# EXACT MULTIPLICITY OF POSITIVE SOLUTIONS IN SEMIPOSITONE PROBLEMS WITH CONCAVE-CONVEX TYPE NONLINEARITIES

Sudhasree Gadam and Joseph A. Iaia

Abstract. We study the existence, multiplicity, and stability of positive solutions to:

$$-u''(x) = \lambda f(u(x)) \text{ for } x \in (-1,1),\ \lambda > 0,$$
$$u(-1) = 0\ = u(1),$$

where $f : [0,\infty) \to \mathbb{R}$ is semipositone ($f(0) < 0$) and superlinear ($\lim_{t\to\infty} f(t)/t = \infty$). We consider the case when the nonlinearity $f$ is of concave-convex type having exactly one inflection point. We establish that $f$ should be appropriately concave (by establishing conditions on $f$) to allow multiple positive solutions. For any $\lambda > 0$, we obtain the exact number of positive solutions as a function of $f(t)/t$ and establish how the positive solution curves to the above problem change. Also, we give examples where our results apply. This work extends the work in [1] by giving a complete classification of positive solutions for concave-convex type nonlinearities.

## 1. Introduction

We study the positive solutions to the two point boundary value problem:

$$-u''(x) = \lambda f(u(x)) \text{ for } x \in (-1,1),\ \lambda > 0, \tag{1.1}$$
$$u(-1) = 0 = u(1), \tag{1.2}$$

where $f : [0,\infty) \to \mathbb{R}$ is a twice differentiable function such that:

$$f(0) < 0 \text{ (semipositone)}, \lim_{t\to\infty} \frac{f(t)}{t} = \infty \text{ (superlinear)}, \text{ and } f \text{ has a unique positive zero } \beta. \tag{1.3}$$

We define $F$ by $F(t) = \int_0^t f(s)\,ds$, and we observe that by (1.3):

$$F \text{ has a unique positive zero } \theta > \beta. \tag{1.4}$$

We also assume that $f$ has exactly one inflection point $t^*$ with:

$$f''(t) < 0 \text{ on } (0,\ t^*),\ f''(t) > 0 \text{ on } (t^*,\ \infty), \text{ and } t^* > \beta. \tag{1.5}$$

Since $(\frac{f(t)}{t})' = \frac{tf'(t)-f(t)}{t^2}$ and $(tf'(t) - f(t))' = tf''(t)$ with $f(0) < 0$, it follows from (1.5) that either:

$$(f(t)/t)' \geq 0 \text{ for all } t > 0, \text{ or} \tag{$1.5_1$}$$

$$(f(t)/t)' > 0 \text{ for } t \in (0,t_1) \cup (t_2,\infty) \text{ and } (f(t)/t)' < 0 \text{ for } t \in (t_1,t_2) \tag{$1.5_2$}$$

for some $t_1, t_2$ with $0 < t_1 < t^* < t_2$.

1991 *Mathematics Subject Classification.* Primary 34B15: Secondary 35J65.
*Key words and phrases.* Semipositone, concave, convex.

For future reference we define:

$$H(t) = F(t) - \frac{1}{2}tf(t) \tag{1.6}$$

and observe that:

$$H'(t) = -\frac{1}{2}t^2(f(t)/t)'. \tag{1.7}$$

Finally, for a positive solution of (1.1)-(1.2), we define:

$$\rho = \sup_{(-1,1)} u(x).$$

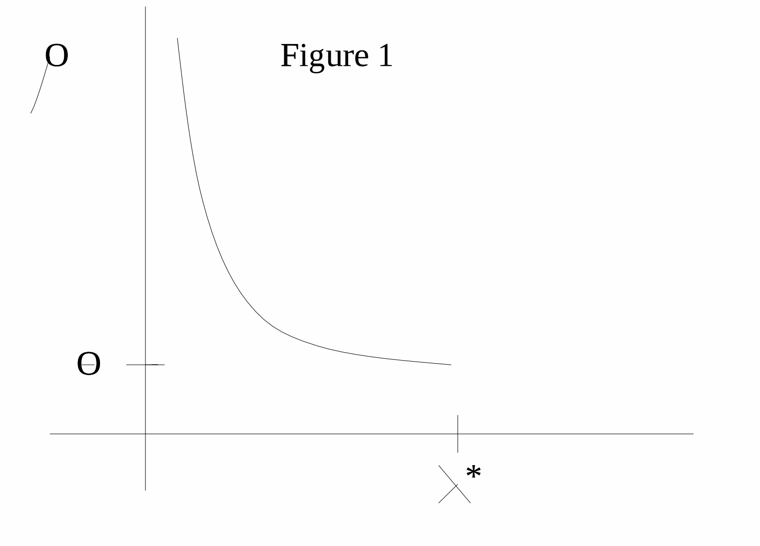
We refer the reader to [2, 3] where the classification $(1.5)_1$, $(1.5)_2$ helps in giving a complete description of positive solution curves for concave nonlinearities. In [7], Shi and Shivaji consider $(1.5)_2$ and obtain a similar result to Theorem 1 section (2) with reasonably different methods from ours.

We also note that in [9], Wang considers the positone problem ($f(0) > 0$) with $f$ initially convex and then concave. Finally, semipositone problems occur in several harvesting models (see [4]) and have been extensively studied in [1-3] and [5-8].

Our main results are:

**Theorem 1.**

(1) *If $f$ satisfies (1.3)-(1.5) and $(1.5)_1$, then there exists $\lambda^*$ with $0 < \lambda^* < \infty$ such that (1.1)-(1.2) has no positive solutions for $\lambda > \lambda^*$ and has a unique positive solution for $\lambda \in (0, \lambda^*]$ (see Fig. 1).*




*In addition, $\rho \equiv \rho_\lambda$ is a decreasing function of $\lambda$ with $\rho_\lambda : (0, \lambda^*] \to [\theta, \infty)$ such that $\rho_{\lambda^*} = \theta$ and $\lim_{\lambda \to 0^+} \rho_\lambda = +\infty$.*

(2) *If $f$ satisfies (1.3)-(1.5), $(1.5)_2$, and $H(t^*) \geq 0$, then there exist $\lambda_1, \lambda_2, \lambda^*$ with $0 < \lambda_1 < \lambda_2 < \infty$ and $\lambda_1 < \lambda^* < \infty$ such that (1.1)-(1.2) has no positive solutions for $\lambda > \max\{\lambda_2, \lambda^*\}$ and has a unique positive solution for $\lambda < \lambda_1$ while for $\lambda = \lambda_1$ it has exactly two positive solutions. Also, $\rho_{\lambda^*} = \theta$ and $\lim_{\lambda \to 0^+} \rho_\lambda = +\infty$.*

Subcase A*: If $\lambda_2 \leq \lambda^*$ then for $\lambda \in (\lambda_1, \lambda_2)$ (1.1)-(1.2) has exactly three positive solutions while for $\lambda = \lambda_2$ it has exactly two positive solutions. Finally, if $\lambda \in (\lambda_2, \lambda^*]$ then (1.1)-(1.2) has exactly one positive solution (see Fig. 2A).*

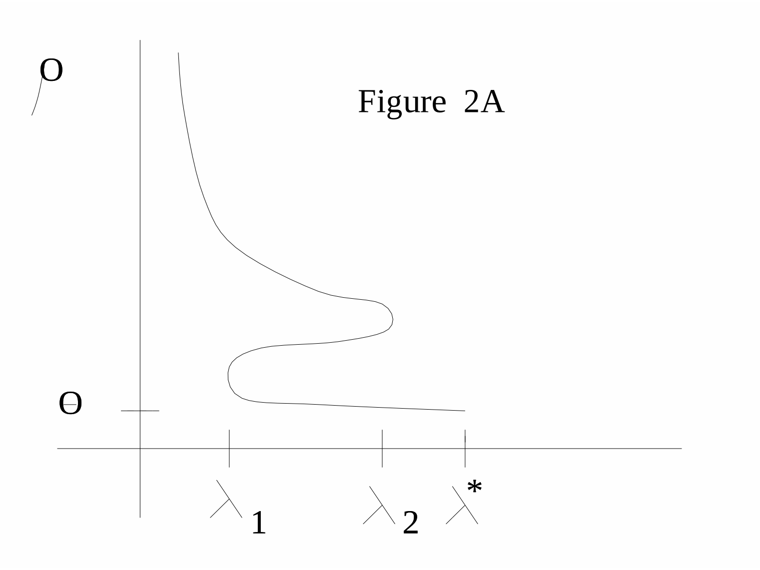


Figure 2A

Subcase b: *If* $\lambda_2 > \lambda^*$ *then for* $\lambda \in (\lambda_1, \lambda^*]$ *(1.1)-(1.2) has exactly three positive solutions while for* $\lambda \in (\lambda^*, \lambda_2)$ *(1.1)-(1.2) has exactly two positive solutions. Finally, for* $\lambda = \lambda_2$ *the problem (1.1)-(1.2) has exactly one positive solution (see Fig. 2B).*

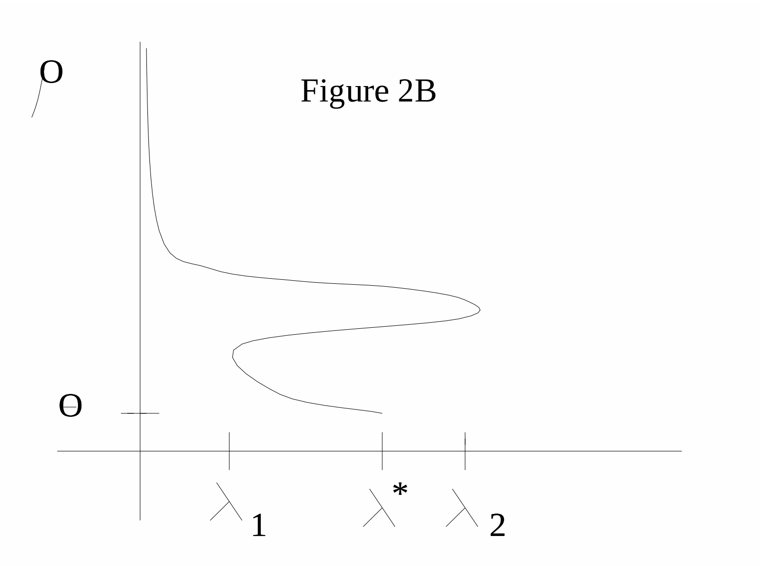


Figure 2B

This paper is organized as follows. In Section 2, we study the variations of the positive solutions with respect to the parameters $\lambda$ and $\rho$. We prove Theorem 1 in Section 3. In Section 4 we give a family of examples which satisfies the hypotheses of Theorem 1.

## 2. First and Second Variations with respect to parameters

We first observe that any positive solution of (1.1)-(1.2) must be symmetric about the origin. To see this, let $x_0 \in (-1, 1)$ be the point at which $u$ attains its maximum. Denote $u(x_0) = \rho > 0$. Thus $u'(x_0) = 0$ and it follows that $u(x_0+x)$ and $u(x_0-x)$ satisfy the differential equation (1.1) as well as the same initial conditions at $x_0$. Therefore, by uniqueness of solutions of initial value problems, we must have $u(x_0 + x) = u(x_0 - x)$. So assuming without loss of generality that $x_0 \geq 0$, we see then that $0 = u(1) = u(2x_0 - 1)$ and since $u > 0$ on $(-1, 1)$, we must have $2x_0 - 1 = -1$ - i.e. $x_0 = 0$ and thus $u$ is symmetric about the origin.

With this result, for any $\rho > 0$ and any $\lambda > 0$ we define $u(x, \lambda, \rho)$ to be the solution to the initial value problem:

$$u''(x) + \lambda f(u(x)) = 0, \ \ \lambda > 0, \tag{2.1}$$

$$u(0) = \rho > 0, \ u'(0) = 0, \tag{2.2}$$

where $'$ denotes differentiation with respect to $x$. Observing that $u(-x, \lambda, \rho)$ also solves (2.1) and (2.2), it follows from the uniqueness of solutions of initial value problems that $u(-x, \lambda, \rho) = u(x, \lambda, \rho)$. Thus we see that the set of positive solutions of (1.1)-(1.2) is precisely the set of solutions of (2.1)-(2.2) for which:

$$u(x, \lambda, \rho) > 0 \text{ for } \ x \in (0, 1) \text{ and } u(1, \lambda, \rho) = 0. \tag{2.3}$$

We now prove some elementary properties of positive solutions of (1.1)-(1.2) (and hence of (2.1)-(2.3) for some $\rho > 0$). Multiplying (2.1) by $u'(x)$, integrating over $(0, x)$, and using (2.2) yields:

$$\frac{1}{2}[u'(x)]^2 + \lambda F(u(x)) = \lambda F(\rho). \tag{2.4}$$

Evaluating this at $x = 1$ gives:

$$0 \le \frac{1}{2}[u'(1)]^2 = \lambda F(\rho). \tag{2.5}$$

Since for $\rho > 0$ we have $F(\rho) \ge 0$ if and only if $\rho \ge \theta$ (by (1.4)), we see from (2.5) that:

$$\text{positive solutions of (1.1)-(1.2) satisfy } \rho \ge \theta, \text{ and} \tag{2.6}$$

$$\text{positive solutions of (1.1)-(1.2) satisfy } u'(1) < 0 \text{ if } \rho > \theta \text{ and } u'(1) = 0 \text{ if } \rho = \theta. \tag{2.7}$$

Also observe that if $u$ is a positive solution to (2.1)-(2.3), then $u''(0) = -\lambda f(\rho) < 0$ (by (1.1), (1.3), and (2.6)) and therefore $u' < 0$ on $(0, \epsilon)$ for some $\epsilon > 0$. In fact $u'(x) < 0$ on $(0, 1)$ for if $u'(x_1) = 0$ at some first $x_1 \in (0, 1)$ then $0 < u(x_1) < \rho$ while from (2.4) and (2.5) we have $F(u(x_1)) = F(\rho) \ge 0$. Thus by (1.4) $\beta < \theta \le u(x_1) < \rho$. But this is impossible since $F$ is increasing for $x > \beta$ (by (1.3)) and thus:

$$\text{positive solutions of (1.1)-(1.2) satisfy } u'(x) < 0 \text{ on } (0, 1). \tag{2.8}$$

Next we observe that $u(xd, \lambda, \rho)$ and $u(x, \lambda d^2, \rho)$ satisfy the same initial value problem and so by uniqueness of solutions of initial value problems we have:

$$u(xd, \lambda, \rho) = u(x, \lambda d^2, \rho).$$

After differentiating this with respect to $d$ and setting $d = 1$, we obtain:

$$xu'(x, \lambda, \rho) = 2\lambda \frac{\partial u}{\partial \lambda}(x, \lambda, \rho). \tag{2.9}$$

Next let $v$ denote the solution to the corresponding linearized problem of (1.1):

$$v''(x) + \lambda f'(u(x))v(x) = 0, \tag{2.10}$$

$$v(0) = 1, \quad v'(0) = 0, \tag{2.11}$$

and let $w$ denote the solution to the problem:

$$w''(x) + \lambda f'(u(x))w(x) + \lambda f''(u(x))v^2(x) = 0, \tag{2.12}$$

$$w(0) = 0, \ w'(0) = 0. \tag{2.13}$$

That is, $v$ and $w$ are the first and second derivatives of $u$ with respect to $\rho$ - i.e. $v \equiv \frac{\partial u}{\partial \rho}(x, \lambda, \rho)$ and $w \equiv \frac{\partial^2 u}{\partial \rho^2}(x, \lambda, \rho)$.

Now observe that by multiplying (2.10) by $u'(x)$ and integrating on $(0, x)$ we obtain:

$$u'(x)v'(x) + \lambda f(u(x))v(x) = \lambda f(\rho). \tag{2.14}$$

Similarly, multiplying (2.12) by $u'(x)$ and integrating on $(0, x)$ gives:

$$u'(x)w'(x) + \lambda f(u(x))w(x) + {v'}^2(x) + \lambda f'(u(x))v^2(x) = \lambda f'(\rho). \tag{2.15}$$

**Lemma 2.1.** *Suppose $f$ satisfies (1.3). Let $u(x,\lambda_0,\rho_0)$ be a positive solution to (1.1)-(1.2). Then $v(x) \equiv \frac{\partial u}{\partial \rho}(x,\lambda_0,\rho_0)$ has at most one zero in [0, 1].*

*Proof.* We first observe that if $v(x_0)=0$ then $v'(x_0)\neq 0$ for if $v'(x_0)=0$ then by uniqueness of solutions of initial value problems, it follows that $v\equiv 0$. On the other hand, $v(0)=1\neq 0$.

Now on to the proof of the lemma. Suppose by the way of contradiction that $x_1$ and $x_2$ are the first two consecutive zeros of $v$. Then by the remarks in the previous paragraph and since $v(0)=1$, we have $v'(x_1)<0$ and $v'(x_2)>0$. Also by (2.14) it follows that $u'(x_2)v'(x_2)=\lambda_0 f(\rho_0)$ and so we see that $u'(x_2)$ and $f(\rho_0)$ have the same sign. But since $\rho_0\geq\theta$ (by (2.6)), it follows from (1.3)-(1.4) that $f(\rho_0)>0$ and hence $u'(x_2)>0$. But this contradicts (2.7)-(2.8). Hence, $v(x)$ can have at most one zero on $[0,1]$. □

**Remark:** Note that the above lemma does not rely on the concavity properties of $f$. □

**Lemma 2.2.** *Suppose $f$ satisfies (1.3)-(1.5). Let $u(x,\lambda_0,\rho_0)$ be a positive solution to (1.1)-(1.2) with $\theta\leq\rho_0\leq t^*$ and suppose also that $v(1)=\frac{\partial u}{\partial\rho}(1,\lambda_0,\rho_0)=0$. Then $w(1)=\frac{\partial^2 u}{\partial\rho^2}(1,\lambda_0,\rho_0)>0$.*

*Proof.* Recall that $v\equiv\frac{\partial u}{\partial\rho}$ satisfies (2.10)-(2.11) and $w\equiv\frac{\partial^2 u}{\partial\rho^2}$ satisfies (2.12)-(2.13). Multiplying (2.10) by $w$ and (2.12) by $v$, subtracting one from the other, integrating over $(0,1)$, and using $v(1)=0$ we obtain:

$$w(1)v'(1)=\int_0^1 \lambda_0 f''(u(x))v^3(x)\,dx. \tag{2.16}$$

Since $v(1)=0$, it follows from lemma 2.1 that we have $v>0$ on $[0,1)$ and it also follows from the uniqueness of solutions to initial value problems that $v'(1)<0$. Since $\theta\leq\rho_0\leq t^*$ and $u(x)$ is decreasing on (0,1) (by (2.8)), it follows that $u(x)<\rho_0\leq t^*$ on $(0,1)$ and so by (1.5) we have $f''(u(x))<0$ on $(0,1)$. These facts and (2.16) imply $w(1)>0$. This proves the lemma. □

**Lemma 2.3.** *If $f$ satisfies (1.3)-(1.5), $(1.5)_2$, and $H(t^*)\geq 0$, then the function defined by $J:[0,\infty)\to\mathbb{R}$, $J(t)=f'(t)F(t)-\frac{1}{2}f^2(t)$ has exactly one positive zero, $t^{**}$, and $\theta<t^*<t^{**}<t_2$.*

*Proof.* By (1.5), $t^*>\beta$. Combining this with the fact that $H(t^*)\geq 0$ implies $F(t^*)\geq\frac{1}{2}t^*f(t^*)>0$ (since $t^*>\beta$) and so $F(t^*)>0$ which implies $t^*>\theta$ (by (1.4)).

Next observe that $J'(t)=f''(t)F(t)$ so $J$ is increasing on $(0,\theta)\cup(t^*,\infty)$ and decreasing on $(\theta,t^*)$. Also, observe $J(\theta)<0$ so that $J<0$ on $[0,t^*]$. Hence $J$ has at most one positive zero.

Also, $J=f'H-fH'$ hence $J(t_2)=f'(t_2)H(t_2)$ and $f(t_2)=t_2f'(t_2)$ (by $(1.5)_2$). Since $t_2>t^*>\beta$ (by $(1.5)_2$), we have $t_2f'(t_2)=f(t_2)>0$ and so $J(t_2)>0$ because $H$ has a maximum at $t_2$ and so $H(t_2)>H(t^*)\geq 0$. Thus, $J$ has exactly one positive zero, $t^{**}$, and $\theta<t^*<t^{**}<t_2$. This completes the proof of the lemma. □

**Lemma 2.4.** *Suppose $f$ satisfies (1.3)-(1.5) and $(1.5)_2$. Let $u(x,\lambda_0,\rho_0)$ be a positive solution of (1.1)-(1.2) with $\rho_0\geq t^{**}$ and suppose also that $v(1)=\frac{\partial u}{\partial\rho}(1,\lambda_0,\rho_0)=0$. Then $w(1)=\frac{\partial^2 u}{\partial\rho^2}(1,\lambda_0,\rho_0)<0$.*

*Proof.* We define:

$$E={v'}^2+\lambda_0 f'(u)v^2$$

and observe (by (2.10)) that:

$$E'=\lambda_0 f''(u)u'v^2.$$

Since $\rho_0\geq t^{**}>t^*$, examining the sign of $E'$ along with (1.5) and (2.8), we see that $E$ is decreasing on $(0,x^*)$ and increasing on $(x^*,1)$ where $x^*$ is the point at which $u(x^*)=t^*$.

Thus, $E$ has exactly one local minimum and no local maxima on (0,1). Hence the maximum of $E$ on [0,1] occurs either at $x=0$ or $x=1$.

Next, we see from lemma 2.3 that $\rho_0\geq t^{**}$ implies $J(\rho_0)\geq 0$. Using (2.4), (2.11), (2.14), and the fact that $v(1)=0$, we obtain:

$$E(0)-E(1)=\frac{\lambda_0}{F(\rho_0)}[f'(\rho_0)F(\rho_0)-\frac{f^2(\rho_0)}{2}]=\frac{\lambda_0}{F(\rho_0)}J(\rho_0)\geq 0.$$

Thus, for $x \in [0,1]$ we have $v'^2 + \lambda_0 f'(u)v^2 = E(x) \le E(0) = \lambda_0 f'(\rho_0)$. Hence, by (2.15):

$$u'w' + \lambda_0 f(u)w \ge 0 \text{ on } [0,1].$$

Now solving (2.4) for $u'$, using (2.8) and substituting into the above inequality gives:

$$w' - \sqrt{\frac{\lambda_0}{2}} \frac{f(u)}{\sqrt{F(\rho_0) - F(u)}} w \le 0 \quad \text{on} \quad (0,1].$$

Multiplying by the appropriate integrating factor and then integrating on $(\epsilon, x) \subset (0,1]$ for $\epsilon > 0$ we have:

$$\int_\epsilon^x \left(we^{-\sqrt{\frac{\lambda_0}{2}}\int_\epsilon^x \frac{f(u)\,dt}{\sqrt{F(\rho_0)-F(u)}}}\right)' \le 0.$$

Now, for $\epsilon$ small enough we have $w(\epsilon) < 0$ because by (2.12)-(2.13) we have $w(0) = 0, w'(0) = 0$, and $w''(0) = -\lambda_0 f''(\rho_0) < 0$ since $\rho_0 \ge t^{**} > t^*$. Therefore:

$$w(x)e^{-\sqrt{\frac{\lambda_0}{2}}\int_\epsilon^x \frac{f(u)\,dt}{\sqrt{F(\rho_0)-F(u)}}} \le w(\epsilon) < 0.$$

Hence $w(x) < 0$ on $(\epsilon, 1]$. In particular, $w(1) < 0$. This completes the proof of the lemma. □

## 3. Proof of Theorem 1

We begin by rewriting (2.4), and we obtain:

$$\frac{-u'(x)}{\sqrt{2}\sqrt{F(\rho) - F(u(x))}} = \sqrt{\lambda} \quad \text{on} \quad (0,1).$$

Thus, after integrating on $(x, 1)$ and using $u(1) = 0$ we obtain:

$$\frac{1}{\sqrt{2}} \int_0^{u(x)} \frac{dt}{\sqrt{F(\rho) - F(t)}} = \sqrt{\lambda}(1-x). \tag{3.1}$$

Letting $x \to 0$ gives:

$$\sqrt{\lambda} = \frac{1}{\sqrt{2}} \int_0^{\rho} \frac{dt}{\sqrt{F(\rho) - F(t)}} \equiv G(\rho). \tag{3.2}$$

Thus, given a positive solution of (1.1)-(1.2) (and hence of (2.1)-(2.3) for some $\rho \ge \theta$), we see that $\lambda$ and $\rho$ are related by equation (3.2).

Conversely, given $\lambda_0 > 0$, if there exists a $\rho_0 \in [\theta, \infty)$ with $G(\rho_0) = \sqrt{\lambda_0}$, then we can obtain a positive solution of (1.1)-(1.2) as follows. Define $K : [0, \rho_0] \to \mathbb{R}$ by:

$$K(x) = \frac{1}{\sqrt{2}} \int_0^x \frac{dt}{\sqrt{F(\rho_0) - F(t)}}.$$

Since $\rho_0 \ge \theta$, it follows from (1.3)-(1.4) that $1/\sqrt{F(\rho_0) - F(t)}$ is integrable on $[0, \rho_0]$. Thus $K$ is continuous on $[0, \rho_0]$ while from (3.2) we have $K(\rho_0) = G(\rho_0) = \sqrt{\lambda_0}$. Also:

$$K'(x) = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{F(\rho) - F(x)}} > 0 \quad \text{on} \quad [0, \rho_0).$$

Thus $K$ is continuous and increasing on $[0, \rho_0]$ and so $K$ has an inverse. In addition,

$$(K^{-1}(x))' = \sqrt{2}\sqrt{F(\rho) - F(K^{-1}(x))}.$$

Taking a hint from (3.1) which says a positive solution of (1.1)-(1.2) satisfies $K(u(x)) = \sqrt{\lambda}(1-x)$, we define

$$u(x) = K^{-1}(\sqrt{\lambda}_0(1 - x)).$$

It is then straightforward to show that $u$ solves (2.1)-(2.3) with $\lambda = \lambda_0$ and $\rho = \rho_0$.

Thus, we see that the set of $\lambda$ for which there is a positive solution of (1.1)-(1.2) is precisely those positive $\lambda$ for which there is a solution - $\rho$ - of $G(\rho) = \sqrt{\lambda}$. Therefore we now turn our attention to a study of the function $G = \sqrt{\lambda}$ defined in (3.2).

We begin by changing variables in (3.2) and obtain:

$$\sqrt{\lambda(\rho)} = G(\rho) = \frac{1}{\sqrt{2}}\int_0^1 \frac{\rho\, dv}{\sqrt{F(\rho) - F(\rho v)}}$$

and from (1.3)-(1.4) it follows $\sqrt{\lambda(\rho)}$ is a positive continuous function on $[\theta, \infty)$. Also, by (1.3)-(1.4):

$$\sqrt{\lambda(\theta)} = G(\theta) = \frac{1}{\sqrt{2}}\int_0^1 \frac{\theta\, dv}{\sqrt{-F(\theta v)}} \equiv \sqrt{\lambda^*} = \text{ finite, positive.}$$

In addition, $\sqrt{\lambda(\rho)}$ is differentiable over $(\theta, \infty)$ and:

(3.3) $$\frac{\lambda'(\rho)}{2\sqrt{\lambda(\rho)}} = G'(\rho) = \frac{1}{\sqrt{2}}\int_0^1 \frac{H(\rho) - H(\rho v)}{[F(\rho) - F(\rho v)]^{3/2}} dv$$

where $H$ is given by (1.6).

Since $u(x, \lambda(\rho), \rho)$ is a positive solution of (1.1)-(1.2), we also have:

$$u(1, \lambda(\rho), \rho) = 0.$$

Differentiating this with respect to $\rho$ gives:

(3.4) $$\frac{\partial u}{\partial \lambda}(1, \lambda(\rho), \rho)\lambda'(\rho) + \frac{\partial u}{\partial \rho}(1, \lambda(\rho), \rho) = 0.$$

We now show that $\lim_{\rho\to\theta^+} \lambda'(\rho) = -\infty$. We know from above that $\lim_{\rho\to\theta^+} \lambda(\rho) = \lambda(\theta) = \lambda^*$ is positive and finite. Also, $\lim_{\rho\to\theta^+} \frac{\partial u}{\partial \lambda}(1, \lambda(\rho), \rho) = \lim_{\rho\to\theta^+} \frac{1}{2\lambda(\rho)} u'(1, \lambda(\rho), \rho) = \frac{1}{2\lambda(\theta)} u'(1, \lambda(\theta), \theta) = 0$ by (2.7) and (2.9). On the other hand, (2.7) and (2.14) imply $\lim_{\rho\to\theta^+} \frac{\partial u}{\partial \rho}(1, \lambda(\rho), \rho) = \frac{f(\theta)}{f(0)} < 0$. It now follows from (3.4) that:

(3.5) $$\lim_{\rho\to\theta^+} \lambda'(\rho) = -\infty.$$

We claim now that $\lambda'(\rho) < 0$ for large $\rho$ and $\lim_{\rho\to\infty} \lambda(\rho) = 0$.

Since $H' = \frac{1}{2}(f - tf') < 0$ for $\rho$ large and $H'' = -\frac{1}{2}tf'' < 0$ for $\rho > t^*$, it follows that $\lim_{\rho\to\infty} H(\rho) = -\infty$. Combining these facts, it follows that for large $\rho$ we have $H(\rho) < H(\rho v)$ for all $v \in (0, 1)$. Therefore, by (3.3)

(3.6) $$\lambda'(\rho) < 0 \text{ for large } \rho.$$

Next, we rewrite $\sqrt{\lambda}$ as:

$$\sqrt{\lambda(\rho)} = G(\rho) = \frac{1}{\sqrt{2}} \int_0^{1/2} \frac{\rho\, dv}{\sqrt{F(\rho) - F(\rho v)}} + \frac{1}{\sqrt{2}} \int_{1/2}^{1} \frac{\rho\, dv}{\sqrt{F(\rho) - F(\rho v)}}$$

From (1.5), $f'' > 0$ for $t > t^*$ and from (1.3) $f(t)/t \to \infty$ as $t \to \infty$, thus $f(= F')$ and $f'$ are positive for large $t$ and $\lim_{t\to\infty} F(t) = \infty$. Therefore, for $0 < v < \frac{1}{2}$ and $\rho$ large we have $F(\rho v) \le F(\frac{1}{2}\rho)$. And so by the mean value theorem:

$$F(\rho) - F(\rho v) \ge F(\rho) - F(\frac{1}{2}\rho) \ge \frac{1}{2}\rho f(\frac{1}{2}\rho).$$

Also for $\frac{1}{2} < v < 1$ and large $\rho$, we have again by the mean value theorem:

$$F(\rho) - F(\rho v) \ge \rho f(\frac{1}{2}\rho)(1 - v).$$

Combining these estimates into the first and second integrals above respectively gives:

$$\sqrt{\lambda(\rho)} = G(\rho) \le \frac{1}{\sqrt{2}} \int_0^{\frac{1}{2}} \frac{\rho}{\sqrt{\frac{1}{2}\rho f(\frac{1}{2}\rho)}} + \frac{1}{\sqrt{2}} \int_{\frac{1}{2}}^{1} \frac{\rho}{\sqrt{\rho f(\frac{1}{2}\rho)}} \frac{1}{\sqrt{1-v}}\, dv = \frac{3}{2}\sqrt{\frac{\rho}{f(\frac{1}{2}\rho)}}.$$

Thus, by the superlinearity of $f$ - (1.3) - we see that

$$\lim_{\rho\to\infty} \lambda(\rho) = 0. \tag{3.7}$$

Consequently, since $\lambda(\rho)$ is continuous on $[\theta, \infty)$ and tends to 0 at infinity (by (3.7)), we see that $\lambda(\rho)$ is a bounded function. Thus, (1.1)-(1.2) has no positive solutions for $\lambda > \max_{[\theta,\infty)} \lambda(\rho)$.

**Case $(1.5)_1$ :** It remains to prove that $\lambda'(\rho) < 0$ for $\rho \in (\theta, \infty)$. From (1.6) we have $H'(t) = \frac{1}{2}[f(t) - tf'(t)]$ and $H''(t) = -\frac{1}{2}tf''(t)$. Since $(1.5)_1$ holds we infer that $H'(t) \le 0$ (in fact, $H'(t) = 0$ for at most one value of $t$) and hence $\lambda'(\rho) < 0$ follows from (3.3).
This together with that $\lambda(\rho)$ is continuous on $[\theta, \infty)$ implies that $\lambda(\rho)$ has an inverse, $\rho_\lambda : (0, \lambda^*] \to [\theta, \infty)$ and $\rho'_\lambda < 0$ on $(\theta, \infty)$ with $\rho_{\lambda^*} = \theta$ and $\lim_{\lambda\to 0^+} \rho_\lambda = \infty$. This completes the proof of Case $(1.5)_1$.

**Case $(1.5)_2$ :** In view of $(1.5)_2$ and (1.7) we have $H'(t) < 0$ on $[0, t_1) \cup (t_2, \infty)$ and $H'(t) > 0$ on $(t_1, t_2)$. Thus for $\rho \in (t^*, t^{**}) \subset (t_1, t_2)$ $H$ is increasing and $H(\rho) > H(t^*) \ge 0$. Also, since $H(0) = 0$ and $H$ is decreasing on $(0, t_1)$, it follows that $H(\rho v) < H(\rho)$ for all $v \in (0, 1)$ and all $\rho \in (t^*, t^{**})$. Hence by (3.3):

$$\lambda'(\rho) > 0 \quad \text{for} \quad \rho \in (t^*, t^{**}). \tag{3.8}$$

Combining this with (3.5) and (3.6) we see that $\lambda(\rho)$ has at least one local minimum on $(\theta, t^*)$ and at least one local maximum on $(t^{**}, \infty)$. To complete the proof of theorem 1 we will show that these are the *only* critical points of $\lambda(\rho)$. First, suppose $\rho_0 \in (\theta, t^*)$ and $\lambda'(\rho_0) = 0$. From (3.4) we see $\frac{\partial u}{\partial \rho}(1, \lambda(\rho_0), \rho_0) = 0$. From lemma 2.2 we see that $\frac{\partial^2 u}{\partial \rho^2}(1, \lambda(\rho_0), \rho_0) > 0$. Differentiating (3.4) and evaluating at $\rho_0$ gives:

$$\frac{\partial u}{\partial \lambda}(1, \lambda(\rho_0), \rho_0)\lambda''(\rho_0) + \frac{\partial^2 u}{\partial \rho^2}(1, \lambda(\rho_0), \rho_0) = 0. \tag{3.9}$$

Since $\frac{\partial u}{\partial \lambda}(1, \lambda(\rho_0), \rho_0) < 0$ by (2.7) and (2.9), we see that $\lambda''(\rho_0) > 0$. Hence, $\rho_0$ *must* be a local minimum of $\lambda(\rho)$. If there were a second critical point, $\rho_1 \in (\theta, t^*)$, of $\lambda(\rho)$, the same argument shows that it too would be a local minimum of $\lambda(\rho)$ and thus between $\rho_0$ and $\rho_1$ there would be a local maximum, $\rho_2$, with $\lambda''(\rho_2) > 0$ but this is clearly impossible. Thus, $\rho_0$ is the *only* critical point of $\lambda(\rho)$ on $(\theta, t^*)$. Similarly, suppose $\rho_0 \in (t^{**}, \infty)$ and $\lambda'(\rho_0) = 0$. Then as before (3.4) implies $\frac{\partial u}{\partial \rho}(1, \lambda(\rho_0), \rho_0) = 0$. Now using lemma 2.4 we see that $\frac{\partial^2 u}{\partial \rho^2}(1, \lambda(\rho_0), \rho_0) < 0$. And as above, using (3.9) we see that $\lambda''(\rho_0) < 0$. Hence, $\rho_0$ *must* be a local maximum of $\lambda(\rho)$ and as above this is the *only* critical point of $\lambda(\rho)$ on $(t^{**}, \infty)$. This completes the proof of theorem 1. □

## 4. Examples

Consider $f(t) = t^3 - 3At^2 + 6Bt - C$ where $A, B$, and $C$ are positive. Then $f$ is semipositone and superlinear. Also, $f$ has exactly one inflection point at $t^* = A$. We have $f'(t) = 3t^2 - 6At + 6B$ hence $f'(t) \geq 0$ for all $t$ if and only if $2B \geq A^2$. Thus if $2B \geq A^2$, $f$ has exactly one zero $\beta$ and since we have $f(t^*) = f(A) = -2A^3 + 6AB - C$, we see that $t^* > \beta$ if $6AB > 2A^3 + C$. Next, $H(t) = F(t) - \frac{1}{2}tf(t) = -\frac{1}{4}t^4 + \frac{A}{2}t^3 - \frac{1}{2}Ct$, $H'(t) = -t^3 + \frac{3A}{2}t^2 - \frac{1}{2}C$, and $H''(t) = -3t^2 + 3At$. Thus, $H'$ has exactly one local maximum at $t^* = A$. If $H'(A) > 0$ then $H'$ has two zeros, while $H' \leq 0$ if $H'(A) \leq 0$. Note that $H'(A) > 0$ if and only if $A^3 > C$ and $H(t^*) = H(A) \geq 0$ if and only if $A^3 \geq 2C$. Thus, (1.3)-(1.5) and $(1.5)_1$ are satisfied if we choose positive $A, B, C$ so that $6B > \frac{C}{A} + 2A^2$, $C \geq A^3$ whereas (1.3)-(1.5) and $(1.5)_2$ are satisfied if $6B > \frac{C}{A} + 2A^2$, $A^3 \geq 2C$, and $2B \geq A^2$.

## References


1. A. Castro and R. Shivaji, *Non-negative solutions for a class of non-positone problems*, Proc. Roy. Soc. Edinburgh. **108A** (1988), 291–302.
2. A. Castro, S. Gadam and R. Shivaji, *Positive solution curves of semipositone problems with concave nonlinearities*, Proc. Roy. Soc. Edinburgh. **127A** (1997), 921–934.
3. A. Castro, S. Gadam and R. Shivaji, *Evolution of positive solution curves in semipositone problems with concave nonlinearities*, to appear in Contemp. Math.
4. M. R. Myerscough, B. F. Gray, W. L. Hogarth, and J. Norbury, *An analysis of an ordinary differential equations model for a two species predator-prey system with harvesting and stocking*, J. Math. Biol. **30** (1992), 389–411.
5. A. Castro and S. Gadam, *Uniqueness of stable and unstable positive solutions for semipositone problems*, Nonlinear Analysis, TMA **22 No. 4** (1994), 425–429.
6. A. Castro, S. Gadam and R. Shivaji, *Branches of radial solutions for semipositone problems*, Jour. Diff. Eqns. **120 No. 1** (1995), 30–45.
7. J. Shi and R. Shivaji, *Exact multiplicity of solutions for classes of semipositone problems with concave-convex type nonlinearity*, Discrete and Continuous Dynamical Systems **7 no. 3** (2001), 559–571.
8. S-H. Wang, *Positive solutions for a class of nonpositone problems with concave nonlinearities*, Proc. Roy. Soc. Edinburgh **124 No. 3** (1994), 507–515.
9. S-H. Wang, *On S-shaped bifurcation curves*, Nonlinear Analysis, TMA **22 No. 12** (1994), 1475–1485.



Mathematics, 'Yashodha', JCR VI Cross, Chitradurga, India 577501

Dept. of Mathematics, University of North Texas, Denton, TX 76203, U.S.A.
*E-mail address*: iaia@unt.edu