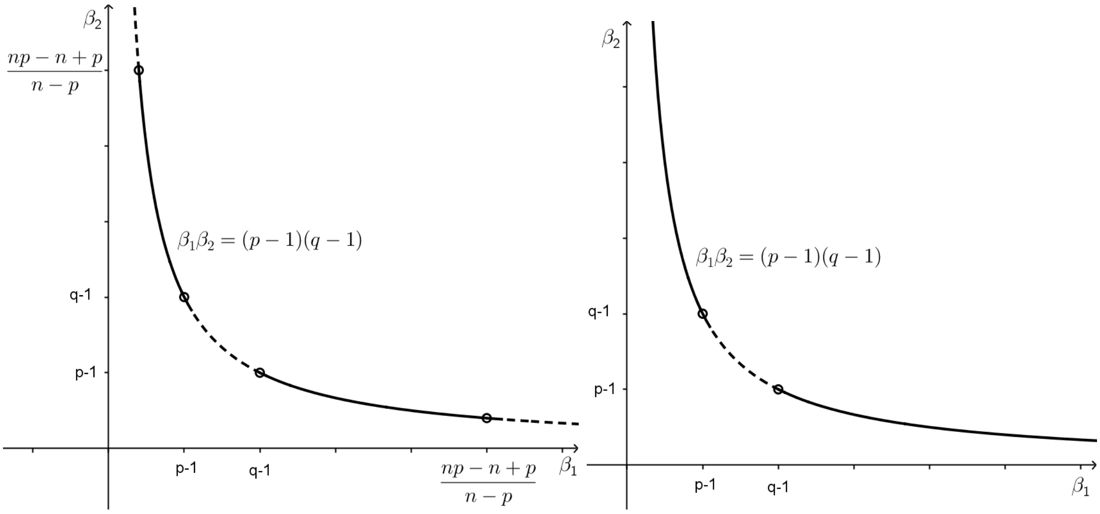
$$\lim_{|\Omega|\downarrow 0} \Lambda'(a,b,\Omega) = \infty.$$

Note that, supposing $p \leq q$, we get an explicit lower estimate for principal eigenvalues of system (1.1) for the range on β_1 and β_2,

$$\frac{(p-1)(q-1)(n-p)}{np-n+p} < \beta_1,\beta_2 \leq p-1 \quad \text{and} \quad q-1 \leq \beta_1,\beta_2 < \frac{np-n+p}{n-p}$$

for $1 < p < n$ and $0 < \beta_1,\beta_2 \leq p-1$ and $q-1 \leq \beta_1,\beta_2 < \infty$ for $p \geq n$. In particular, the result holds for all hyperbole $\beta_1\beta_2 = (p-1)(q-1)$ if $p = q \geq n$. The problem remains open in other remaining cases (see Figure 1.1). Clearly, the case $q < p$ follows similarly.

Our approach is inspired by the papers [2,7,23,29]. By mean of topological arguments, strong maximum principle, Hopf's lemma and (weak and strong) comparison principles related to the p-Laplacian operator, we prove five properties of $\mathcal{C}_1(a,b,\Omega)$ which will be presented in Section 2. In Section 3, by using the Faber–Krahn inequality for the first eigenvalue of $-\Delta_p$, variational characterization of $\lambda_{1,p}(\Omega)$, Hölder, Young, interpolation and Sobolev inequalities, we show Theorem 1.2.



(a) Case $1 < p < n$ and $p \le q < p^*$ (b) Case $p \ge n$ and $p \le q$

Figure 1.1: Couples (β_1, β_2).

2 Proof of Theorem 1.1

In this section we provide some essential properties satisfied by the principal curve $\mathcal{C}_1(a,b,\Omega)$ which is organized into five propositions.

We first show the strict monotonicity of the principal eigenvalues with respect to the domain stated in the part (i) of Theorem 1.1. Precisely:

Proposition 2.1. *Let D and Ω be two bounded domain in $\mathbb{R}^n$ with a C^2-boundary, such that $\overline{D} \subset \Omega$ and $\mathcal{C}_1(a,b,\Omega)$ and $\mathcal{C}_1(a,b,D)$ principal curves. Then, $\Lambda'(a,b,\Omega) < \Lambda'(a,b,D)$.*

Proof. Assume by contradiction that $\Lambda'(a,b,\Omega) \ge \Lambda'(a,b,D)$. Let $(\lambda_1,\mu_1) \in \mathcal{C}_1(a,b,\Omega)$ and $(\tilde\lambda_1,\tilde\mu_1) \in \mathcal{C}_1(a,b,D)$ be such that $\frac{\lambda_1}{\mu_1} = \frac{\tilde\lambda_1}{\tilde\mu_1}$. Thus, $\lambda_1 \ge \tilde\lambda_1$ and $\mu_1 \ge \tilde\mu_1$. Let $(\varphi,\psi), (\tilde\varphi,\tilde\psi)$ be positive eigenfunctions associated to the principal eigenvalues $(\lambda_1,\mu_1), (\tilde\lambda_1,\tilde\mu_1)$, respectively. Define

$$c := \min\left\{\min_{x\in\overline{D}} \varphi(x), \min_{x\in\overline{D}} \psi(x)\right\} > 0.$$

We claim that $\varphi \ge \tilde\varphi$ and $\psi \ge \tilde\psi$ in D. In fact, assume by contradiction that $\varphi < \tilde\varphi$ or $\psi < \tilde\psi$ somewhere in D. In this case, the set $\Gamma := \{\gamma > 0 : \varphi > \gamma\tilde\varphi \text{ and } \psi > \gamma^\omega\tilde\psi \text{ in } D\}$ is upper bounded, where $\omega := \frac{p-1}{\beta_1}$. In addition, the positivity of φ and ψ in $\overline{D}$ imply that Γ is nonempty. Define $0 < \overline{\gamma} := \sup\Gamma < 1$. It is clear that $\varphi \ge \overline{\gamma}\tilde\varphi$ and $\psi \ge \overline{\gamma}^\omega\tilde\psi$ in D, with $\varphi \not\equiv \overline{\gamma}\tilde\varphi$ and $\psi \not\equiv \overline{\gamma}^\omega\tilde\psi$ in D. Moreover, $\varphi \ge \overline{\gamma}\tilde\varphi + c$ and $\psi \ge \overline{\gamma}^\omega\tilde\psi + c$ on ∂D. So, we get

$$\begin{cases} -\Delta_p(\overline{\gamma}\tilde\varphi + c) = -\Delta_p(\overline{\gamma}\tilde\varphi) = \tilde\lambda_1 a(x)(\overline{\gamma}^\omega\tilde\psi)^{\beta_1} \le (\not\equiv)\ \lambda_1 a(x)\psi^{\beta_1} = -\Delta_p(\varphi) \\ -\Delta_q(\overline{\gamma}^\omega\tilde\psi + c) = -\Delta_q(\overline{\gamma}^\omega\tilde\psi) = \tilde\mu_1 b(x)(\overline{\gamma}\tilde\varphi)^{\beta_2} \le (\not\equiv)\ \mu_1 b(x)\psi^{\beta_2} = -\Delta_q(\psi) \end{cases} \quad \text{in } D.$$

Then, applying the weak comparison principle to each above equation (see [18] or [34, Lemma 3.1]), we derive $\varphi \ge \overline{\gamma}\tilde\varphi + c$ and $\psi \ge \overline{\gamma}^\omega\tilde\psi + c$ in D. Thus, $\varphi > \overline{\gamma}\tilde\varphi$ and $\psi > \overline{\gamma}^\omega\tilde\psi$ in D. So, we can find $0 < \varepsilon < 1$ such that $\varphi > (\overline{\gamma}+\varepsilon)\tilde\varphi$ and $\psi > (\overline{\gamma}+\varepsilon)^\omega\tilde\psi$ in D, contradicting the definition of $\overline{\gamma}$. Therefore, $\varphi \ge \tilde\varphi$ and $\psi \ge \tilde\psi$ in D. Note that $(\kappa\tilde\varphi, \kappa^\omega\tilde\psi)$, $\kappa > 0$, are also eigenfunctions associated to $(\tilde\lambda_1,\tilde\mu_1)$. Then, $\varphi \ge \kappa\tilde\varphi$ and $\psi \ge \kappa^\omega\tilde\psi$ in D for all $\kappa > 0$; and from there we arrive at a contradiction. This concludes the desired proof. $\square$

We now show the monotonicity of principal eigenvalues with respect to the weights which corresponds to the part (ii) of Theorem 1.1.

Proposition 2.2. *Let a, b, $\tilde{a}$ and $\tilde{b}$ be functions in $L^\infty(\Omega)$ satisfying (1.2) such that $a \leq \tilde{a}$ and $b \leq \tilde{b}$ in Ω. Then, $\Lambda'(a,b,\Omega) \geq \Lambda'(\tilde{a},\tilde{b},\Omega)$. Moreover, if $(a,b) \not\equiv (\tilde{a},\tilde{b})$ then $\Lambda'(a,b,\Omega) > \Lambda'(\tilde{a},\tilde{b},\Omega)$.*

Proof. Assume by contradiction that $\Lambda'(a,b,\Omega) < \Lambda'(\tilde{a},\tilde{b},\Omega)$. Let $(\lambda_1(a,b),\mu_1(a,b)) \in \mathcal{C}_1(a,b,\Omega)$ and $(\lambda_1(\tilde{a},\tilde{b}),\mu_1(\tilde{a},\tilde{b})) \in \mathcal{C}_1(\tilde{a},\tilde{b},\Omega)$ be such that $\frac{\lambda_1(a,b)}{\mu_1(a,b)} = \frac{\lambda_1(\tilde{a},\tilde{b})}{\mu_1(\tilde{a},\tilde{b})}$. Thus,

$$\lambda_1(a,b) < \lambda_1(\tilde{a},\tilde{b}) \quad \text{and} \quad \mu_1(a,b) < \mu_1(\tilde{a},\tilde{b}).$$

Let (φ,ψ) and $(\tilde{\varphi},\tilde{\psi})$ be positive eigenfunctions associated to the principal eigenvalues

$$(\lambda_1(a,b),\mu_1(a,b)) \quad \text{and} \quad (\lambda_1(\tilde{a},\tilde{b}),\mu_1(\tilde{a},\tilde{b})),$$

respectively. Consider the set $\Gamma = \{\gamma > 0 : \tilde{\varphi} > \gamma\varphi \text{ and } \tilde{\psi} > \gamma^\omega \psi \text{ in } \Omega\}$, where $\omega := \frac{p-1}{\beta_1}$. Note that Γ is upper bounded, and by strong maximum principle (see [18, 34, 36]) Γ is nonempty. Define $\overline{\gamma} = \sup \Gamma > 0$. Note that, $\tilde{\varphi} \geq \overline{\gamma}\varphi$ and $\tilde{\psi} \geq \overline{\gamma}^\omega \psi$ in Ω.

Since $(-\Delta_p)^{-1}$ and $(-\Delta_q)^{-1}$ are strictly order preserving, we can find $0 < \varepsilon < 1$ such that $\tilde{\varphi} > (\overline{\gamma}+\varepsilon)\varphi$ and $\tilde{\psi} > (\overline{\gamma}+\varepsilon)^\omega\psi$ in Ω which clearly contradicts the definition of $\overline{\gamma}$. Therefore, $\Lambda'(a,b,\Omega) \geq \Lambda'(\tilde{a},\tilde{b},\Omega)$.

Finally, assume that $(a,b) \not\equiv (\tilde{a},\tilde{b})$. Arguing again by contradiction, assume that

$$\Lambda'(a,b,\Omega) = \Lambda'(\tilde{a},\tilde{b},\Omega).$$

Let (φ,ψ) and $(\tilde{\varphi},\tilde{\psi})$ be positive eigenfunctions corresponding to the principal eigenvalues $(\lambda_1(a,b),\mu_1(a,b)) = (\lambda_1(\tilde{a},\tilde{b}),\mu_1(\tilde{a},\tilde{b}))$. Proceeding similarly to the first part of the proof, we obtain $\Lambda'(a,b,\Omega) > \Lambda'(\tilde{a},\tilde{b},\Omega)$. This ends the proof. □

The two next propositions are dedicated the local isolation above and below the principal curve $\mathcal{C}_1(a,b,\Omega)$. These correspond to the parts (iii) and (iv) of Theorem 1.1, respectively. Precisely:

Proposition 2.3. *The curve $\mathcal{C}_1(a,b,\Omega)$ is locally isolated above.*

Proof. Assume by contradiction that the claim is false. Thus, there are $(\lambda_1,\mu_1) \in \mathcal{C}_1(a,b,\Omega)$ and a sequence of eigenvalues $((\lambda_k,\mu_k))_{k\geq 1}$ contained in $B_{\varepsilon_k}(\lambda_1,\mu_1) \cap \overline{\mathcal{R}_1(a,b,\Omega)}^c$, where $\varepsilon_k \to 0$ with $\varepsilon_k > 0$ for all $k \in \mathbb{N}$. Let (φ_k,ψ_k) an eigenfunction associated to (λ_k,μ_k); that is, a weak solution of the system

$$\begin{cases} -\Delta_p\varphi_k = \lambda_k a(x)|\psi_k|^{\beta_1-1}\psi_k & \text{in } \Omega; \\ -\Delta_q\psi_k = \mu_k b(x)|\varphi_k|^{\beta_2-1}\varphi_k & \text{in } \Omega; \\ \varphi_k = \psi_k = 0 & \text{on } \partial\Omega, \end{cases}$$

where at least one of $-\varphi_k$ or $-\psi_k$ does not belong to $\mathring{X}_+$. Define the functions

$$u_k := \frac{\varphi_k}{\|\psi_k\|_{L^\infty(\Omega)}^{\frac{\beta_1}{p-1}}}, \quad \tilde{u}_k := \frac{\varphi_k}{\|\varphi_k\|_{L^\infty(\Omega)}}, \quad \tilde{v}_k := \frac{\psi_k}{\|\psi_k\|_{L^\infty(\Omega)}} \quad \text{and} \quad v_k := \frac{\psi_k}{\|\varphi_k\|_{L^\infty(\Omega)}^{\frac{\beta_2}{q-1}}}.$$

Then, we have $0 \leq |\tilde{u}_k|, |\tilde{v}_k| \leq 1$ in Ω. Therefore, the right-hand side of the following system

$$\begin{cases} -\Delta_p u_k = \lambda_k a(x)|\tilde{v}_k|^{\beta_1 - 1}\tilde{v}_k & \text{in } \Omega; \\ -\Delta_q v_k = \mu_k b(x)|\tilde{u}_k|^{\beta_2 - 1}\tilde{u}_k & \text{in } \Omega; \\ \varphi_k = \psi_k = 0 & \text{on } \partial\Omega; \end{cases} \tag{2.1}$$

is uniformly bounded in $(L^\infty(\Omega))^2$. It follows the sequences $(u_k)_{k\geq 1}$ and $(v_k)_{k\geq 1}$ are bounded in $C_0^{1,\alpha}(\overline{\Omega})$, by regularity and, in addition, also bounded in $L^\infty(\Omega)$; i.e., there exists a constant $C > 0$ such that $\|u_k\|_{L^\infty(\Omega)}, \|v_k\|_{L^\infty(\Omega)} \leq C$ for all $k \in \mathbb{N}$. Therefore, $\|\varphi_k\|_{L^\infty(\Omega)}$ is uniformly bounded if, and only if, $\|\psi_k\|_{L^\infty(\Omega)}$ is uniformly bounded.

First, we assume that, both $\|\varphi_k\|_{L^\infty(\Omega)}$ and $\|\psi_k\|_{L^\infty(\Omega)}$ are uniformly bounded. Applying the regularity result in $C_0^{1,\alpha}(\overline{\Omega})$, we get $(\varphi_k)_{k\geq 1}$ and $(\psi_k)_{k\geq 1}$, are bounded in $C_0^{1,\alpha}(\overline{\Omega})$. Since Ω is bounded, by Arzelà–Ascoli Theorem, up to a subsequence, we derive the convergence

$$\varphi_k \to \varphi \quad \text{and} \quad \psi_k \to \psi \quad \text{in } C_0^1(\overline{\Omega}) \quad \text{as } k \to \infty. \tag{2.2}$$

Thus, $(\varphi, \psi) \in (C_0^1(\Omega))^2$ is a weak solution of the system

$$\begin{cases} -\Delta_p \varphi = \lambda_1 a(x)|\psi|^{\beta_1 - 1}\psi & \text{in } \Omega; \\ -\Delta_q \psi = \mu_1 b(x)|\varphi|^{\beta_2 - 1}\varphi & \text{in } \Omega; \\ \varphi = \psi = 0 & \text{on } \partial\Omega. \end{cases}$$

By simplicity in X property (c), we must have either $(\varphi, \psi) \in \mathring{X}_+$ or $(-\varphi, -\psi) \in \mathring{X}_+$. If $(\varphi, \psi) \in \mathring{X}_+$, from the convergence in (2.2), we obtain $(\varphi_k, \psi_k) \in \mathring{X}_+$ for k sufficiently large. So, by uniqueness property (a), we have $(\lambda_k, \mu_k) \in \mathcal{C}_1(a, b, \Omega)$ for k large enough, contradicting that $(\lambda_k, \mu_k) \in \overline{\mathcal{R}_1(a, b, \Omega)}^c$ for all $k \in \mathbb{N}$. Then, we must have $(-\varphi, -\psi) \in \mathring{X}_+$. We now obtain $(-\varphi_k, -\psi_k) \in \mathring{X}_+$ for k sufficiently large, by convergence in (2.2). But this contradicts our hypothesis that at least one of $-\varphi_k$ or $-\psi_k$ doesn't belong to $\mathring{X}_+$ for all $k \in \mathbb{N}$.

Now, we assume that, $\|\varphi_k\|_{L^\infty(\Omega)} \to \infty$ and $\|\psi_k\|_{L^\infty(\Omega)} \to \infty$ as $k \to \infty$. For a subsequence indicated again by $((\varphi_k, \psi_k))_{k\geq 1}$, there is a function $(\tilde{\varphi}, \tilde{\psi}) \in (C_0^1(\overline{\Omega}))^2$, such that $\|\tilde{\varphi}\|_{L^\infty(\Omega)} = \|\tilde{\psi}\|_{L^\infty(\Omega)} = 1$,

$$\tilde{u}_k \to \tilde{\varphi} \quad \text{and} \quad \tilde{v}_k \to \tilde{\psi} \quad \text{in } C_0^1(\overline{\Omega}) \quad \text{as } k \to \infty. \tag{2.3}$$

Moreover, there are $\tilde{\lambda}, \tilde{\mu} \in \mathbb{R}$ such that $\tilde{\lambda}^{\beta_2}\tilde{\mu}^{p-1} = 1$,

$$\|u_k\|_{L^\infty(\Omega)}^{\beta_2} \to \tilde{\mu} \quad \text{and} \quad \|v_k\|_{L^\infty(\Omega)}^{\beta_1} \to \tilde{\lambda} \quad \text{as } k \to \infty.$$

Letting $k \to \infty$ in problem (2.1), we obtain $(\tilde{\varphi}, \tilde{\psi}) \in (C_0^1(\overline{\Omega}))^2$ is a weak solution of the problem

$$\begin{cases} -\Delta_p \tilde{\varphi} = \lambda_1 \tilde{\lambda} a(x)|\tilde{\psi}|^{\beta_1 - 1}\tilde{\psi} & \text{in } \Omega; \\ -\Delta_q \tilde{\psi} = \mu_1 \tilde{\mu} b(x)|\tilde{\varphi}|^{\beta_2 - 1}\tilde{\varphi} & \text{in } \Omega; \\ \tilde{\varphi} = \tilde{\psi} = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore, $(\lambda_1\tilde{\lambda}, \mu_1\tilde{\mu}) \in \mathcal{C}_1(a, b, \Omega)$. By simplicity in X property (c), we must have either $(\tilde{\varphi}, \tilde{\psi}) \in \mathring{X}_+$ or $(-\tilde{\varphi}, -\tilde{\psi}) \in \mathring{X}_+$. Again, we obtain a contradiction in an analogous way, instead of convergence in (2.2), we invoke convergence in (2.3). This ends the proof. □

Proposition 2.4. *The system* (1.1) *does not admit any eigenvalues in* $\mathcal{R}_1(a, b, \Omega)$. *In particular, the curve* $\mathcal{C}_1(a, b, \Omega)$ *is locally isolated below.*

Proof. Arguing by contradiction, assume that the system (1.1) has an eigenvalue $(\lambda,\mu) \in \mathcal{R}_1(a,b,\Omega)$. Let $(\lambda_1,\mu_1) \in \mathcal{C}_1(a,b,\Omega)$ be such that $\frac{\mu}{\lambda} = \frac{\mu_1}{\lambda_1}$. So, we have $\lambda < \lambda_1$ and $\mu < \mu_1$. Consider a positive eigenfunction (φ,ψ) corresponding to (λ_1,μ_1) and an eigenfunction (u,v) to (λ,μ). Now, we can assume that u or v is positive somewhere in Ω. Otherwise, we take $(-u,-v)$ in place of (u,v). Consider the set $\Gamma = \{\gamma > 0 : \varphi > \gamma u \text{ and } \psi > \gamma^\omega v \text{ in } \Omega\}$, where $\omega := \frac{p-1}{\beta_1}$. Notice that Γ is upper bounded. Moreover, by strong maximum principle, Γ is nonempty. Define the positive number $\overline{\gamma} = \sup\Gamma$. Note that, $\varphi \geq \overline{\gamma}u$ and $\psi \geq \overline{\gamma}^\omega v$ in Ω.

Since $\lambda < \lambda_1$, $\mu < \mu_1$ and $(-\Delta_p)^{-1}$ and $(-\Delta_q)^{-1}$ are strictly order preserving, we can find $0 < \varepsilon < 1$ such that $\varphi > (\overline{\gamma}+\varepsilon)u$ and $\psi > (\overline{\gamma}+\varepsilon)^\omega v$ in Ω. But this contradicts the definition of $\overline{\gamma}$. This concludes the proof. □

The last proposition establishes the continuity of the principal eigenvalue with respect to the weight functions a and b which corresponds to the part (v) of Theorem 1.1.

Proposition 2.5. *Let $(a_k)_{k\geq 1}$ and $(b_k)_{k\geq 1}$ be sequences of weight functions in $L^\infty(\Omega)$ which are positive in Ω. Assume that $a_k \to a$ and $b_k \to b$ uniformly in Ω. If $a,b > 0$ in $\overline{\Omega}$, then $\Lambda'(a_k,b_k,\Omega) \to \Lambda'(a,b,\Omega)$.*

Proof. Given a fixed number $r_0 > 0$, let $(\lambda_1(a,b),\mu_1(a,b)) \in \mathcal{C}_1(a,b,\Omega)$ and $(\lambda_1(a_k,b_k),\mu_1(a_k,b_k)) \in \mathcal{C}_1(a_k,b_k,\Omega)$ be such that

$$\frac{\lambda_1(a,b)}{\mu_1(a,b)} = \frac{\lambda_1(a_k,b_k)}{\mu_1(a_k,b_k)} = \frac{1}{r_0}, \quad \text{for all } k \in \mathbb{N}. \tag{2.4}$$

By definitions of $\Lambda'(a_k,b_k,\Omega)$ and $\Lambda'(a,b,\Omega)$ and equalities in (2.4), it suffices to prove only that $\lambda_1(a_k,b_k) \to \lambda_1(a,b)$ as $k \to \infty$. Assume by contradiction that there is a number $\varepsilon > 0$ such that

$$|\lambda_1(a_k,b_k) - \lambda_1(a,b)| \geq \varepsilon$$

for $k \in \mathbb{N}$. Without loss of generality, we can assume

$$\lambda_1(a_k,b_k) - \lambda_1(a,b) \geq \varepsilon\,.$$

Since a and b are positive on $\overline{\Omega}$, we can define $\delta \in \mathbb{R}$ to be such that

$$0 < \delta < \frac{\varepsilon}{\lambda_1(a,b)+\varepsilon}\ \min\left\{\inf_{x\in\Omega} a(x), \inf_{x\in\Omega} b(x)\right\}\,.$$

By uniform convergence of the sequences $(a_k)_{k\geq 1}$ and $(b_k)_{k\geq 1}$, up to a subsequence, we can assume without loss of generality that

$$a_k(x) \geq a(x) - \delta, \qquad b_k(x) \geq b(x) - \delta$$

for all $x \in \Omega$ and $k \in \mathbb{N}$. Let (φ_k,ψ_k) and (φ,ψ) be positive eigenfunctions associated to the principal eigenvalues

$$(\lambda_1(a_k,b_k),\mu_1(a_k,b_k)) \quad \text{and} \quad (\lambda_1(a,b),\mu_1(a,b))\,,$$

respectively. Then, by strong maximum principle, the usual set $\Gamma = \{\gamma > 0 : \varphi_k > \gamma\varphi \text{ and } \psi_k > \gamma^\omega\psi \text{ in } \Omega\}$ is nonempty and upper bounded, where $\omega := \frac{p-1}{\beta_1}$. Set $\overline{\gamma} := \sup\Gamma > 0$. Using the definitions of r_0, ε and δ and the above inequalities, we get

$$\begin{aligned}-\Delta_p(\overline{\gamma}\varphi) &= \lambda_1(a,b)a(x)(\overline{\gamma}^{\omega}\psi)^{\beta_1}\\ &= (\lambda_1(a,b)+\varepsilon)(a(x)-\delta)(\overline{\gamma}^{\omega}\psi)^{\beta_1} + (-\varepsilon a(x)+\lambda_1(a,b)\delta+\varepsilon\delta)(\overline{\gamma}^{\omega}\psi)^{\beta_1}\\ &< \lambda_1(a_k,b_k)a_k(x)\psi_k^{\beta_1} = -\Delta_p(\varphi_k)\,;\end{aligned}$$

$$\begin{aligned}-\Delta_q(\overline{\gamma}^{\omega}\psi) &= \mu_1(a,b)b(x)(\overline{\gamma}\varphi)^{\beta_2}\\ &= r_0(\lambda_1(a,b)+\varepsilon)(b(x)-\delta)(\overline{\gamma}\varphi)^{\beta_2} + r_0(-\varepsilon b(x)+\lambda_1(a,b)\delta+\varepsilon\delta)(\overline{\gamma}\varphi)^{\beta_2}\\ &< r_0\lambda_1(a_k,b_k)b_k(x)\varphi_k^{\beta_2} = \mu_1(a_k,b_k)b_k(x)\varphi_k^{\beta_2} = -\Delta_q(\psi_k);\end{aligned}$$

and $\varphi_k = \overline{\gamma}\varphi = \psi_k = \overline{\gamma}^{\omega}\psi = 0$ on $\partial\Omega$. Applying the strong comparison principle to each above equation (see [7, Theorem A.1]), we derive

$$\varphi_k > \overline{\gamma}\varphi,\ \psi_k > \overline{\gamma}^{\omega}\psi \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial\varphi_k}{\partial\nu} < \frac{\partial\overline{\gamma}\varphi}{\partial\nu},\ \frac{\partial\psi_k}{\partial\nu} < \frac{\partial\overline{\gamma}^{\omega}\psi}{\partial\nu} \quad \text{on } \partial\Omega.$$

Then, $\varphi_k > (\overline{\gamma}+\varepsilon)\varphi$ and $\psi_k > (\overline{\gamma}+\varepsilon)^{\omega}\psi$ in Ω for $0<\varepsilon<1$. But this contradicts the definition of $\overline{\gamma}$, and so concluding the proof. □

3 Proof of Theorem 1.2

We first prove the case $1 < p,q < n$. Let (φ,ψ) denote a principal eigenfunction corresponding to (λ,μ). Since

$$-\Delta_p\varphi = \lambda a(x)\psi^{\beta_1}$$

in the weak sense, then applying the equality (1.4) with $\Phi = \varphi$, we obtain

$$\lambda\int_\Omega a(x)\psi^{\beta_1}\varphi dx = \int_\Omega |\nabla\varphi|^p dx.$$

Moreover, by using Hölder and Young inequalities, we get

$$\int_\Omega a(x)\psi^{\beta_1}\varphi dx \le \|a\|_{L^\infty(\Omega)}\left(\frac{1}{p}\|\varphi\|^p_{L^{\beta_1+1}(\Omega)} + \frac{p-1}{p}\|\psi\|^{p\beta_1/(p-1)}_{L^{\beta_1+1}(\Omega)}\right).$$

Consequently,

$$\lambda D_1\left(\|\varphi\|^p_{L^{\beta_1+1}(\Omega)} + \|\psi\|^{p\beta_1/(p-1)}_{L^{\beta_1+1}(\Omega)}\right) \ge \int_\Omega |\nabla\varphi|^p dx, \tag{3.1}$$

where

$$D_1 = \max\left\{\frac{1}{p}\|a\|_{L^\infty(\Omega)}, \frac{p-1}{p}\|a\|_{L^\infty(\Omega)}, \frac{1}{q}\|b\|_{L^\infty(\Omega)}, \frac{q-1}{q}\|b\|_{L^\infty(\Omega)}\right\}.$$

Similarly, it follows from

$$-\Delta_q\psi = \mu b(x)\varphi^{\beta_2}$$

in the weak sense that

$$\mu\|b\|_{L^\infty(\Omega)}|\Omega|^{\frac{\beta_1-\beta_2}{\beta_1+1}}\left(\frac{q-1}{q}\|\varphi\|^{q\beta_2/(q-1)}_{L^{\beta_1+1}(\Omega)} + \frac{1}{q}\|\psi\|^q_{L^{\beta_1+1}(\Omega)}\right) \ge \int_\Omega |\nabla\psi|^q dx.$$

Now, since $|\Omega| \le 1$ and $\frac{p(q-1)}{q\beta_2} \ge 1$, we have

$$(D_1\mu)^{\frac{p(q-1)}{q\beta_2}} D_2\left(\|\varphi\|^p_{L^{\beta_1+1}(\Omega)} + \|\psi\|^{p\beta_1/(p-1)}_{L^{\beta_1+1}(\Omega)}\right) \ge \left(\int_\Omega |\nabla\psi|^q dx\right)^{\frac{p(q-1)}{q\beta_2}}, \tag{3.2}$$

where $D_2 = 2^{\frac{p(q-1)}{q\beta_2}-1}$. Thus, adding up (3.1) and (3.2) inequalities shows that

$$\lambda + \mu^{\frac{p(q-1)}{q\beta_2}} \geq \frac{1}{D_3}\left(\frac{\int_\Omega |\nabla\varphi|^p dx + \left(\int_\Omega |\nabla\psi|^q dx\right)^{\frac{p(q-1)}{q\beta_2}}}{\|\varphi\|^p_{L^{\beta_1+1}(\Omega)} + \|\psi\|^{p\beta_1/(p-1)}_{L^{\beta_1+1}(\Omega)}}\right),$$

where $D_3 = \max\left\{D_2 D_1^{\frac{p(q-1)}{q\beta_2}}, D_1\right\}$.

On the other hand, by interpolation inequality, inequality (1.5) and variational characterization of $\lambda_{1,p}(\Omega)$, we obtain

$$\frac{\int_\Omega |\nabla\varphi|^p dx}{\|\varphi\|^p_{L^{\beta_1+1}(\Omega)}} \geq \left(c_{n,p}\right)^{(\theta_p-1)p} \lambda_{1,p}(\Omega)^{\theta_p},$$

where

$$\frac{1}{\beta_1+1} = \frac{\theta_p}{p} + \frac{1-\theta_p}{p^*}$$

and

$$\frac{\left(\int_\Omega |\nabla\psi|^q dx\right)^{\frac{p(q-1)}{q\beta_2}}}{\|\psi\|^{p\beta_1/(p-1)}_{L^{\beta_1+1}(\Omega)}} \geq \left(c_{n,q}\right)^{(\theta_q-1)\frac{p\beta_1}{p-1}} \lambda_{1,q}(\Omega)^{\theta_q\frac{p\beta_1}{(p-1)q}},$$

where

$$\frac{1}{\beta_1+1} = \frac{\theta_q}{q} + \frac{1-\theta_q}{q^*}.$$

Furthermore, by Faber-Krahn inequality for the first eigenvalue of $-\Delta_p$ and inequality (1.8), we get

$$\lambda_{1,p}(\Omega) \geq \lambda_{1,p}(B_1)|B_1|^{\frac{p}{n}}|\Omega|^{-\frac{p}{n}} \geq \left(\frac{n}{p}\right)^p |B_1|^{\frac{p}{n}}|\Omega|^{-\frac{p}{n}}.$$

Then, using that $p \leq q$, $|\Omega| \leq 1$ and $\beta_1 \geq p-1$, we obtain

$$\lambda_{1,p}(\Omega)^{\theta_p}, \lambda_{1,q}(\Omega)^{\theta_q\frac{p\beta_1}{(p-1)q}} \geq \left(\frac{n}{q}\right)^{p\theta_p} |B_1|^{\theta_p\frac{p}{n}}|\Omega|^{-\theta_p\frac{p}{n}}.$$

Therefore,

$$\lambda + \mu^{\frac{p(q-1)}{q\beta_2}} \geq C\left(\frac{n}{q}\right)^{p\theta_p} |B_1|^{\theta_p\frac{p}{n}}|\Omega|^{-\theta_p\frac{p}{n}},$$

where $C = \frac{1}{D_3}\min\left\{\left(c_{n,p}\right)^{(\theta_p-1)p}, \left(c_{n,q}\right)^{(\theta_q-1)\frac{p\beta_1}{p-1}}\right\}$.

The rest of proof is analogue, by using interpolation inequality with θ_p and θ_q appropriate and instead of inequality (1.5), we invoke inequalities (1.6) and (1.7). This concludes the proof of the theorem.

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