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Positive solutions of a derivative dependent second-order problem subject to Stieltjes integral boundary conditions

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Abstract. In this paper, we investigate the derivative dependent second-order problem subject to Stieltjes integral boundary conditions

$$\begin{cases} -u''(t) = f(t, u(t), u'(t)), \quad t \in [0,1], \\ au(0) - bu'(0) = \alpha[u],\ cu(1) + du'(1) = \beta[u], \end{cases}$$

where $f\colon [0,1] \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ is continuous, $\alpha[u]$ and $\beta[u]$ are linear functionals involving Stieltjes integrals. Some inequality conditions on nonlinearity f and the spectral radius condition of linear operator are presented that guarantee the existence of positive solutions to the problem by the theory of fixed point index. Not only is the general case considered but a large range of coefficients can be chosen to weaken the conditions in previous work for some special cases. The conditions allow that $f(t, x_1, x_2)$ has superlinear or sublinear growth in x_1, x_2. Two examples are provided to illustrate the theorems under multi-point and integral boundary conditions with sign-changing coefficients.

Keywords: positive solution, fixed point index, cone, spectral radius.

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1 Introduction

The existence of solutions for second-order boundary value problem (BVP) with dependence on derivative in nonlinearity

$$\begin{cases} -u''(t) = f(t, u(t), u'(t)), \quad t \in [0,1], \\ u(0) = u(1) = 0 \end{cases} \tag{1.1}$$

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was considered by Li [8], where $f\colon [0,1] \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ is continuous. The results of [8] extend those of [16] in which only sublinear problem was treated. Recently, the authors in [17] studied the existence of positive solutions for BVP

$$\begin{cases} -u''(t) = f(t, u(t), u'(t)), & t \in [0,1], \\ u(0) = \alpha[u], \quad u'(1) = 0, \end{cases} \tag{1.2}$$

where $f\colon [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and $\alpha[u] = \int_0^1 u(t)dA(t)$ is a Stieltjes integral with the function A of bounded variation. In [8,17], the theory of fixed point index is applied and the nonlinearity $f(t, x_1, x_2)$ has superlinear or sublinear growth on x_1 and x_2. Zima [18] studied the problem with $au(0) - bu'(0) = \alpha[u], u'(1) = \beta[u]$ for positive measures and also allows f to be singular in u, u'.

In this paper, we discuss the existence of positive solutions for the general derivative dependent BVP subject to Stieltjes integral boundary conditions

$$\begin{cases} -u''(t) = f(t, u(t), u'(t)), & t \in [0,1], \\ au(0) - bu'(0) = \alpha[u], \ cu(1) + du'(1) = \beta[u], \end{cases} \tag{1.3}$$

where a, b, c and d are nonnegative constants with $\rho = ac + ad + bc > 0$, α and β denote linear functionals given by

$$\alpha[u] = \int_0^1 u(t)dA(t), \qquad \beta[u] = \int_0^1 u(t)dB(t)$$

involving Stieltjes integrals with suitable functions A, B of bounded variation. The problem (1.3) but with f not positive was studied by Webb in [11].

The features of the paper are stated in the following three aspects.

1. The method in [8] depends essentially on the zero boundary conditions $u(0) = u(1) = 0$, and in [17] the problem is only considered under the boundary assumption $u'(1) = 0$. We study the more general case with the terms $\alpha[u]$ and $\beta[u]$ included in this paper.

2. The sign of the derivative with respect to t of the corresponding Green's function does not change in [17] so that the monotonicity is led into constructing the cones. However for BVP (1.3) the derivative of the Green's function may be sign-changing.

3. Not only is the general case investigated but a large range of coefficients can be chosen to weaken the conditions in [8] for special cases, see Remarks 3.6, 3.9 and 4.3 behind. The spectral radius conditions of associated linear operators are also used in [17] similar to the ones here, but those operators involve the term u' and are defined on the space C^1 which are different from here. Actually for BVP (1.2) the conditions in [17] do not be covered here, and vice versa, see Remarks 3.10 and 3.11 for details.

We first apply the method due to Webb and Infante [13] to give the corresponding Green's function and discuss the inequalities about it and its derivative. Meanwhile two cones are constructed, the large one induces the partial ordering and the small is employed to compute the fixed point index later. Then the theory of fixed point index is used to establish the existence of positive solutions to BVP (1.3) under some inequality conditions on nonlinearity f and the spectral radius condition of linear operator. Finally, two examples are provided to illustrate the theorems under multi-point and integral boundary conditions with sign-changing coefficients. Some relevant articles are referred to for nonlocal boundary problems, for example, [4, 9, 10, 13–15], and for BVPs with dependence on the first-order derivative in nonlinearities such as [5, 6, 12].

2 Preliminaries

Let $C^1[0,1]$ denote the Banach space of all continuously differentiable functions on $[0,1]$ with the norm

$$\|u\|_{C^1} = \max\{\|u\|_C, \|u'\|_C\} = \max\left\{\max_{0\le t\le 1}|u(t)|, \max_{0\le t\le 1}|u'(t)|\right\}.$$

We first make the assumption:

(C_1) $f\colon [0,1]\times\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}^+$ is continuous, here $\mathbb{R}^+ = [0,\infty)$.

As shown by Webb and Infante [13], BVP (1.3) has a solution if and only if there exists a solution in $C^1[0,1]$ for the following integral equation

$$u(t) = \gamma_1(t)\alpha[u] + \gamma_2(t)\beta[u] + \int_0^1 k(t,s)f(s,u(s),u'(s))ds =: (Tu)(t), \tag{2.1}$$

where

$$\gamma_1(t) = \frac{c(1-t)+d}{\rho}, \qquad \gamma_2(t) = \frac{b+at}{\rho},$$

$$k(t,s) = \frac{1}{\rho}\begin{cases}(as+b)(c+d-ct), & 0\le s\le t\le 1,\\ (at+b)(c+d-cs), & 0\le t\le s\le 1.\end{cases} \tag{2.2}$$

We also impose the following hypotheses:

(C_2) A and B are of bounded variation and for $s\in[0,1]$,

$$\mathcal{K}_A(s) := \int_0^1 k(t,s)dA(t)\ge 0,\ \mathcal{K}_B(s) := \int_0^1 k(t,s)dB(t)\ge 0;$$

(C_3) $0\le\alpha[\gamma_1]<1$, $\beta[\gamma_1]\ge 0$, $0\le\beta[\gamma_2]<1$, $\alpha[\gamma_2]\ge 0$, and

$$D := (1-\alpha[\gamma_1])(1-\beta[\gamma_2]) - \alpha[\gamma_2]\beta[\gamma_1] > 0.$$

Adopting the notations and ideas in [13], define the operator S as

$$\begin{aligned}(Su)(t) = {}& \frac{\gamma_1(t)}{D}\left[(1-\beta[\gamma_2])\int_0^1 \mathcal{K}_A(s)f(s,u(s),u'(s))ds + \alpha[\gamma_2]\int_0^1 \mathcal{K}_B(s)f(s,u(s),u'(s))ds\right]\\ &+\frac{\gamma_2(t)}{D}\left[\beta[\gamma_1]\int_0^1 \mathcal{K}_A(s)f(s,u(s),u'(s))ds + (1-\alpha[\gamma_1])\int_0^1 \mathcal{K}_B(s)f(s,u(s),u'(s))ds\right]\\ &+\int_0^1 k(t,s)f\left(s,u(s),u'(s)\right)ds\\ =:{}& \int_0^1 k_S(t,s)f(s,u(s),u'(s))ds\end{aligned}$$

i.e.,

$$(Su)(t) = \int_0^1 k_S(t,s)f(s,u(s),u'(s))ds, \tag{2.3}$$

where

$$\begin{aligned}k_S(t,s) = {}& \frac{\gamma_1(t)}{D}[(1-\beta[\gamma_2])\mathcal{K}_A(s) + \alpha[\gamma_2]\mathcal{K}_B(s)]\\ &+\frac{\gamma_2(t)}{D}[\beta[\gamma_1]\mathcal{K}_A(s) + (1-\alpha[\gamma_1])\mathcal{K}_B(s)] + k(t,s).\end{aligned} \tag{2.4}$$

By direct calculation, we easily get the inequalities about Green's function in Lemma 2.1.

Lemma 2.1. *If* (C_2) *and* (C_3) *hold, then there exists a nonnegative continuous function* $\Phi(s)$ *satisfying*

$$c(t)\Phi(s) \le k_S(t,s) \le \Phi(s) \quad \text{for } t,s \in [0,1],$$

where $c(t) = \min\{t, 1-t\}$ *and*

$$\Phi(s) = \frac{c+d}{\rho D}[(1-\beta[\gamma_2])\mathcal{K}_A(s) + \alpha[\gamma_2]\mathcal{K}_B(s)] + \frac{a+b}{\rho D}[\beta[\gamma_1]\mathcal{K}_A(s) + (1-\alpha[\gamma_1])\mathcal{K}_B(s)] + k(s,s).$$

By (2.4)

$$\begin{aligned}\left|\frac{\partial k_S(t,s)}{\partial t}\right| &\le \left|\frac{-c}{\rho D}[(1-\beta[\gamma_2])\mathcal{K}_A(s) + \alpha[\gamma_2]\mathcal{K}_B(s)] + \frac{a}{\rho D}[\beta[\gamma_1]\mathcal{K}_A(s) + (1-\alpha[\gamma_1])\mathcal{K}_B(s)]\right| \\ &\quad + \left|\frac{\partial k(t,s)}{\partial t}\right| \\ &\le \left|\frac{-c}{\rho D}[(1-\beta[\gamma_2])\mathcal{K}_A(s) + \alpha[\gamma_2]\mathcal{K}_B(s)] + \frac{a}{\rho D}[\beta[\gamma_1]\mathcal{K}_A(s) + (1-\alpha[\gamma_1])\mathcal{K}_B(s)]\right| \\ &\quad + \frac{1}{\rho}\max\{a(c+d-cs), c(as+b)\} =: \Phi_1(s), \end{aligned} \tag{2.5}$$

where

$$\frac{\partial k(t,s)}{\partial t} = \frac{1}{\rho}\begin{cases} -c(as+b), & 0 \le s \le t \le 1, \\ a(c+d-cs), & 0 \le t \le s \le 1. \end{cases}$$

Define two cones in $C^1[0,1]$ and two linear operators in $C[0,1]$ as follow:

$$P = \{u \in C^1[0,1] : u(t) \ge 0, \ \forall t \in [0,1]\}, \tag{2.6}$$

$$K = \Big\{u \in P : u(t) \ge c(t)\|u\|_C, \ \forall t \in [0,1]; \ \alpha[u] \ge 0, \ \beta[u] \ge 0\Big\}, \tag{2.7}$$

$$(Lu)(t) = \int_0^1 k_S(t,s)u(s)ds, \ u \in C[0,1], \tag{2.8}$$

$$(L^*u)(s) = \int_0^1 k_S(t,s)u(t)dt, \ u \in C[0,1]. \tag{2.9}$$

We write $u \preceq v$ equivalently $v \succeq u$ if and only if $v - u \in P$, to denote the cone ordering induced by P.

Lemma 2.2. *If* (C_1)–(C_3) *hold, then* $S\colon P \to K$ *and* $L, \ L^*\colon C[0,1] \to C[0,1]$ *are completely continuous operators with* $L(P) \subset K$.

Proof. From (2.3), (2.4) and (C_1)–(C_3) we have for $u \in P$ that $(Su)(t) \ge 0$. It is easy to see from (C_1) that $S : P \to C^1[0,1]$ is continuous. Let F be a bounded set in P, then there exists $M > 0$ such that $\|u\|_{C^1} \le M$ for all $u \in F$. By (C_1) and Lemma 2.1 we have that $\forall u \in F$ and $t \in [0,1]$,

$$(Su)(t) \le \Big(\max_{(s,x,y)\in[0,1]\times[0,M]\times[-M,M]} f(s,x,y)\Big)\int_0^1 \Phi(s)ds,$$

$$\begin{aligned}|(Su)'(t)| &\le \Big(\max_{(s,x,y)\in[0,1]\times[0,M]\times[-M,M]} f(s,x,y)\Big)\int_0^1 \left|\frac{\partial k_S(t,s)}{\partial t}\right|ds \\ &\le \max_{(s,x,y)\in[0,1]\times[0,M]\times[-M,M]} f(s,x,y)\int_0^1 \Phi_1(s)ds,\end{aligned}$$

then $S(F)$ is uniformly bounded in $C^1[0,1]$. Moreover $\forall u \in F$ and $t_1, t_2 \in [0,1]$ with $t_1 < t_2$,

$$\begin{aligned}|(Su)(t_1) - (Su)(t_2)| &\le \int_0^1 |k_S(t_1,s) - k_S(t_2,s)| f(s,u(s),u'(s))ds \\ &\le \Big(\max_{(s,x,y)\in[0,1]\times[0,M]\times[-M,M]} f(s,x,y) \Big) \int_0^1 |k_S(t_1,s) - k_S(t_2,s)| ds,\end{aligned}$$

$$\begin{aligned}|(Su)'(t_1) - (Su)'(t_2)| &\le \int_0^1 \big|k'_S(t_1,s) - k'_S(t_2,s)\big| f(s,u(s),u'(s))ds \\ &= \int_{t_1}^{t_2} \big|k'_S(t_1,s) - k'_S(t_2,s)\big| f(s,u(s),u'(s))ds \\ &\le 2\Big(\max_{(s,x,y)\in[0,1]\times[0,M]\times[-M,M]} f(s,x,y) \Big) \int_{t_1}^{t_2} \Phi_1(s) ds,\end{aligned}$$

thus $S(F)$ and $S'(F) =: \{v' : v'(t) = (Su)'(t), u \in F\}$ are equicontinuous.

Therefore $S : P \to C^1[0,1]$ is completely continuous by the Arzelà–Ascoli theorem.

For $u \in P$ it follows from Lemma 2.1 that

$$\|Su\|_C = \max_{0\le t\le 1} \left(\int_0^1 k_S(t,s) f(s,u(s),u'(s))ds \right) \le \int_0^1 \Phi(s) f(s,u(s),u'(s))ds,$$

and hence for $t \in [0,1]$,

$$(Su)(t) = \int_0^1 k_S(t,s) f(s,u(s),u'(s))ds \ge c(t) \int_0^1 \Phi(s) f(s,u(s),u'(s))ds \ge c(t)\|Su\|_C.$$

From (C_1)–(C_3) it can easily be checked that $\alpha[Su] \ge 0$ and $\beta[Su] \ge 0$. Thus $S : P \to K$.

Similarly, $L,\ L^* : C[0,1] \to C[0,1]$ are completely continuous operators with $L(P) \subset K$. □

Lemma 2.3 ([13]). *If (C_1)–(C_3) hold, then S and T have the same fixed points in K. As a result, BVP (1.3) has a solution if and only if S has a fixed point.*

3 Main results

In order to prove the main theorems, we need the following properties of fixed point index, see [1,2,7].

Lemma 3.1. *Let Ω be a bounded open subset of X with $0 \in \Omega$ and K be a cone in X. If $A\colon K \cap \overline{\Omega} \to K$ is a completely continuous operator and $\mu A u \ne u$ for $u \in K \cap \partial\Omega$ and $\mu \in [0,1]$, then the fixed point index $i(A, K \cap \Omega, K) = 1$.*

Lemma 3.2. *Let Ω be a bounded open subset of X and K be a cone in X. If $A\colon K \cap \overline{\Omega} \to K$ is a completely continuous operator and there exists $v_0 \in K \setminus \{0\}$ such that $u - Au \ne \nu v_0$ for $u \in K \cap \partial\Omega$ and $\nu \ge 0$, then the fixed point index $i(A, K \cap \Omega, K) = 0$.*

Recall that a cone P in Banach space X is said to be total if $X = \overline{P - P}$.

Lemma 3.3 (Krein–Rutman). *Let P be a total cone in Banach space X and $L\colon X \to X$ be a completely continuous linear operator with $L(P) \subset P$. If the spectral radius $r(L) > 0$, then there exists $\varphi \in P \setminus \{0\}$ such that $L\varphi = r(L)\varphi$, where 0 denotes the zero element in X.*

The following lemma comes from [7, Theorem 2.5] and is useful for later calculations of $r(L)$.

Lemma 3.4. *Let P be a cone in Banach space X and $L\colon X \to X$ be a completely continuous linear operator with $L(P) \subset P$. If there exist $v_0 \in P \setminus \{0\}$ and $\lambda_0 > 0$ such that $Lv_0 \geq \lambda_0 v_0$ in the sense of partial ordering induced by P, then there exist $u_0 \in P \setminus \{0\}$ and $\lambda_1 \geq \lambda_0$ such that $Lu_0 = \lambda_1 u_0$.*

In the sequel, let $X = C^1[0,1]$ and denote $\Omega_r = \{u \in X : \|u\|_{C^1} < r\}$ for $r > 0$.

Theorem 3.5. *Under the hypotheses (C_1)–(C_3) suppose that*

(F_1) *there exist nonnegative constants a_1, b_1, c_1 satisfying*

$$a_1 \int_0^1 \Phi(s)ds + b_1 \int_0^1 \Phi_1(s)ds < 1 \tag{3.1}$$

such that

$$f(t,x_1,x_2) \leq a_1 x_1 + b_1|x_2| + c_1, \tag{3.2}$$

for all $(t,x_1,x_2) \in [0,1] \times \mathbb{R}^+ \times \mathbb{R}$;

(F_2) *there exist constants $a_2 > 0$ and $r > 0$ such that*

$$f(t,x_1,x_2) \geq a_2 x_1, \tag{3.3}$$

for all $(t,x_1,x_2) \in [0,1] \times [0,r] \times [-r,r]$, moreover the spectral radius $r(L) \geq 1/a_2$, where L is defined by (2.8).

Then BVP (1.3) has at least one positive solution.

Proof. Let $W = \{u \in K : u = \mu Su,\ \mu \in [0,1]\}$ where S and K are respectively defined in (2.3) and (2.7).

We first assert that W is a bounded set. In fact, if $u \in W$, then $u = \mu Su$ for some $\mu \in [0,1]$. From Lemma 2.1 and (3.2) we have that

$$\begin{aligned}
\|u\|_C &= \mu \max_{0\leq t\leq 1} \left(\int_0^1 k_S(t,s) f(s,u(s),u'(s))ds \right) \\
&\leq \int_0^1 \Phi(s)\left[a_1 u(s) + b_1|u'(s)| + c_1\right]ds \\
&\leq \left(a_1\|u\|_C + b_1\|u'\|_C + c_1\right) \int_0^1 \Phi(s)ds,
\end{aligned}$$

$$\begin{aligned}
\|u'\|_C &= \mu \max_{0\leq t\leq 1} \left| \int_0^1 \frac{\partial k_S(t,s)}{\partial t} f(s,u(s),u'(s))ds \right| \\
&\leq \int_0^1 \Phi_1(s)\left[a_1 u(s) + b_1|u'(s)| + c_1\right]ds \\
&\leq \left(a_1\|u\|_C + b_1\|u'\|_C + c_1\right) \int_0^1 \Phi_1(s)ds,
\end{aligned}$$

thus

$$\|u\|_C \leq \left(1 - a_1 \int_0^1 \Phi(s)ds\right)^{-1} \left(b_1\|u'\|_C + c_1\right) \int_0^1 \Phi(s)ds, \tag{3.4}$$

$$
\begin{aligned}
\|u'\|_C \le{}& \frac{a_1 b_1}{1 - a_1 \int_0^1 \Phi(s)ds} \|u'\|_C \Big(\int_0^1 \Phi(s)ds \Big) \Big(\int_0^1 \Phi_1(s)ds \Big) \\
&+ \frac{a_1 c_1}{1 - a_1 \int_0^1 \Phi(s)ds} \Big(\int_0^1 \Phi(s)ds \Big) \Big(\int_0^1 \Phi_1(s)ds \Big) \\
&+ b_1 \|u'\|_C \int_0^1 \Phi_1(s)ds + c_1 \int_0^1 \Phi_1(s)ds. \qquad (3.5)
\end{aligned}
$$

From (3.1), (3.4) and (3.5) it follows that

$$
\|u\|_C \le \frac{c_1 \int_0^1 \Phi(s)ds}{1 - a_1 \int_0^1 \Phi(s)ds - b_1 \int_0^1 \Phi_1(s)ds},
$$

$$
\|u'\|_C \le \frac{c_1 \int_0^1 \Phi_1(s)ds}{1 - a_1 \int_0^1 \Phi(s)ds - b_1 \int_0^1 \Phi_1(s)ds},
$$

and hence W is bounded.

Now select $R > \max\{r, \sup W\}$, then $\mu Su \neq u$ for $u \in K \cap \partial\Omega_R$ and $\mu \in [0,1]$, and $i(S, K \cap \Omega_R, K) = 1$ follows from Lemma 3.1.

It is easy to see that $L(C^+[0,1]) \subset P \subset C^+[0,1]$, where $C^+[0,1] = \{u \in C[0,1] : u(t) \ge 0, \ \forall t \in [0,1]\}$ is a total cone in $C[0,1]$. Since $r(L) \ge 1/a_2 > 0$, it follows from Lemma 3.3 that there exists $\varphi_0 \in C^+[0,1] \setminus \{0\}$ such that $L\varphi_0 = r(L)\varphi_0$. Furthermore, $\varphi_0 = (r(L))^{-1} L\varphi_0 \in K$ by Lemma 2.2.

We may suppose that S has no fixed points in $K \cap \partial\Omega_r$ and will show that $u - Su \neq \nu\varphi_0$ for $u \in K \cap \partial\Omega_r$ and $\nu \ge 0$.

Otherwise, there exist $u_0 \in K \cap \partial\Omega_r$ and $\nu_0 \ge 0$ such that $u_0 - Su_0 = \nu_0\varphi_0$, and it is clear that $\nu_0 > 0$. Since $u_0 \in K \cap \partial\Omega_r$, we have $0 \le u_0(t) \le r, -r \le u_0'(t) \le r, \forall t \in [0,1]$. It follows from (3.3) that $(Su_0)(t) \ge a_2(Lu_0)(t)$ which implies that

$$
u_0 = \nu_0\varphi_0 + Su_0 \succeq \nu_0\varphi_0 + a_2 Lu_0 \succeq \nu_0\varphi_0. \qquad (3.6)
$$

Set $\nu^* = \sup\{\nu > 0 : u_0 \succeq \nu\varphi_0\}$, then $\nu_0 \le \nu^* < +\infty$ and $u_0 \succeq \nu^*\varphi_0$. Thus it follows from (3.6) that

$$
u_0 \succeq \nu_0\varphi_0 + a_2 Lu_0 \succeq \nu_0\varphi_0 + a_2\nu^* L\varphi_0 = \nu_0\varphi_0 + a_2\nu^* r(L)\varphi_0.
$$

But $r(L) \ge 1/a_2$, so $u_0 \succeq (\nu_0 + \nu^*)\varphi_0$, which is a contradiction to the definition of ν^*. Therefore $u - Su \neq \nu\varphi_0$ for $u \in K \cap \partial\Omega_r$ and $\nu \ge 0$.

From Lemma 3.2 it follows that $i(S, K \cap \Omega_r, K) = 0$.

Making use of the properties of fixed point index, we have that

$$
i(S, K \cap (\Omega_R \setminus \overline{\Omega}_r), K) = i(S, K \cap \Omega_R, K) - i(S, K \cap \Omega_r, K) = 1
$$

and hence S has at least one fixed point in K. Therefore, BVP (1.3) has at least one positive solution by Lemma 2.3. □

Remark 3.6. For the case $\alpha[u] = \beta[u] = 0$ and $a = c = 1, b = d = 0$ considered in [8], we have that $\Phi(s) = s(1-s)$, $\Phi_1(s) = \max\{1-s, s\}$, thus $\int_0^1 \Phi(s)ds = 1/6$, $\int_0^1 \Phi_1(s)ds = 3/4$. Moreover, the spectral radius $r(L) = 1/\pi^2$. Therefore, (3.1) and $r(L) \ge 1/a_2$ are satisfied when $a_1 + b_1 < 1$ and $a_2 > \pi^2$ are required in [8]. This means that the result of Theorem 3.5 extends Theorem 1.2 of [8].

Lemma 3.7 ([3, Lemma 5.1 of Chapter XII])**.** *Let $R > 0$, and let $\varphi\colon [0, \infty) \to (0, \infty)$ be continuous and satisfy*

$$\int_0^\infty \frac{\rho d\rho}{\varphi(\rho)} = \infty. \tag{3.7}$$

Then there exists a number $M > 0$, depending only on φ, R such that if $v \in C^2[0, 1]$ which satisfies $\|v\|_C \leq R$ and $|v''(t)| \leq \varphi(|v'(t)|), t \in [0, 1]$, then $\|v'\|_C \leq M$.

Theorem 3.8. *Under the hypotheses (C_1)–(C_3) suppose that*

(F_3) *there exist nonnegative constants a_1, b_1 and $r > 0$ satisfying*

$$(a_1 + b_1) \max\left\{ \int_0^1 \Phi(s)ds,\ \int_0^1 \Phi_1(s)ds \right\} < 1 \tag{3.8}$$

such that

$$f(t, x_1, x_2) \leq a_1 x_1 + b_1 |x_2|, \tag{3.9}$$

for all $(t, x_1, x_2) \in [0, 1] \times [0, r] \times [-r, r]$;

(F_4) *there exist positive constants a_2, c_2 such that*

$$f(t, x_1, x_2) \geq a_2 x_1 - c_2, \tag{3.10}$$

for all $(t, x_1, x_2) \in [0, 1] \times \mathbb{R}^+ \times \mathbb{R}$, moreover the spectral radii $r(L) \geq 1/a_2$, $r(L^*) > 1/a_2$, where L, L^* are defined by (2.8) and (2.9) respectively;

(F_5) *for any $M > 0$ there is a positive continuous function $\varphi(\rho)$ on $\mathbb{R}^+$ satisfying (3.7) such that*

$$f(t, x, y) \leq \varphi(|y|) - c_2, \quad \forall (t, x, y) \in [0, 1] \times [0, M] \times \mathbb{R}, \tag{3.11}$$

then BVP (1.3) has at least one positive solution.

Proof. (i) First we prove that $\mu S u \neq u$ for $u \in K \cap \partial\Omega_r$ and $\mu \in [0, 1]$. In fact, if there exist $u_1 \in K \cap \partial\Omega_r$ and $\mu_0 \in [0, 1]$ such that $u_1 = \mu_0 S u_1$, then we deduce from Lemma 2.1, (2.5), (3.8), (3.9) and $0 \leq u_1(t) \leq r$, $-r \leq u_1'(t) \leq r$, $\forall t \in [0, 1]$ that

$$\begin{aligned}
\|u_1\|_C &= \mu_0 \max_{0 \leq t \leq 1} \Big(\int_0^1 k_S(t, s) f(s, u_1(s), u_1'(s)) ds \Big) \\
&\leq \int_0^1 \Phi(s)[a_1 u_1(s) + b_1 |u_1'(s)|] ds \\
&\leq (a_1 + b_1) \Big(\int_0^1 \Phi(s) ds \Big) \|u_1\|_{C^1} < \|u_1\|_{C^1} = r,
\end{aligned}$$

$$\begin{aligned}
\|u_1'\|_C &= \mu_0 \max_{0 \leq t \leq 1} \Big| \int_0^1 \frac{\partial k_S(t, s)}{\partial t} f(s, u_1(s), u_1'(s)) ds \Big| \\
&\leq \int_0^1 \Phi_1(s)[a_1 u_1(s) + b_1 |u_1'(s)|] ds \\
&\leq (a_1 + b_1) \Big(\int_0^1 \Phi_1(s) ds \Big) \|u_1\|_{C^1} < \|u_1\|_{C^1} = r.
\end{aligned}$$

Hnece $\|u_1\|_{C^1} < r$ which contradicts $u_1 \in K \cap \partial\Omega_r$.

Therefore, $i(S, K \cap \Omega_r, K) = 1$ follows from Lemma 3.1.

(ii) It is easy to see that $L^*(C^+[0,1]) \subset C^+[0,1]$. Since $r(L^*) \geq 1/a_2 > 0$, it follows from Lemma 3.3 that there exists $\varphi^* \in C^+[0,1] \setminus \{0\}$ such that $L^*\varphi^* = r(L^*)\varphi^*$.

Let

$$M = \frac{c_2 \int_0^1 \varphi^*(t)dt \int_0^1 k_S(t,s)ds}{(a_2 r(L^*) - 1) \int_0^1 c(t)\varphi^*(t)dt}, \tag{3.12}$$

where $c(t)$ comes from Lemma 2.1.

(iii) For $u \in P$ define

$$(S_1 u)(t) = \int_0^1 k_S(t,s)(f(s,u(s),u'(s)) + c_2)ds. \tag{3.13}$$

Similar to the proof in Lemma 2.2, we know that $S_1 : P \to K$ is completely continuous.

If there exist $u_2 \in K$ and $\lambda_0 \in [0,1]$ such that

$$(1 - \lambda_0)Su_2 + \lambda_0 S_1 u_2 = u_2, \tag{3.14}$$

thus by (3.10) and (3.14) we obtain that

$$\begin{aligned}
\int_0^1 \varphi^*(t)u_2(t)dt &= (1-\lambda_0)\int_0^1 \varphi^*(t)dt \int_0^1 k_S(t,s)f(s,u_2(s),u_2'(s))ds \\
&\quad + \lambda_0 \int_0^1 \varphi^*(t)dt \int_0^1 k_S(t,s)(f(s,u_2(s),u_2'(s)) + c_2)ds \\
&= \int_0^1 \varphi^*(t)dt \int_0^1 k_S(t,s)(f(s,u_2(s),u_2'(s)) + \lambda_0 c_2)ds \\
&\geq \int_0^1 \varphi^*(t)dt \int_0^1 k_S(t,s)(a_2 u_2(s) - c_2 + \lambda_0 c_2)ds \\
&\geq a_2 \int_0^1 \varphi^*(t)dt \int_0^1 k_S(t,s)u_2(s)ds - c_2 \int_0^1 \varphi^*(t)dt \int_0^1 k_S(t,s)ds \\
&= a_2 \int_0^1 u_2(s)ds \int_0^1 k_S(t,s)\varphi^*(t)dt - c_2 \int_0^1 \varphi^*(t)dt \int_0^1 k_S(t,s)ds \\
&= a_2 \int_0^1 u_2(s)(L^*\varphi^*)(s)ds - c_2 \int_0^1 \varphi^*(t)dt \int_0^1 k_S(t,s)ds \\
&= a_2 r(L^*) \int_0^1 \varphi^*(s)u_2(s)ds - c_2 \int_0^1 \varphi^*(t)dt \int_0^1 k_S(t,s)ds,
\end{aligned}$$

which implies that

$$\|u_2\|_C \int_0^1 c(t)\varphi^*(t)dt \leq \int_0^1 \varphi^*(t)u_2(t)dt \leq \frac{c_2 \int_0^1 \varphi^*(t)dt \int_0^1 k_S(t,s)ds}{a_2 r(L^*) - 1}$$

and thus

$$\|u_2\|_C \leq \frac{c_2 \int_0^1 \varphi^*(t)dt \int_0^1 k_S(t,s)ds}{(a_2 r(L^*) - 1)\int_0^1 c(t)\varphi^*(t)dt} = M. \tag{3.15}$$

We can derive from (3.11), (3.14) and (3.15) that

$$\begin{aligned}
|u_2''(t)| &= (1-\lambda_0)f(t,u_2(t),u_2'(t)) + \lambda_0(f(t,u_2(t),u_2'(t)) + c_2) \\
&= f(t,u_2(t),u_2'(t)) + \lambda_0 c_2 \leq f(t,u_2(t),u_2'(t)) + c_2 \\
&\leq \varphi(|u_2'(t)|).
\end{aligned} \tag{3.16}$$

By Lemma 3.7, there exists a constant $M_1 > 0$ such that $\|u_2'\|_C \le M_1$.

Let $R > \max\{r, M, M_1\}$, then

$$(1-\lambda)Su + \lambda S_1 u \neq u, \quad \forall u \in K \cap \partial\Omega_R,\ \lambda \in [0,1]. \tag{3.17}$$

From (3.17) it follows that

$$i(S, K\cap\Omega_R, K) = i(S_1, K\cap\Omega_R, K) \tag{3.18}$$

by the homotopy invariance property of fixed point index.

(iv) Since $L(C^+[0,1]) \subset P \subset C^+[0,1]$ and $r(L) \ge 1/a_2 > 0$, it follows from Lemma 3.3 that there exists $\varphi_0 \in C^+[0,1] \setminus \{0\}$ such that $L\varphi_0 = r(L)\varphi_0$. Furthermore, $\varphi_0 = (r(L))^{-1}L\varphi_0 \in K$ by Lemma 2.2. Now we prove that $u - S_1 u \neq \nu\varphi_0$ for $u \in K\cap\partial\Omega_R$ and $\nu \ge 0$ and hence

$$i(S_1, K\cap\Omega_R, K) = 0 \tag{3.19}$$

holds by Lemma 3.2.

If there exist $u_0 \in K\cap\partial\Omega_R$ and $\nu_0 \ge 0$ such that $u_0 - S_1u_0 = \nu_0\varphi_0$. Obviously $\nu_0 > 0$ by (3.17) and

$$u_0 = S_1u_0 + \nu_0\varphi_0 \succeq \nu_0\varphi_0. \tag{3.20}$$

Set

$$\nu^* = \sup\{\nu > 0 : u_0 \succeq \nu\varphi_0\},$$

then $\nu_0 \le \nu^* < +\infty$ and $u_0 \succeq \nu^*\varphi_0$. From (3.10) and (3.20) we have

$$\begin{aligned} u_0 &= S_1u_0 + \nu_0\varphi_0 \succeq a_2Lu_0 + \nu_0\varphi_0 \\ &\succeq a_2\nu^*L\varphi_0 + \nu_0\varphi_0 = a_2\nu^*r(L)\varphi_0 + \nu_0\varphi_0. \end{aligned}$$

But $r(L) \ge 1/a_2$, so $u_0 \succeq (\nu^* + \nu_0)\varphi_0$, which is a contradiction to the definition of ν^*.

(vi) From (3.18) and (3.19) it follows that $i(S, K\cap\Omega_R, K) = 0$ and

$$i(S, K\cap(\Omega_R \setminus \overline{\Omega}_r), K) = i(S, K\cap\Omega_R, K) - i(S, K\cap\Omega_r, K) = -1.$$

Hence S has at least one fixed point in K and BVP (1.3) has at least one positive solution by Lemma 2.3. □

Remark 3.9. For the case $\alpha[u] = \beta[u] = 0$ and $a = c = 1, b = d = 0$ considered in [8], we have that $\max\{\int_0^1 \Phi(s)ds, \int_0^1 \Phi_1(s)ds\} = 3/4$ and (3.8) is satisfied when $a_1 + b_1 < 1$ is required in [8]. Moreover, since $k_S(t,s) = k_S(s,t)$ is symmetric and $L = L^*$, we know that the spectral radii $r(L) = r(L^*) = 1/\pi^2$ and $r(L) \ge 1/a_2$, $r(L^*) > 1/a_2$ are satisfied when $a_2 > \pi^2$ is required in [8]. This means that the result of Theorem 3.8 extends Theorem 1.1 of [8].

Remark 3.10. In [17] the following two cones in $C^1[0,1]$ and two linear operators are defined:

$$\begin{aligned} \widetilde{P} &= \{u \in C^1[0,1] : u(t) \ge 0,\ u'(t) \ge 0,\ \forall t \in [0,1]\}, \\ \widetilde{K} &= \Big\{u \in P : u(t) \ge t\|u\|_C,\ \forall t \in [0,1],\ \alpha[u] \ge 0,\ u'(1) = 0\Big\}, \\ (L_iu)(t) &= \int_0^1 k_S(t,s)(a_iu(s) + b_iu'(s))ds \qquad (i = 1,2). \end{aligned}$$

[17, Lemma 2.2] tells us that $L_i : C^1[0,1] \to C^1[0,1]$ are completely continuous operators with $L_i(\widetilde{P}) \subset \widetilde{K}$ $(i = 1,2)$.

Now we compare the conditions of Theorem 3.4 in [17] with ones in Theorem 3.5. In [17, Theorem 3.4] a assumption is described as follows: There exist constants $a_2 > 0$, $b_2 \geq 0$ and $r > 0$ such that

$$f(t, x_1, x_2) \geq a_2 x_1 + b_2 x_2, \tag{3.21}$$

for all $(t, x_1, x_2) \in [0, 1] \times [0, r]^2$, moreover it is assumed that the spectral radius $r(L_2) \geq 1$.

Note that L_2 acts in $C^1[0, 1]$ and $r(L_2)$ is for that space, and it is for the BC $u'(1) = 0$ so in the BC in (1.3) we have $c = 0$ and $\beta[u] \equiv 0$. In this special case $\frac{\partial}{\partial t} k_S(t, s) \geq 0$ so for $u \in \widetilde{K}$ we get $a_2(Lu)(t) \leq (L_2 u)(t)$ and $a_2(Lu)'(t) \leq (L_2 u)'(t)$. Then taking φ to be the eigenfunction in $C[0, 1]$ of L corresponding to the eigenvalue $r(L)$, since $L : C[0, 1] \to C^1[0, 1]$ then $\varphi \in \widetilde{K}$ in this special case, and we get $a_2 r(L)\varphi = a_2 L\varphi \preceq L_2\varphi$ [cone ordering of $\widetilde{K}$] and Lemma 3.4 gives $r(L_2) \geq a_2 r(L)$. If the spectral radius $r(L) \geq 1/a_2$ (see Theorem 3.5), then $r(L_2) \geq a_2 r(L) \geq 1$. However (3.21) implies (3.3).

By comparing the conditions of [17, Theorem 3.4] and Theorem 3.5 in special case, we show that the conditions of one of these two theorems cannot contain the conditions of the other.

Remark 3.11. For the convenience of comparison let $\alpha[u] \equiv 0$ in (1.2), then $\Phi(s) = s$ and $k_S(t, s) = k(t, s) = \min\{t, s\}$ in this case. In [17, Theorem 3.5] two assumptions are described as follow:

I. There exist constants $a_1 > 0$, $b_1 \geq 0$ and $r > 0$ such that $f(t, x_1, x_2) \leq a_1 x_1 + b_1 x_2$ for all $(t, x_1, x_2) \in [0, 1] \times [0, r]^2$, moreover the spectral radius $r(L_1) < 1$.

II. There exist positive constants a_2, c_2 satisfying

$$a_2 \int_0^1 s\Phi(s)ds > 1 \tag{3.22}$$

such that $f(t, x_1, x_2) \geq a_2 x_1 - c_2$ for all $(t, x_1, x_2) \in [0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+$.

If (3.8) holds, for $u \in C^1[0, 1]$ we have $(L_1 u)(t) = \int_0^1 k_S(t, s)(a_1 u(s) + b_1 u'(s))ds$, then

$$|(L_1 u)(t)| \leq \int_0^1 \Phi(s)(a_1|u(s)| + b_1|u'(s)|)ds \leq (a_1 + b_1)\|u\|_{C^1} \int_0^1 \Phi(s)ds,$$

$$|(L_1 u)'(t)| \leq \int_0^1 \Phi_1(s)(a_1|u(s)| + b_1|u'(s)|)ds \leq (a_1 + b_1)\|u\|_{C^1} \int_0^1 \Phi_1(s)ds.$$

Therefore $r(L_1) \leq \|L_1\|_{C^1} < 1$. However, (3.22) (i.e. $a_2 > 3$) implies that $r(L) \geq 1/a_2$, $r(L^*) > 1/a_2$. In fact, for $u_0(t) = t$ we have

$$(Lu_0)(t) = \int_0^1 k(t, s)s ds = \int_0^t s^2 ds + t\int_t^1 s ds = \frac{t}{2} - \frac{t^3}{6} \geq \frac{t}{3},$$

i.e., $Lu_0 \succeq u_0/3$. Consequently, $r(L) \geq 1/3 > 1/a_2$ by Lemma 3.4. Using the same method, we have $r(L^*) > 1/a_2$.

These mean also that the conditions of one of [17, Theorem 3.5] and Theorem 3.8 cannot contain the conditions of the other.

4 Examples

We consider second-order problem under mixed boundary conditions involving multi-point with coefficients of both signs and integral with sign-changing kernel

$$\begin{cases} -u''(t) = f(t, u(t), u'(t)), \quad t \in [0,1], \\ u(0) = \frac{1}{4}u(\frac{1}{4}) - \frac{1}{12}u(\frac{3}{4}), \\ u(1) = \int_0^1 u(t)\Big(\cos \pi t + \frac{2}{\pi}\Big)dt, \end{cases} \tag{4.1}$$

that is, $\alpha[u] = \frac{1}{4}u(\frac{1}{4}) - \frac{1}{12}u(\frac{3}{4})$, $\beta[u] = \int_0^1 u(t)\big(\cos \pi t + \frac{2}{\pi}\big)dt$ and $a = c = 1, b = d = 0$. Hence

$$k(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1. \end{cases}$$

For $s \in [0,1]$,

$$\begin{aligned} 0 \le \mathcal{K}_A(s) &= \frac{1}{4}k\Big(\frac{1}{4}, s\Big) - \frac{1}{12}k\Big(\frac{3}{4}, s\Big) \\ &= \begin{cases} \frac{s}{6}, & 0 \le s \le \frac{1}{4}, \\ \frac{3-4s}{48}, & \frac{1}{4} < s \le \frac{3}{4}, \\ 0, & \frac{3}{4} < s \le 1, \end{cases} \end{aligned}$$

$$\mathcal{K}_B(s) = \int_0^1 k(t,s)\Big(\cos \pi t + \frac{2}{\pi}\Big)dt = \frac{\cos(\pi s) + 2s - 1}{\pi^2} + \frac{s - s^2}{\pi} \ge 0,$$

then (C_2) is satisfied. Since

$$0 \le \alpha\left[\gamma_1\right] = \alpha\left[1-t\right] = \frac{1}{6} < 1, \;\; \alpha\left[\gamma_2\right] = \alpha\left[t\right] = 0,$$

$$\beta\left[\gamma_1\right] = \beta\left[1-t\right] = \frac{1}{\pi} + \frac{2}{\pi^2} \ge 0, \;\; 0 \le \beta\left[\gamma_2\right] = \beta\left[t\right] = \frac{1}{\pi} - \frac{2}{\pi^2} < 1$$

and

$$D = (1 - \alpha\left[\gamma_1\right])\left(1 - \beta\left[\gamma_2\right]\right) - \alpha\left[\gamma_2\right]\beta\left[\gamma_1\right] = \frac{5(\pi^2 - \pi + 2)}{6\pi^2} > 0,$$

(C_3) is also satisfied. Furthermore,

$$\begin{aligned} \Phi(s) &= \frac{1}{D}\left[\frac{\pi^2+4}{\pi^2}\mathcal{K}_A(s) + \frac{5}{6}\mathcal{K}_B(s)\right] + s(1+s), \\ \Phi_1(s) &= \frac{1}{D}\left|\frac{2-\pi}{\pi}\mathcal{K}_A(s) + \frac{5}{6}\mathcal{K}_B(s)\right| + \max\{s, 1-s\}. \end{aligned}$$

Example 4.1. If $f(t, x_1, x_2) = (1+t)x_1^{\frac{1}{3}} + x_2^{\frac{2}{3}}$, take $a_1 = \frac{2}{3}$, $b_1 = \frac{1}{2}$ and thus

$$a_1\int_0^1 \Phi(s)ds + b_1\int_0^1 \Phi_1(s)ds = \frac{2}{3} \times \frac{89\pi^2 + 196}{480\,(\pi^2 - \pi + 2)} + \frac{1}{2} \times \frac{351\pi^2 - 262\pi + 720}{480\,(\pi^2 - \pi + 2)} < 1.$$

So (F_1) holds for c_1 large enough. In addition, take $a_2 = 15$, $r = \frac{1}{15\sqrt{15}}$. From Lemma 2.1 and Lemma 2.2 we have that $c(t) \in C^+[0,1]$ and for $t \in [0,1]$,

$$Lc(t) \ge c(t)\int_0^1 \Phi(s)c(s)ds,$$

then by Lemma 3.4, the spectral radius

$$r(L) \geq \int_0^1 \Phi(s)c(s)ds = \frac{111\pi^2 + 244}{1920\,(\pi^2 - \pi + 2)} > \frac{1}{a_2}. \tag{4.2}$$

Therefore, (F_2) holds since (3.3) can be inferred easily. By Theorem 3.5 we know that BVP (4.1) has at least one positive solution.

Example 4.2. If

$$f(t, x_1, x_2) = \frac{(1+t){x_1}^4 + 2x_2^4}{4(1 + x_1^2 + x_2^2)},$$

take $a_1 = \frac{2}{3}, b_1 = \frac{1}{2}$ and thus

$$(a_1 + b_1) \int_0^1 \Phi(s)ds = \frac{7}{6} \times \frac{89\pi^2 + 196}{480\,(\pi^2 - \pi + 2)} < 1,$$

$$(a_1 + b_1) \int_0^1 \Phi_1(s)ds = \frac{7}{6} \times \frac{351\pi^2 - 262\pi + 720}{480\,(\pi^2 - \pi + 2)} < 1.$$

Therefore, (F_3) holds since (3.9) can be inferred easily for $r = 1$.

Now take $a_2 = 15$. From Lemma 2.1 and Lemma 2.2 we have that $\Phi \in C^+[0,1]$ and for $s \in [0,1]$,

$$(L^*\Phi)(s) \geq \Phi(s) \int_0^1 c(t)\Phi(t)dt,$$

then by Lemma 3.4, the spectral radius

$$r(L^*) \geq \int_0^1 c(t)\Phi(t)dt = \frac{111\pi^2 + 244}{1920\,(\pi^2 - \pi + 2)} > \frac{1}{a_2}.$$

It is easy to see that (3.10) holds for c_2 large enough. Therefore, (F_4) is satisfied if (4.2) is combined with. As for (F_5), one can let $\varphi(\rho) = M^2 + \rho^2 + c_2$. By Theorem 3.8 we know that BVP (4.1) has at least one positive solution.

Remark 4.3. Here we intentionally take $a_1 + b_1 \geq 1$ in order to compare with the conditions in [8].

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