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Barriers in impulsive antiperiodic problems

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Abstract. Some real world models are described by means of impulse control of nonlinear BVPs, where time instants of impulse actions depend on intersection points of solutions with given barriers. For $i = 1, \dots, m$, and $[a, b] \subset \mathbb{R}$, continuous functions $\gamma_i : \mathbb{R} \to [a, b]$ determine barriers $\Gamma_i = \{(t, z) : t = \gamma_i(z), z \in \mathbb{R}\}$. A solution (x, y) of a planar BVP on $[a, b]$ is searched such that the graph of its first component $x(t)$ has exactly one intersection point with each barrier, i.e. for each $i \in \{1, \dots, m\}$ there exists a unique root $t = t_{ix} \in [a, b]$ of the equation $t = \gamma_i(x(t))$. The second component $y(t)$ of the solution has impulses (jumps) at the points $t_{1x}, \dots, t_{mx}$. Since a size of jumps and especially the points $t_{1x}, \dots, t_{mx}$ depend on x, impulses are called *state-dependent*.

Here we focus our attention on an antiperiodic solution (x, y) of the van der Pol equation with a positive parameter μ and a Lebesgue integrable antiperiodic function f

$$x'(t) = y(t),\ y'(t) = \mu \left(x(t) - \frac{x^3(t)}{3} \right)' - x(t) + f(t) \quad \text{for a.e. } t \in \mathbb{R},\ t \notin \{t_{1x}, \dots, t_{mx}\},$$

where y has impulses at the points from the set $\{t_{1x}, \dots, t_{mx}\}$,

$$y(t+) - y(t-) = \mathcal{J}_i(x), \quad t = t_{ix}, \quad i = 1, \dots, m,$$

and $\mathcal{J}_i$ are continuous functionals defining a size of jumps.

Previous results in the literature for this antiperiodic problem assume that impulse points are values of given continuous functionals. Such formulation is certain handicap for applications to real world problems where impulse instants depend on barriers. The paper presents conditions which enable to find such functionals from given barriers. Consequently the existence results for impulsive antiperiodic problem to the van der Pol equation formulated in terms of barriers are reached.

Keywords: van der Pol equation, state-dependent impulses, barriers, existence, antiperiodic solution.

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1 Introduction

Some models of real world problems are characterized by the occurrence of abrupt changes of their behavior at certain time instants depending on the state and situation of a model. A natural assumption for differential models is that these instants (impulse points) are determined by means of intersections of a solution of a model with given barriers. For periodic problems see [1,2,8,9,14,16] and for boundary value problems with various linear boundary conditions see [10] and [15]. Existence theorems in these papers are not applicable to equations of van der Pol type. On the other hand we can find existence theorems for impulsive periodic or antiperiodic solutions to equations of van der Pol type, but these results are proved under the assumption that impulse points are values of given continuous functionals [3–6,11–13]. This brings difficulties in applications where impulse instants depend on barriers.

The aim of this paper is to overcome this handicap. In particular:

- For positive numbers K and L, an appropriate function set Ω_{KL} (see (2.1)) is determined.
- Conditions for barriers $\Gamma_i = \{(t,z) : t = \gamma_i(z), z \in \mathbb{R}\}$, $i = 1, \dots, m$, are found such that a graph of each function $x \in \Omega_{KL}$ has exactly one intersection point $(t_{ix}, x(t_{ix}))$ with each of the barriers (see Lemma 2.2).
- The conditions imply in addition that points t_{ix} depend continuously on x (see Lemma 2.3).
- Conditions formulated in terms of barriers and guaranteeing the solvability of an impulsive antiperiodic problem to the van der Pol equation are found (see Theorem 1.1).

More precisely, for $T > 0$ and given continuous functions $\gamma_1, \dots, \gamma_m$, we prove the existence of a T-antiperiodic solution (x, y) of the van der Pol equation with a positive parameter μ and a Lebesgue integrable T-antiperiodic function f

$$\begin{aligned} x'(t) &= y(t), \\ y'(t) &= \mu \left(x(t) - \frac{x^3(t)}{3} \right)' - x(t) + f(t) \end{aligned} \qquad \text{for a.e. } t \in [0,T],\ t \notin \{t_{1x}, \dots, t_{mx}\}, \tag{1.1}$$

where y has impulses at the points $t_{1x}, \dots, t_{mx} \in (0, T)$ determined by the barriers $\Gamma_1, \dots, \Gamma_m$ through the equalities

$$t_{ix} = \gamma_i(x(t_{ix})), \quad i = 1, \dots, m, \tag{1.2}$$

and y is continuous anywhere else in $[0, T]$. The impulse conditions have the form

$$y(t+) - y(t-) = \mathcal{J}_i(x), \quad t = t_{ix}, \quad i = 1, \dots, m, \tag{1.3}$$

where $\mathcal{J}_i$ are continuous bounded functionals defining a size of jumps.

Notations

T-antiperiodic function x (satisfying (1.1), (1.2), (1.3)) will be found in the set of $2T$-periodic real-valued functions. To do it functional sets defined below are used.

- L^1 consists of $2T$-periodic Lebesgue integrable functions on $[0, 2T]$ with the norm $\|x\|_{\mathrm{L}^1} := \frac{1}{2T} \int_0^{2T} |x(t)| \, \mathrm{d}t$,

- BV consists of $2T$-periodic functions of bounded variation on $[0, 2T]$,
- $\mathrm{var}(x)$ for $x \in \mathrm{BV}$ is the total variation of x on $[0, 2T]$,
- $\|x\|_\infty := \sup\{|x(t)| : t \in [0, 2T]\}$ for $x \in \mathrm{BV}$,
- NBV consists of normalized functions $x \in \mathrm{BV}$ in the sense that $x(t) = \frac{1}{2}(x(t+) + x(t-))$,
- $\bar{x} := \frac{1}{2T} \int_0^{2T} x(t)\,\mathrm{d}t = 0$ is the mean value of $x \in \mathrm{BV}$,
- $\widetilde{\mathrm{NBV}}$ consists from functions $x \in \mathrm{NBV}$ with $\bar{x} = 0$; $\widetilde{\mathrm{NBV}}$ with the norm $\mathrm{var}(x)$ is the Banach space,
- $\mathrm{AC}(J)$ consists of $2T$-periodic absolutely continuous functions on $J \subset [0, 2T]$ and if $J = [0, 2T]$ we write AC,
- $\widetilde{\mathrm{AC}} := \mathrm{AC} \cap \widetilde{\mathrm{NBV}}$.
- A couple $(x, y) \in \widetilde{\mathrm{AC}} \times \widetilde{\mathrm{NBV}}$ satisfying (1.1), (1.2), (1.3) is a $2T$-periodic *solution* of problem (1.1)–(1.3). If in addition
$$x(0) = -x(T), \quad y(0) = -y(T), \tag{1.4}$$
then (x, y) is a *T-antiperiodic solution* of problem (1.1)–(1.3).

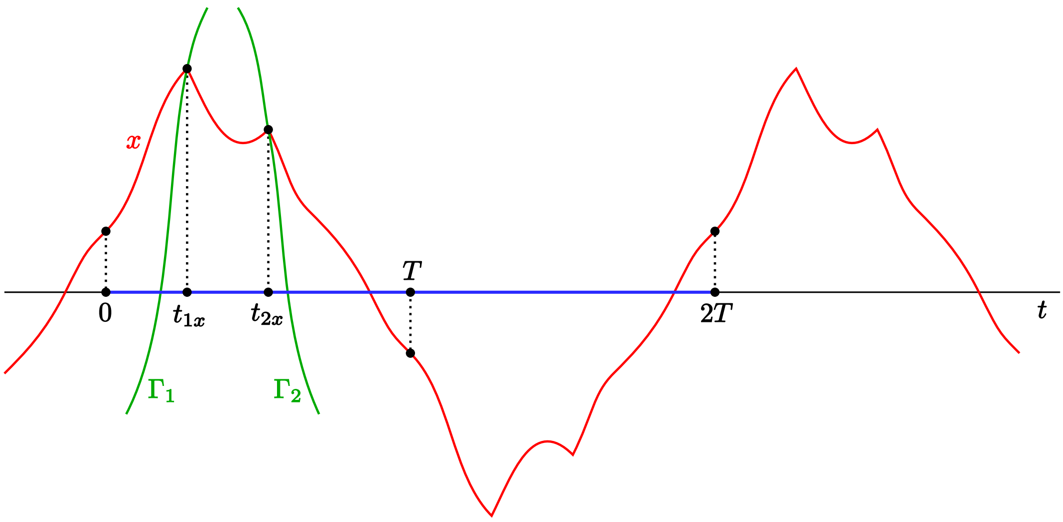

Figure 1.1: The first component x of T-antiperiodic solution (x, y) of a problem with two barriers Γ_1 and Γ_2

The main existence result is contained in the next theorem.

Theorem 1.1 (Main result). *Let $T \in (0, \sqrt{3})$, $K, L \in (0, \infty)$, let $\mathcal{J}_i$, $i = 1, \dots, m$, be contiuous bounded functionals on $\widetilde{\mathrm{NBV}}$, and let $f \in \mathrm{L}^1$ be T-antiperiodic, i.e. $f(t + T) = -f(t)$ for a.e. $t \in \mathbb{R}$. Assume that there exist $a, b \in (0, T)$ such that functions $\gamma_1, \dots, \gamma_m$ satisfy*
$$0 < a \le \gamma_1(z) < \gamma_2(z) < \cdots < \gamma_m(z) \le b < T, \quad z \in [-K, K]. \tag{1.5}$$

Further, assume that $L_i \in (0, 1/L)$, $i = 1, \dots, m$, *are such that*

$$|\gamma_i(z_1) - \gamma_i(z_2)| \le L_i |z_1 - z_2|, \quad z_1, z_2 \in [-K, K],\ i = 1, \dots, m. \tag{1.6}$$

Then there exists $\mu_0 > 0$ *such that for each* $\mu \in (0, \mu_0]$ *problem* (1.1)–(1.3) *has a* T*-antiperiodic solution* (x, y)*, where* y *has* m *jumps at the points* $t_{1x}, \dots, t_{mx} \in [a, b]$ *and* y *is continuous anywhere else in* $[0, T]$*. Moreover the estimate*

$$|x(t)| \le \operatorname{var}(x) \le K, \quad |y(t)| \le L, \quad t \in [0, T], \tag{1.7}$$

is valid.

We can find the optimal (maximal) value μ_0 as follows. Since $\mathcal{J}_i$ are bounded, it holds

$$\mathcal{J}_i : \widetilde{\mathrm{NBV}} \to [-a_i, a_i], \quad i = 1, \dots, m,$$

for some $a_i \in (0, \infty)$. Denote

$$c_1 := T\|f\|_{\mathrm{L}^1} + \sum_{i=1}^{m} a_i, \tag{1.8}$$

and define a function φ by

$$\varphi(\mu) := \frac{1 - \mu T - \frac{T^2}{3}}{3} \sqrt{\frac{1 - \mu T - \frac{T^2}{3}}{\mu T}}, \quad \mu \in (0, 1/T - T/3]. \tag{1.9}$$

Then, according to the proof of Theorem 1.1, $\mu_0 = \varphi^{-1}(Tc_1) \in (0, 1/T - T/3)$.

Auxiliary results

Denote

$$(x * y)(t) := \frac{1}{2T} \int_0^{2T} x(t - s) y(s)\, \mathrm{d}s, \quad t \in [0, 2T] \quad \text{for } x, y \in \mathrm{L}^1,$$

and remind the inequalities

$$\operatorname{var}(x * y) \le \operatorname{var}(x)\|y\|_\infty, \quad x, y \in \mathrm{NBV}, \tag{1.10}$$

$$\operatorname{var}(x * f) \le \operatorname{var}(x)\|f\|_{\mathrm{L}^1}, \quad x \in \mathrm{NBV},\ f \in \mathrm{L}^1, \tag{1.11}$$

$$\|x\|_{\mathrm{L}^1} \le \|x\|_\infty \le \operatorname{var}(x), \quad x \in \widetilde{\mathrm{NBV}}. \tag{1.12}$$

Further, using the function

$$E_1(t) = \begin{cases} T - t & \text{for } t \in (0, 2T), \\ 0 & \text{for } t = 0, \end{cases}$$

which fulfils

$$\operatorname{var}(E_1) = 4T, \quad \|E_1\|_\infty = T, \tag{1.13}$$

we introduce antiderivative operators I and I^2 by

$$Iu := E_1 * u \in \widetilde{\mathrm{AC}}, \quad I^2 u := I(Iu) \in \widetilde{\mathrm{AC}}, \quad u \in \mathrm{L}^1. \tag{1.14}$$

For $\tau \in \mathbb{R}$ we define a distribution ε_τ by the Fourier series

$$\varepsilon_\tau := \sum_{n \in \mathbb{Z}} (1 - (-1)^n) \mathrm{e}^{\frac{in\pi}{T}(t - \tau)}, \quad t \in \mathbb{R}. \tag{1.15}$$

Then it holds

$$I\varepsilon_\tau \in \widetilde{\mathrm{NBV}}, \quad I^2\varepsilon_\tau \in \widetilde{\mathrm{AC}}, \quad \|I\varepsilon_\tau\|_\infty = T. \tag{1.16}$$

See [11] for more details. Using this we investigated in [11] the van del Pol equation

$$x'(t) = y(t), \quad y'(t) = \mu \left(x(t) - \frac{x^3(t)}{3} \right)' - x(t) + f(t) \quad \text{for a.e. } t \in \mathbb{R}, \tag{1.17}$$

with a positive parameter μ, a Lebesgue integrable T-antiperiodic function f, and with the state-dependent impulse conditions

$$\lim_{t\to\tau_i(x)+} y(t) - \lim_{t\to\tau_i(x)-} y(t) = \mathcal{J}_i(x), \quad i = 1, \dots, m, \tag{1.18}$$

where $\mathcal{J}_i$ and also τ_i, $i = 1, \dots, m$, are given continuous and bounded real-valued functionals on $\widetilde{\mathrm{NBV}}$. For such setting we proved the existence result contained in Theorem 1.2.

Theorem 1.2 ([11, Theorem 1.1])**.** *Assume that $T \in (0, \sqrt{3})$, and the functionals $\tau_1, \dots, \tau_m$ have values in $(0, T)$. Further, let*

$$i \neq j \quad \Longrightarrow \quad \tau_i(x) \neq \tau_j(x), \quad x \in \widetilde{\mathrm{AC}}, \quad i, j = 1, \dots, m. \tag{1.19}$$

Then there exists $\mu_0 > 0$ such that for each $\mu \in (0, \mu_0]$ the problem (1.17), (1.18) has a T-antiperiodic solution (x, y).

2 Existence of continuous functionals

If we study an impulsive boundary value problem which is formulated by means of barriers $\Gamma_1, \dots, \Gamma_m$, then a number of impulse points for some solution (x, y) is equal to a number of values of t satisfying the equations $t - \gamma_i(x(t)) = 0$, $i = 1, \dots, m$. In general, such equations need not be solvable, or they can have finite or infinite number of roots. In Theorem 1.1 we present conditions imposed on barriers which yield for each $i \in \{1, \dots, m\}$ a unique solution $t = t_{ix}$ of the equation $t = \gamma_i(x(t))$ provided x belongs to some suitable set Ω_{KL}.

For positive numbers K and L, we define a set Ω_{KL}

$$\Omega_{KL} := \{x \in \widetilde{\mathrm{AC}} : \mathrm{var}(x) \leq K, \quad |x'(t)| \leq L \text{ for a.e. } t \in [0, 2T], \ x \text{ is T-antiperiodic}\}, \tag{2.1}$$

and prove its properties.

Lemma 2.1. *The set Ω_{KL} is nonempty, bounded, convex and closed in $\widetilde{\mathrm{NBV}}$.*

Proof. Ω_{KL} is nonempty because the zero function belongs to Ω_{KL} and if $K \leq LT$, then $x(t) = \frac{K}{4}\sin(\pi t/T) \in \Omega_{KL}$, if $K > LT$, then $x(t) = \frac{LT}{4}\sin(\pi t/T) \in \Omega_{KL}$. In addition, we see that Ω_{KL} is bounded and convex. It remains to prove that Ω_{KL} is closed. Consider a sequence $\{x_n\}_{n=1}^\infty \subset \Omega_{KL}$ and let $x \in \widetilde{\mathrm{NBV}}$ is such that

$$\lim_{n\to\infty} \mathrm{var}(x - x_n) = 0. \tag{2.2}$$

We need to prove that $x \in \Omega_{KL}$. From $\mathrm{var}(x_n) \leq K$, $n \in \mathbb{N}$, and (2.2) it follows that $\mathrm{var}(x) \leq K$. Further, there exists a unique function $x^{AC} \in \widetilde{\mathrm{AC}}$ such that $x = x^{AC} + x^S$, where $x^S \in \widetilde{\mathrm{NBV}}$

is a singular part of x having zero derivative for a.e. $t \in [0, 2T]$. Moreover, since $x_n \in \widetilde{AC}$, $n \in \mathbb{N}$, we have by [7, Theorem 3.3.5],

$$\operatorname{var}(x - x_n) = \operatorname{var}(x^{AC} - x_n) + \operatorname{var}(x^S), \ n \in \mathbb{N},$$

and letting $n \to \infty$, we get $x^S \equiv 0$ due to (2.2). Consequently $x \in \widetilde{AC}$ and there exists a set $M \subset (0, 2T)$ of a zero measure such that $x'(t)$ is defined for all $t \in (0, 2T) \setminus M$. Choose an arbitrary $t \in (0, 2T) \setminus M$. We can find $\varepsilon > 0$ such that $(t - \varepsilon, t + \varepsilon) \subset (0, 2T)$. Having in mind that $|x_n'(t)| \le L$ for a.e. $t \in [0, 2T]$ and all $n \in \mathbb{N}$, we get for $h \in (-\varepsilon, \varepsilon)$

$$|x_n(t+h) - x_n(t)| \le \left| \int_t^{t+h} |x_n'(s)| \, \mathrm{d}s \right| \le L|h|.$$

This yields $|x(t+h) - x(t)| \le L|h|$, and after the limit $h \to 0$ we get $|x'(t)| \le L$ for a.e. $t \in [0, 2T]$. Finally, for each $n \in \mathbb{N}$, the function x_n is T-antiperiodic which implies by (1.12) and (2.2) that x is T-antiperiodic, as well. □

Lemma 2.2. *Let $K, L \in (0, \infty)$. Assume that there exist $a, b \in (0, T)$ and $L_i \in (0, 1/L)$, $i = 1, \dots, m$, such that (1.5) and (1.6) are fulfilled. Then for each $x \in \Omega_{KL}$ and $i \in \{1, \dots, m\}$ the equation*

$$t = \gamma_i(x(t)) \tag{2.3}$$

has a unique solution $t_{ix} \in [a, b]$.

Proof. Choose $x \in \Omega_{KL}$, $i \in \{1, \dots, m\}$, and put $\sigma_x(t) = t - \gamma_i(x(t))$ for $t \in [0, T]$. Then $|x|_\infty \le K$, σ_x is continuous and by (1.5), $\sigma_x(0) < 0$, $\sigma_x(T) > 0$. This yields $t_x \in (0, T)$ such that $\sigma_x(t_x) = 0$. Let $t_x, s_x \in (0, T)$ satisfy $\gamma_i(x(t_x)) = t_x$, $\gamma_i(x(s_x)) = s_x$. Then, by (1.6) and (2.1),

$$\begin{aligned} |s_x - t_x| &= |\gamma_i(x(s_x)) - \gamma_i(x(t_x))| \le L_i |x(s_x) - x(t_x)| \\ &= L_i \left| \int_{t_x}^{s_x} x'(\xi) \, \mathrm{d}\xi \right| \le L_i L |s_x - t_x| < |s_x - t_x|, \end{aligned}$$

which gives $t_x = s_x$. □

Lemma 2.3. *Let the assumptions of Lemma 2.2 be fulfilled. Then for $i \in \{1, \dots, m\}$, the functional*

$$\tau_i : \Omega_{KL} \to [a, b], \quad \tau_i(x) = t_{ix}, \tag{2.4}$$

where t_{ix} is a solution of (2.3), is continuous.

Proof. Choose $x, v \in \Omega_{KL}$ and $i \in \{1, \dots, m\}$. Then

$$|\tau_i(x) - \tau_i(v)| = |t_{ix} - t_{iv}| = |\gamma_i(x(t_{ix})) - \gamma_i(v(t_{iv}))| \le L_i \left(|x(t_{ix}) - v(t_{ix})| + \left| \int_{t_{iv}}^{t_{ix}} v'(\xi) \, \mathrm{d}\xi \right| \right),$$

and so

$$|\tau_i(x) - \tau_i(v)| \le L_i \operatorname{var}(x - v) + L_i L |t_x - t_v|) = L_i \operatorname{var}(x - v) + L_i L |\tau_i(x) - \tau_i(v)|.$$

Therefore

$$|\tau_i(x) - \tau_i(v)| \le \frac{L_i \operatorname{var}(x - v)}{1 - L_i L}, \quad x, v \in \Omega_{KL},$$

which yields the continuity of τ_i on Ω_{KL}. □

3 Proof of Theorem 1.1

Proof. (i) Having continuous functionals $\tau_1, \ldots, \tau_m$ from Lemma 2.3, we can argue similarly as in [11] because (1.5) implies (1.19). Since $\mathcal{J}_i$ are bounded, there exist $a_i \in (0, \infty)$, $i = 1, \ldots, m$, such that

$$\mathcal{J}_i : \widetilde{\mathrm{NBV}} \to [-a_i, a_i], \quad i = 1, \ldots, m. \tag{3.1}$$

Choose

$$T \in (0, \sqrt{3}), \quad \mu \in (0, 1/T - T/3), \tag{3.2}$$

introduce constants K, L by

$$K := \frac{1}{2}\sqrt{\frac{1 - \mu T - T^2/3}{\mu T}}, \tag{3.3}$$

$$L := \mu K + \frac{2\mu}{3}K^3 + TK + 2T\|f\|_{\mathrm{L}^1} + \frac{1}{2}\sum_{i=1}^{m} a_i, \tag{3.4}$$

and consider the set Ω_{KL} from (2.1). Similarly as in [11] we define an operator $\mathcal{F}$ by

$$\mathcal{F}x = \mu I\left(x - \frac{x^3}{3}\right) + I^2\left(-x + f + \frac{1}{2T}\sum_{i=1}^{m}\mathcal{J}_i(x)\varepsilon_{\tau_i(x)}\right), \quad x \in \Omega_{KL}, \tag{3.5}$$

where τ_i is from (2.4), $\varepsilon_{\tau_i(x)}$ is from (1.15), I, I^2 are from (1.14). It follows from [11, Lemma 4.2] that $\mathcal{F}$ is compact on Ω_{KL}.

(ii) Let us show that $\mathcal{F}$ maps Ω_{KL} to Ω_{KL}. Since the definition of the set Ω_{KL} in (2.1) is different from the definition of the corresponding set Ω in [11], we need to prove the estimate

$$|(\mathcal{F}x)'(t)| \le L \quad \text{for a.e. } t \in [0, 2T], \quad \text{and all } x \in \Omega_{KL}. \tag{3.6}$$

Differentiating (3.5) we get

$$(\mathcal{F}x)'(t) = \mu\left(x(t) - \frac{x^3(t)}{3} - \bar{x} + \frac{\overline{x^3}}{3}\right) + I\left(f(t) - x(t)\right) + \frac{1}{2T}\sum_{i=1}^{m}\mathcal{J}_i(x)\left(I\varepsilon_{\tau_i(x)}\right)(t),$$

and, by

$$\begin{aligned}
|(\mathcal{F}x)'(t)| &\le \mu|x(t)| + \frac{\mu}{3}|x^3(t) - \overline{x^3}| + \operatorname{var}\left(E_1 * (f - x)\right) + \frac{1}{2T}\sum_{i=1}^{m}|\mathcal{J}_i(x)|\|I\varepsilon_{\tau_i(x)}\|_\infty \\
&\le \mu\|x\|_\infty + \frac{2\mu}{3}\|x\|_\infty^3 + \operatorname{var}(E_1)\|f\|_{\mathrm{L}^1} + \|E_1\|_\infty \operatorname{var}(-x) + \frac{1}{2T}\sum_{i=1}^{m} a_i\|I\varepsilon_{\tau_i(x)}\|_\infty \\
&\le \mu K + \frac{2\mu}{3}K^3 + 2T\|f\|_{\mathrm{L}^1} + TK\frac{1}{2}\sum_{i=1}^{m} a_i = L.
\end{aligned}$$

Now, consider c_1 and φ from (1.8) and (1.9) and assume that

$$Tc_1 \le \varphi(\mu). \tag{3.7}$$

Then, using the arguments from the proof in [11, Theorem 4.4], we get

$$\operatorname{var}(\mathcal{F}x) \le K \quad \text{for all } x \in \Omega_{KL}. \tag{3.8}$$

In addition, by (1.16) and (1.14), $\mathcal{F}x \in \widetilde{AC}$ and it is antiperiodic for $x \in \Omega_{KL}$. Therefore $\mathcal{F}(\Omega_{KL}) \subset \Omega_{KL}$.

(iii) Consequently, by the Schauder fixed point theorem there exists a fixed point $x \in \Omega_{KL}$ of the operator $\mathcal{F}$. By [11, Lemma 4.1, Lemma 3.4], if we put $y(t) = x'(t)$ for a.e. $t \in \mathbb{R}$, then (x, y) is a T-antiperiodic solution of problem (1.1)–(1.3). Having in mind that φ is continuous and decreasing on $(0, 1/T - T/3]$ and $\lim_{\mu\to 0+} \varphi(\mu) = \infty$, $\varphi(1/T - T/3) = 0$, we get a unique $\mu_0 \in (0, 1/T - T/3)$ satisfying $Tc_1 = \varphi(\mu_0)$. Clearly, if $\mu \le \mu_0$, then (3.7) holds. Consequently we get a T-antiperiodic solution of problem (1.1)–(1.3) for each $\mu \in (0, \mu_0]$. □

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