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On S-shaped and reversed S-shaped bifurcation curves for singular problems

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Abstract

We analyze the positive solutions to the singular boundary value problem

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda \frac{g(u)}{u^\beta}\ ;\ (0,1), \\ u(0) = 0 = u(1), \end{cases}$$

where $p > 1, \beta \in (0,1), \lambda > 0$ and $g : [0,\infty) \to \mathbb{R}$ is a C^1 function. In particular, we discuss examples when $g(0) > 0$ and when $g(0) < 0$ that lead to S-shaped and reversed S-shaped bifurcation curves, respectively.

1 Introduction

We consider the singular boundary value problem involving the p-Laplacian operator of the form:

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda \frac{g(u)}{u^\beta}\ ;\ (0,1), \\ u(0) = 0 = u(1), \end{cases} \tag{1.1}$$

where $p > 1, \beta \in (0,1), \lambda > 0$ is a parameter and $g : [0,1] \to \mathbb{R}$ is a C^1 function. Problem (1.1) arises in the study of non-Newtonian fluids ([6]) and nonlinear diffusion problems. The quantity p is a characteristic of the medium, and for $p > 2$ the fluids medium are called dilatant fluids, while those with $p < 2$ are called pseudoplastics. When $p = 2$ they are Newtonian fluids ([5]).

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In this paper, we study the following two examples:

$$(A)\ \ g(u) = e^{\frac{\alpha u}{\alpha+u}}; \alpha > 0,$$

$$(B)\ \ \ g(u) = u^3 - au^2 + bu - c\,; a > 0, b > 0 \text{ and } c > 0.$$

Note that in Case (A) $\lim_{u\to+0} \frac{g(u)}{u^\beta} = +\infty$ (Infinite Positone Case) and in Case (B) $\lim_{u\to+0} \frac{g(u)}{u^\beta} = -\infty$ (Infinite Semipositone Case). When $p = 2$ and $\beta = 0$, Case (A) is generally referred as the one-dimensional perturbed Gelfand problem ([1]).

In Case (A) we will prove that for α large, the bifurcation curve of positive solution is at least S-shaped, while in Case (B) for certain ranges of a, b and c, we will prove that the bifurcation curve of positive solution is at least reversed S-shaped. For $p = 2$ and $\beta = 0$, results on S-shaped bifurcation curves have been studied by many authors ([3], [7], [8], [11] and [12]) and results on a reversed S-shaped bifurcation curve have been studied by Castro and Shivaji in ([4]). We will establish the results via the quadrature method which we will describe in Section 2. In Section 3, we will discuss Case (A), and in Section 4 we will discuss Case (B). In Section 5, we provide computational results describing the exact shapes of the two bifurcation curves.

2 Preliminaries

In this section we give some preliminaries. Let $f(u) = \frac{g(u)}{u^\beta}$ and we rewrite (1.1) as:

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda f(u)\ ;\ (0,1), \\ \ \ u(0) = 0 = u(1). \end{cases} \tag{2.1}$$

It follows easily that if u is a strictly positive solution of (2.1), then necessarily u must be symmetric about $x = \frac{1}{2}, u' > 0; (0, \frac{1}{2})$ and $u' < 0; (\frac{1}{2}, 1)$. To prove our main results, we will first state some lemmas that follow from the quadrature method described in [2] and [10] for the one dimensional $p-$Laplacian problem for $p > 1$. See also $[3]$, $[4]$ and $[9]$ for the description of the quadrature method in the case $p = 2$. Define $F : R_+ \to R$ by $F(u) := \int_0^u f(s)\, ds$ and $G : D \subseteq R_+ \to R_+$ be defined by

$$G(\rho) := 2(\frac{p-1}{p})^{\frac{1}{p}} \int_0^\rho \frac{ds}{(F(\rho) - F(s))^{\frac{1}{p}}}, \tag{2.2}$$

where $D = \{\rho > 0 | f(\rho) > 0 \text{ and } F(\rho) > F(s), \ \forall\ 0 \le s < \rho\}$.

Lemma 2.1. *(See [10])* (u, λ) *is a positive solution of* (2.1) *with* $\lambda > 0$ *if and only if* $\lambda(\rho)^{\frac{1}{p}} = G(\rho)$, *where* $\rho = \|u\| = \sup_{s\in(0,1)} u(s) = u(\frac{1}{2})$.

Now we also state an important lemma that can be easily deduced from the results in [3] for the $p-$Laplacian problem.

Lemma 2.2. *$G(\rho)$ is differentiable on D and*

$$\frac{dG(\rho)}{d\rho} = 2(\frac{p-1}{p})^{\frac{1}{p}} \int_0^1 \frac{H(\rho) - H(\rho v)}{[F(\rho) - F(\rho v)]^{\frac{p+1}{p}}} dv, \tag{2.3}$$

where $H(s) = F(s) - \frac{1}{p} s f(s)$.

We will deduce information on the nature of the bifurcation curve by analyzing the sign of $\frac{dG(\rho)}{d\rho}$. It is clear that $\frac{dG(\rho)}{d\rho}$ has the same sign as $\frac{d}{d\rho}\left(\lambda(\rho)^{\frac{1}{p}}\right)$. From (2.3), a sufficient condition for $\frac{dG(\rho)}{d\rho}$ to be positive is:

$$H(\rho) > H(s) \;\; \forall\, s \in [0, \rho) \tag{2.4}$$

and a sufficient condition for $\frac{dG(\rho)}{d\rho}$ to be negative is:

$$H(\rho) < H(s) \;\; \forall\, s \in [0, \rho). \tag{2.5}$$

Hence, if $H'(s) > 0$ for all $s > 0$, then $G(\rho) = (\lambda(\rho))^{\frac{1}{p}}$ is a strictly increasing function, i.e. the bifurcation curve is neither S-shaped nor reversed S-shaped.

In Section 3, for the Case A, we will show that if $\alpha \gg 1$, then there exist $\rho_0 > 0$ and $\rho_1 > \rho_0$ such that $H'(s) > 0; 0 < s < \rho_0$ and $H(\rho_1) < 0$ (see Figure 1).

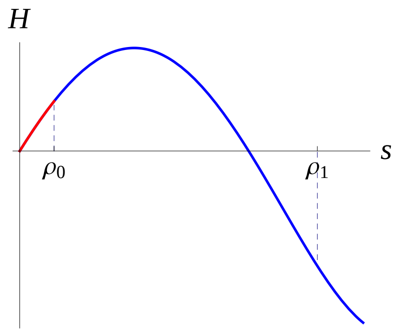

Figure 1: Function H for the Case A

Here, $D = (0, \infty), \lim_{\rho \to 0^+} G(\rho) = 0$ and since $\lim_{s \to \infty} \frac{f(s)}{s^{p-1}} = 0$ we obtain $\lim_{\rho \to \infty} G(\rho) = \infty$. (See [2], Theorem 7). Now, using $(2.4) - (2.5)$, $G'(\rho) > 0$ for $0 < \rho \leq \rho_0$ and $G'(\rho_1) < 0$. Hence this will establish that the bifurcation curve is at least S-shaped (see Figure 2).

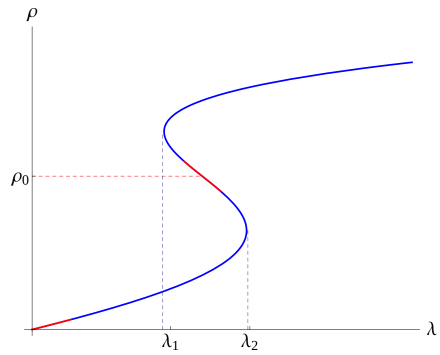

Figure 2: $S-$shaped bifurcaiton curve

In Section 4, for the Case B, for certain ranges of a, b, c and p we will show that f and F take the following shapes (see Figure 3) and $f'(s) > 0; s \geq 0$.

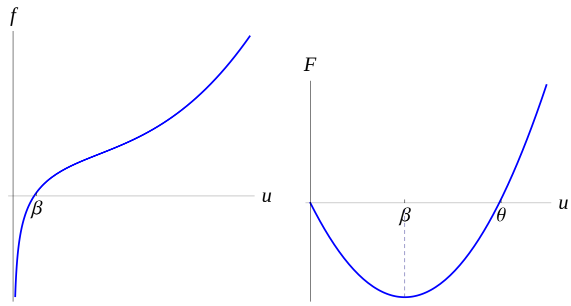

Figure 3: Functions $f(u)$ and $F(u)$

Here β and θ are the unique positive zeros of f and F, respectively. Further, we will show that $H'(s) < 0; 0 < s \leq \theta$ and there exists $\rho_2 > \theta$ such that $H(\rho_2) > 0$ (see Figure 4).

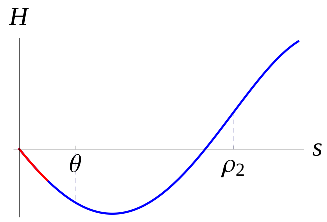

Figure 4: Function H for the Case B

Here $D = (\theta, \infty)$, $\lim_{\rho\to\theta^+} G(\rho) > 0$ and since $\lim_{s\to\infty} \frac{f(s)}{s^{p-1}} = \infty$, we obtain that $\lim_{\rho\to\infty} G(\rho) = 0$. (See [2], Theorem 7). Now using $(2.4)-(2.5)$, $G'(\rho) < 0$ for $\rho \in (\theta, \theta+\epsilon)$ for $\epsilon \approx 0$ and $G'(\rho_2) > 0$. Hence this will establish that the bifurcation curve is at least reversed S-shaped (see Figure 5).

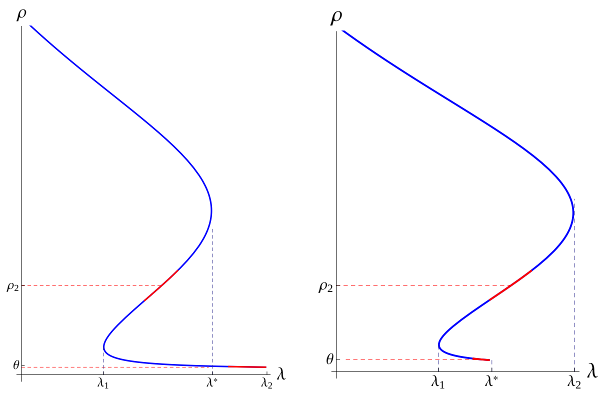

Figure 5: Reversed $S-$shaped bifurcation curves

Finally, in Section 5, we will use Mathematica computations to provide the exact shape of the bifurcation curves for certain values of the parameters involved.

3 Infinite Positone Case A

Here we study the Case A, namely the boundary value problem :

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda \frac{e^{\frac{\alpha u}{\alpha+u}}}{u^\beta}; (0,1), \\ u(0) = 0 = u(1), \end{cases} \tag{3.1}$$

where $p > 1, \alpha > 0$ and $0 < \beta < 1$. We prove:

Theorem 3.1. $\forall\lambda > 0$, *the problem* (3.1) *has a solution. Further, there exist* $\lambda_1 > 0$ *and* $\lambda_2 > 0$ *such that* (3.1) *has at least three solutions for* $\lambda \in (\lambda_1, \lambda_2)$ *for* $\alpha \gg 1$.

Proof . To prove Theorem 3.1, from our discussion in Section 2 it is enough to show that when $\alpha \gg 1$ H has the shape in Figure 1 : namely

- $\lim_{s\to 0^+} H'(s) > 0$.
- there exists $\rho_1 > 0$ such that $H(\rho_1) < 0$.

Here $f(s) = \frac{e^{\frac{\alpha s}{\alpha+s}}}{s^\beta}$. Recall that $F(u) = \int_0^u f(s)\,ds$ and $H(s) = F(s) - \frac{1}{p} s f(s)$. Clearly $H(0) = 0$. Since $f'(s) = e^{\frac{\alpha s}{\alpha+s}} \{\frac{\alpha^2}{s^\beta(\alpha+s)^2} - \frac{\beta}{s^{\beta+1}}\}$, we have

$$\begin{aligned}
H'(s) &= \frac{1}{p}\big[(p-1)f(s) - sf'(s)\big] \\
&= \frac{1}{p}\left[(p-1)\frac{e^{\frac{\alpha s}{\alpha+s}}}{s^\beta} - s e^{\frac{\alpha s}{\alpha+s}}\left(\frac{\alpha^2}{s^\beta(\alpha+s)^2} - \frac{\beta}{s^{\beta+1}}\right)\right] \\
&= \frac{e^{\frac{\alpha s}{\alpha+s}}}{ps^\beta}\left[\frac{(\beta+p-1)(\alpha+s)^2 - \alpha^2 s}{(\alpha+s)^2}\right].
\end{aligned}$$

and hence $\lim_{s\to 0^+} H'(s) = +\infty$. Next, we show that there exists $\rho_1 > 0$ such that $H(\rho_1) < 0$. Take $\rho_1 = \alpha$. Then we have that $H(\alpha) = \int_0^\alpha f(s)ds - \frac{\alpha}{p} f(\alpha)$. Since

$$\begin{aligned}
\frac{dH(\alpha)}{d\alpha} &= \left(1 - \frac{1}{p}\right) f(\alpha) - \frac{\alpha}{p} f'(\alpha) \\
&= \left(1 - \frac{1}{p}\right)\frac{e^{\frac{\alpha}{2}}}{\alpha^\beta} - \frac{\alpha}{p}\frac{e^{\frac{\alpha}{2}}}{\alpha^\beta}\left(\frac{1}{4} - \frac{\beta}{\alpha}\right) \\
&= \frac{e^{\frac{\alpha}{2}}}{\alpha^\beta}\left[\left(1 - \frac{1}{p}\right) - \frac{\alpha}{4p} + \frac{\beta}{p}\right] \\
&= \frac{1}{p} e^{\frac{\alpha}{2}} \alpha^{1-\beta}\left[\frac{\beta+p-1}{\alpha} - \frac{1}{4}\right],
\end{aligned}$$

we obtain that $\frac{dH(\alpha)}{d\alpha} \to -\infty$ as $\alpha \to \infty$. Hence $H(\alpha) < 0$ for $\alpha \gg 1$. Hence, $H(s)$ has the shape in Figure 3 for $\alpha \gg 1$, and Theorem 3.1 is proven.

4 Infinite Semipositone Case B

Here we study the Case B, namely the boundary value problem :

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda \frac{u^3 - au^2 + bu - c}{u^\beta}; (0,1), \\ u(0) = 0 = u(1), \end{cases} \tag{4.1}$$

where $p > 1, a, b$ and c are positive real numbers and $0 < \beta < 1$. We establish:

Theorem 4.1. *Let $a > 0$ be fixed and let $p \in [2-\beta, 3-2\beta)$. Then there exist positive quantites $b^*(a), c^*(a), \lambda_1, \lambda^*$ and λ_2 such that for $b > b^*(a)$ and $c < c^*(a)$ the followings are true:*

(1) *for $\lambda \le \lambda_2$, (4.1) has at least one solution.*

(2) *for $\lambda > \lambda_2$, (4.1) has no solution.*

(3) *for $\lambda_1 < \lambda < \lambda^*$, (4.1) has at least three solutions.*

Proof. To prove Theorem 4.1 from our discussion in Section 2, it is enough to show that for certain parameter values H has the shape in Figure 4 : namely

- $f'(s) > 0$ for all $s \ge 0$.
- $H'(s) < 0; 0 < s \le \theta$.
- there exists $\rho_2 > \theta$ such that $H(\rho_2) > 0$.

Here $f(u) = \frac{u^3 - au^2 + bu - c}{u^\beta}$. First, we show that $f'(s) > 0$ for all $s \ge 0$. Indeed, if $b > \frac{(2-\beta)^2 a^2}{4(3-\beta)(1-\beta)} := b_1$,

$$\begin{aligned} f'(s) &= (3-\beta)s^{2-\beta} - a(2-\beta)s^{1-\beta} + b(1-\beta)s^{-\beta} + \beta c s^{-\beta-1} \\ &> s^{-\beta}\left[(3-\beta)s^2 - a(2-\beta)s + b(1-\beta)\right] \\ &= s^{-\beta}(3-\beta)\left[\left(s - \frac{(2-\beta)a}{2(3-\beta)}\right)^2 - \frac{(2-\beta)^2a^2}{4(3-\beta)^2} + \frac{(1-\beta)b}{3-\beta}\right] \\ &> s^{-\beta}(3-\beta)\left[-\frac{(2-\beta)^2a^2}{4(3-\beta)^2} + \frac{(1-\beta)b}{3-\beta}\right] \\ &= s^{-\beta}(1-\beta)\left[-\frac{(2-\beta)^2a^2}{4(3-\beta)(1-\beta)} + b\right] \\ &> 0. \end{aligned}$$

Next, since f is increasing on $(0,\infty)$, $\lim_{s\to 0^+} f(s) = -\infty$ and $\lim_{s\to\infty} f(s) = +\infty$, there exists a unique $\beta > 0$ such that $f(\beta) = 0$ and a unique $\theta > \beta$ such that $F(\theta) = 0$. Now recall that $H(s) = F(s) - \frac{1}{p}sf(s)$. Clearly $H(0) = 0$. We will now show that $H'(s) < 0\,; 0 < s \le \theta$. First note that

$$F(s) = \frac{1}{4-\beta}s^{4-\beta} - \frac{a}{3-\beta}s^{3-\beta} + \frac{b}{2-\beta}s^{2-\beta} - \frac{c}{1-\beta}s^{1-\beta} < 0\,; 0 < s \le \theta.$$

Hence

$$cs^{-\beta} > \frac{1-\beta}{4-\beta}s^{3-\beta} - \frac{a(1-\beta)}{3-\beta}s^{2-\beta} + \frac{b(1-\beta)}{2-\beta}s^{1-\beta}\ ;\ 0 < s \le \theta. \tag{4.2}$$

Now since $p < 3-2\beta$, if $b > \frac{(2-\beta)(4-\beta)(p-4+2\beta)^2a^2}{3(3-\beta)^2(p-5+2\beta)(p-3+2\beta)} := b_2$, by using (4.2), we obtain that

$$
\begin{aligned}
pH'(s) &= (p-1)f(s) - sf'(s) \\
&= (p-4+\beta)s^{3-\beta} - a(p-3+\beta)s^{2-\beta} + b(p-2+\beta)s^{1-\beta} \\
&\quad - c(p-1+\beta)s^{-\beta} \\
&< (p-4+\beta)s^{3-\beta} - a(p-3+\beta)s^{2-\beta} + b(p-2+\beta)s^{1-\beta} \\
&\quad - (p-1+\beta)[\frac{1-\beta}{4-\beta}s^{3-\beta} - \frac{a(1-\beta)}{3-\beta}s^{2-\beta} + \frac{b(1-\beta)}{2-\beta}s^{1-\beta}] \\
&= [\frac{3(p-5+2\beta)}{4-\beta}s^2 - \frac{2a(p-4+2\beta)}{3-\beta}s + \frac{b(p-3+2\beta)}{2-\beta}]s^{1-\beta} \\
&= [\ \frac{3(p-5+2\beta)}{4-\beta}\left(s - \frac{a(4-\beta)(p-4+2\beta)}{3(3-\beta)(p-5+2\beta)}\right)^2 \\
&\quad - \frac{a^2(4-\beta)(p-4+2\beta)^2}{3(3-\beta)^2(p-5+2\beta)} + \frac{b(p-3+2\beta)}{2-\beta}\ ]s^{1-\beta} \\
&< [\ - \frac{a^2(4-\beta)(p-4+2\beta)^2}{3(3-\beta)^2(p-5+2\beta)} + \frac{b(p-3+2\beta)}{2-\beta}\ ]s^{1-\beta} \\
&= \frac{p-3+2\beta}{2-\beta}[\ - \frac{a^2(2-\beta)(4-\beta)(p-4+2\beta)^2}{3(3-\beta)^2(p-5+2\beta)(p-3+2\beta)} + b\ ]s^{1-\beta} \\
&< 0\,; 0 < s \le \theta.
\end{aligned}
$$

Next, we show that there exists $\rho_2 > \theta$ such that $H(\rho_2) > 0$. Let $\rho_2 = \mu a$, where $\mu = \frac{(2-\beta)(p-3+\beta)}{(3-\beta)(p-4+\beta)}$. Since $p \ge 2-\beta$, we obtain that

$$
\begin{aligned}
H(\rho_2) &= F(\rho_2) - \frac{\rho_2}{p}f(\rho_2) \\
&= \frac{1}{4-\beta}\rho_2^{4-\beta} - \frac{a}{3-\beta}\rho_2^{3-\beta} + \frac{b}{2-\beta}\rho_2^{2-\beta} - \frac{c}{1-\beta}\rho_2^{1-\beta} \\
&\quad - \frac{\rho_2}{p}[\ \rho_2^{3-\beta} - a\rho_2^{2-\beta} + b\rho_2^{1-\beta} - c\rho_2^{-\beta}\ ] \\
&= \frac{\rho_2^{1-\beta}}{p}[\ \frac{p-4+\beta}{4-\beta}\rho_2^3 - \frac{a(p-3+\beta)}{3-\beta}\rho_2^2 + \frac{b(p-2+\beta)}{2-\beta}\rho_2 \\
&\quad - \frac{c(p-1+\beta)}{1-\beta}] \\
&\ge \frac{\rho_2^{1-\beta}}{p}[\ \frac{p-4+\beta}{4-\beta}\rho_2^3 - \frac{a(p-3+\beta)}{3-\beta}\rho_2^2 - \frac{c(p-1+\beta)}{1-\beta}] \\
&= \frac{\rho_2^{1-\beta}}{p}[\ \left(\frac{(p-4+\beta)\mu}{4-\beta} - \frac{p-3+\beta}{3-\beta}\right)\mu^2a^3 - \frac{c(p-1+\beta)}{1-\beta}\ ] \\
&= \frac{\rho_2^{1-\beta}(p-1+\beta)}{p(1-\beta)}[\ \frac{(1-\beta)(-2(p-3+\beta))}{(p-1+\beta)(3-\beta)(4-\beta)}\mu^2a^3 - c\,].
\end{aligned}
$$

Thus we have $H(\rho_2) > 0$ if $c < \frac{(1-\beta)(-2(p-3+\beta))}{(p-1+\beta)(3-\beta)(4-\beta)}\mu^2 a^3 := c^*(a)$. Since $H(0) = 0, H'(s) < 0\ ; 0 < s \leq \theta$ and $H(\rho_2) > 0$, clearly $\rho_2 > \theta$. Taking $b^*(a) = \max\{b_1, b_2\}$, it follows that for $b > b^*(a)$ and $c < c^*(a)$, Theorem 4.1 holds.

Remark 4.1. *The range restriction on p here helps us prove analytically that the bifurcation curve is reversed S-shaped. However, this is not a necessary condition as seen from our computational result. (See Example (d) in Section 5)*

5 Computational Results

Here using Mathematica computations of (2.2), we derive the exact bifurcation curves for the following examples:

(a) Case A with $p = 1.6, \alpha = 10$ and $\beta = 0.5$

(b) Case A with $p = 10, \alpha = 50$ and $\beta = 0.5$

(c) Case B with $p = 2.5, a = 10, b = 72, c = 1$ and $\beta = 0.1$

(d) Case B with $p = 3, a = 10, b = 50, c = 20$ and $\beta = 0.1$

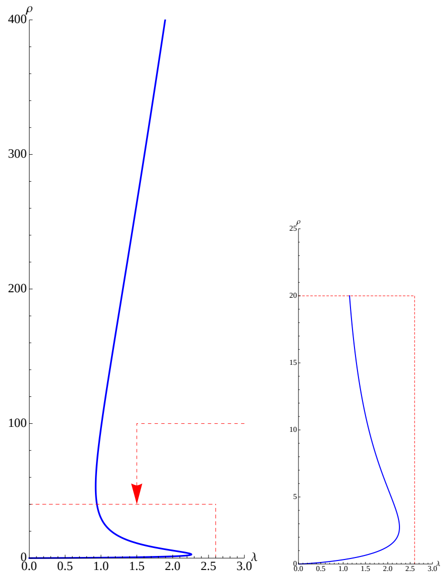

Figure 6: Example (a) $p = 1.6, \alpha = 10$ and $\beta = 0.5$

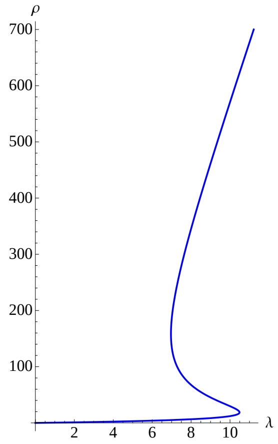

Figure 7: Example (b) $p = 10, \alpha = 50$ and $\beta = 0.5$

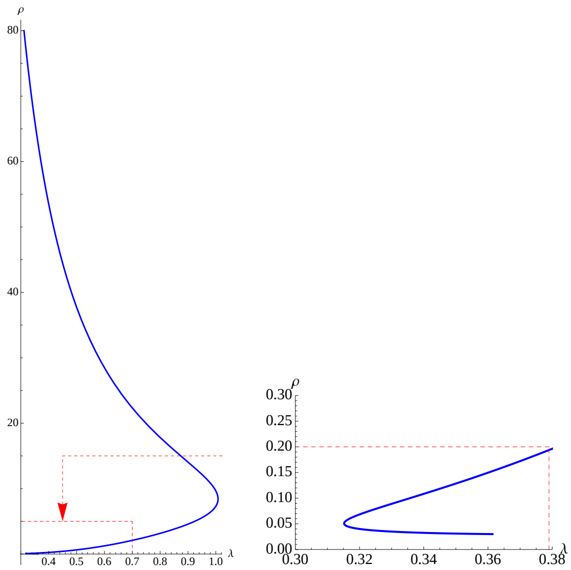

Figure 8: Example (c) $p = 2.5, a = 10, b = 72, c = 1$ and $\beta = 0.1$

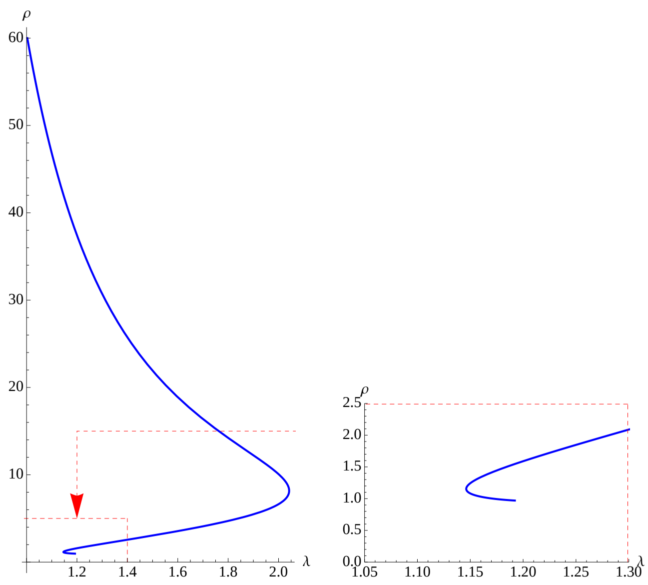

Figure 9: Example (d) $p = 3, a = 10, b = 50, c = 20$ and $\beta = 0.1$

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