



Oscillatory property of solutions to nonlinear eigenvalue problems

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Abstract. This paper is concerned with the nonlinear eigenvalue problem

$$-u''(t) = \lambda (u(t) + g(u(t))), \quad u(t) > 0, \quad t \in I := (-1, 1), \quad u(\pm 1) = 0,$$

where $g(u) = u^p \sin(u^q)$ ($0 \leq p < 1, 0 < q \leq 1$) and $\lambda > 0$ is a bifurcation parameter. It is known that, for a given $\alpha > 0$, there exists a unique solution pair $(\lambda(\alpha), u_\alpha) \in \mathbb{R}_+ \times C^2(I)$ satisfying $\alpha = \|u_\alpha\|_\infty (= u_\alpha(0))$. We establish the precise asymptotic formula for L^r -norm $\|u_\alpha\|_r$ ($1 \leq r < \infty$) of the solution u_α as $\alpha \rightarrow \infty$ to show the evidence that $u_\alpha(t)$ is oscillatory as $\alpha \rightarrow \infty$. We also obtain the asymptotic formula for λ in L^r -framework, which has different property from that for diffusive logistic equation of population dynamics.

Keywords: global structure of bifurcation curves, oscillatory nonlinear terms.

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1 Introduction

This paper is concerned with the following nonlinear eigenvalue problems

$$-u''(t) = \lambda (u(t) + g(u(t))), \quad t \in I := (-1, 1), \tag{1.1}$$

$$u(t) > 0, \quad t \in I, \tag{1.2}$$

$$u(-1) = u(1) = 0, \tag{1.3}$$

where $g(u)$ is an oscillatory nonlinear term and $\lambda > 0$ is a parameter. We know from [11] that if $u + g(u) > 0$ for $u > 0$, then for any given $\alpha > 0$, there exists a unique classical solution pair (λ, u_α) of (1.1)–(1.3) satisfying $\alpha = \|u_\alpha\|_\infty (= u_\alpha(0))$. Furthermore, λ is parametrized by α as $\lambda = \lambda(\alpha)$ and is continuous in $\alpha > 0$.

In this paper, we study the oscillatory behavior of $u_\alpha(t)$ as $\alpha \rightarrow \infty$ by establishing the asymptotic formula for $\|u_\alpha\|_r$, where $\|u_\alpha\|_r$ ($1 \leq r < \infty$) is L^r -norm of u_α . Furthermore, we establish the asymptotic formula for $\lambda(\beta)$ ($\beta := \|u_\alpha\|_r$) as $\alpha \rightarrow \infty$.

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A lot of investigation on the global behavior of the bifurcation curves have been made for a long time. Indeed, many topics come from mathematical biology, engineering, etc., and have been investigated intensively by many authors. We refer to [1–3, 5, 6, 12] and the references therein. On the other hand, there seems to be few works about the oscillatory properties of bifurcation curves. The important point is that, if bifurcation curves have the oscillatory structures, it is reasonable to suppose that the equations contain some oscillatory nonlinear terms. Therefore, there is a close relationship between oscillatory phenomena of bifurcation curves and inverse bifurcation problems. We refer to [7, 9, 10, 13, 14] and the references therein.

Our equation here is motivated by the work of Cheng [4], which was proposed as a model case of oscillatory bifurcation phenomenon. In [4], the equation (1.1)–(1.3) with $g(u) = \sin \sqrt{u}$ has been treated. It has been shown there that there are arbitrary many solutions near the line $\lambda = \pi^2/4$.

Theorem 1.1 ([4, Theorem 6]). *Let $g(u) = \sin \sqrt{u}$ ($u \geq 0$). Then for any integer $n \geq 1$, there is $\delta > 0$ such that if $\lambda \in (\pi^2/4 - \delta, \pi^2/4 + \delta)$, then (1.1)–(1.3) has at least n distinct solutions.*

Theorem 1.1 suggests that $\lambda(\alpha)$ oscillates and intersects the line $\lambda = \pi^2/4$ infinitely many times for $\alpha \gg 1$ if $g(u) = \sin \sqrt{u}$. Motivated by this result, the following asymptotic formula has been obtained recently in [13].

Theorem 1.2 ([13, Theorem 1.1]). *Let $g(u) = \sin \sqrt{u}$. Then as $\alpha \rightarrow \infty$,*

$$\lambda(\alpha) = \frac{\pi^2}{4} - \pi^{3/2} \alpha^{-5/4} \sin\left(\sqrt{\alpha} - \frac{\pi}{4}\right) + o(\alpha^{-5/4}). \quad (1.4)$$

It should be mentioned that the proof of Theorem 1.2 depends on a very long calculation of the time-map, and it seems that the method in [13] is not applicable to the case where $g(u)$ is a relevant nonlinear term, such as $g(u) = \sin(u^q)$. We remark that the case $g(u) = \sin u$ was considered in [8] and found that stationary phase method is applicable to understand the oscillatory bifurcation.

Motivated by [8] and [13] by using the time-map argument and careful use of the stationary phase method, the precise asymptotic formulas for $\lambda(\alpha)$ with $g(u) = u^p \sin(u^q)$ as $\alpha \rightarrow \infty$ were obtained in [15].

Theorem 1.3 ([15]). *Let $g(u) = u^p \sin(u^q)$, where $0 \leq p < 1$ and $0 < q \leq 1$ are fixed constants. Then as $\alpha \rightarrow \infty$,*

$$\lambda(\alpha) = \frac{\pi^2}{4} - \frac{\pi^{3/2}}{\sqrt{2q}} \alpha^{p-1-(q/2)} \sin\left(\alpha^q - \frac{\pi}{4}\right) + o(\alpha^{p-1-(q/2)}). \quad (1.5)$$

Nevertheless to say, Theorem 1.3 coincides with Theorem 1.2 if $p = 0$ and $q = 1/2$.

On the other hand, as far as the author knows, the precise asymptotic behavior of $u_\alpha(t)$ itself as $\alpha \rightarrow \infty$ is not known yet. It is easy to see from Theorem 1.3 that as $\alpha \rightarrow \infty$,

$$\frac{u_\alpha(t)}{\alpha} \rightarrow \cos\left(\frac{\pi}{2}t\right) \quad (1.6)$$

in $C(\bar{I})$. In other words, the leading term of $u_\alpha(t)$ is equal to $\alpha \cos(\frac{\pi}{2}t)$. Therefore, it seems interesting to clarify how $u_\alpha(t)$ oscillates as $\alpha \rightarrow \infty$. In this paper, since it is difficult to obtain the explicit second term of u_α , we establish the precise asymptotic formula for $\|u_\alpha\|_r$ to show that $u_\alpha(t)$ certainly oscillates as $\alpha \rightarrow \infty$.

Now we state our main result.

Theorem 1.4. Let $1 \leq r < \infty$ be fixed, and $g(u) = u^p \sin(u^q)$, where $0 \leq p < 1$, $0 < q \leq 1$ are fixed constants satisfying $p = 0$, $q = 1$ or $p + 1 \geq 2q$.

(i) The following asymptotic formulas hold as $\alpha \rightarrow \infty$.

$$\|u_\alpha\|_r^r = \left\| \alpha \cos\left(\frac{\pi}{2}t\right) \right\|_r^r + \frac{4}{\pi} \left(A_r \sqrt{\frac{2}{\pi q}} - \sqrt{\frac{\pi}{2q}} \right) \alpha^{p-1-q/2+r} \sin\left(\alpha^q - \frac{\pi}{4}\right) + o(\alpha^{p-1-q/2+r}), \quad (1.7)$$

where

$$A_r := \int_0^1 \frac{s^r}{\sqrt{1-s^2}} ds = \int_0^{\pi/2} \cos^r t dt. \quad (1.8)$$

(ii) Let $\beta_r(\alpha) := \left(\frac{\pi}{4A_r}\right)^{1/r} \|u_\alpha\|_r$. Then as $\alpha \rightarrow \infty$

$$\lambda(\beta_r(\alpha)) = \frac{\pi^2}{4} - \frac{\pi^{3/2}}{\sqrt{2q}} \beta_r(\alpha)^{p-1-(q/2)} \sin\left(\beta_r(\alpha)^q - \frac{\pi}{4}\right) + o(\beta_r(\alpha)^{p-1-(q/2)}). \quad (1.9)$$

Corollary 1.5. Let $v_\alpha(t) := u_\alpha(t) - \alpha \cos\left(\frac{\pi}{2}t\right)$. Assume $p = 0$, $q = 1/2$. Then as $\alpha \rightarrow \infty$,

$$\int_{-1}^1 v_\alpha(t) dt = \frac{4}{\pi} \left(\frac{2}{\sqrt{\pi}} - \sqrt{\pi} \right) \alpha^{-1/4} \sin\left(\sqrt{\alpha} - \frac{\pi}{4}\right) + o(\alpha^{-1/4}). \quad (1.10)$$

Therefore, we see that $u_\alpha(t)$ is eventually oscillatory as $\alpha \rightarrow \infty$. The restriction of p and q in Theorem 1.4 comes from the lack of regularity when we use the stationary phase method in the proof.

It should be mentioned that the asymptotic formula (1.9) for $\alpha \gg 1$ coincides with (1.5) up to the second term. Such phenomenon for λ in L^∞ -framework and L^r -framework does not occur when we consider the diffusive logistic equations of population dynamics (cf. [12]). If we consider the asymptotic behavior of λ in L^r -framework, then usually, its second term is affected by the growth rate of the slope of boundary layer $u'_\alpha(\pm 1)$, and in the case of diffusive logistic equation, it is greater than that of $\|u_\alpha\|_\infty$. On the other hand, in our problem, the the growth rate of $u'_\alpha(\pm 1)$ is the same as that of $\|u_\alpha\|_\infty$. This is the reason why (1.9) is the same as (1.5).

2 Proof of Theorem 1.4

Let $\alpha \gg 1$ in this section. We denote by C the various positive constants independent of α . Let $g(u) = u^p \sin(u^q)$ for $u \geq 0$ and

$$G(u) := \int_0^u g(s) ds. \quad (2.1)$$

If $(u_\alpha, \lambda(\alpha)) \in C^2(\bar{I}) \times \mathbb{R}_+$ satisfies (1.1)–(1.3), then

$$u_\alpha(t) = u_\alpha(-t), \quad 0 \leq t \leq 1, \quad (2.2)$$

$$u_\alpha(0) = \max_{-1 \leq t \leq 1} u_\alpha(t) = \alpha, \quad (2.3)$$

$$u'_\alpha(t) > 0, \quad -1 \leq t < 0. \quad (2.4)$$

We introduce the standard time-map argument (cf. [15]). By (1.1), we obtain

$$(u_\alpha''(t) + \lambda(u_\alpha(t) + g(u_\alpha(t)))) u_\alpha'(t) = 0.$$

This along with (2.3) implies that, by putting $t = 0$, we obtain

$$\frac{1}{2} u_\alpha'(t)^2 + \lambda \left(\frac{1}{2} u_\alpha(t)^2 + G(u_\alpha(t)) \right) = \text{constant} = \lambda \left(\frac{1}{2} \alpha^2 + G(\alpha) \right).$$

This along with (2.4) implies that for $-1 \leq t \leq 0$,

$$u_\alpha'(t) = \sqrt{\lambda} \sqrt{\alpha^2 - u_\alpha(t)^2 + 2(G(\alpha) - G(u_\alpha(t)))}. \quad (2.5)$$

For $0 \leq s \leq 1$, we have

$$\left| \frac{G(\alpha) - G(\alpha s)}{\alpha^2(1-s^2)} \right| = \left| \frac{\int_{\alpha s}^{\alpha} g(t) dt}{\alpha^2(1-s^2)} \right| \leq C \frac{\alpha^{p+1}(1-s^{p+1})}{\alpha^2(1-s^2)} \leq C \alpha^{p-1} \ll 1. \quad (2.6)$$

By (2.2), (2.4), (2.5), (2.6), putting $s := u_\alpha(t)/\alpha$ and Taylor expansion, we obtain

$$\begin{aligned} \|u_\alpha\|_r^r &= 2 \int_{-1}^0 u_\alpha(t)^r dt \\ &= \frac{2}{\sqrt{\lambda}} \int_{-1}^0 \frac{u_\alpha(t)^r u_\alpha'(t)}{\sqrt{\alpha^2 - u_\alpha(t)^2 + 2(G(\alpha) - G(u_\alpha(t)))}} dt \\ &= \frac{2}{\sqrt{\lambda}} \int_0^\alpha \frac{\theta^r}{\sqrt{\alpha^2 - \theta^2 + 2(G(\alpha) - G(\theta))}} d\theta \\ &= \frac{2\alpha^r}{\sqrt{\lambda}} \int_0^1 \frac{s^r}{\sqrt{1-s^2} \sqrt{1 + \frac{2(G(\alpha) - G(\alpha s))}{\alpha^2(1-s^2)}}} ds \\ &= \frac{2\alpha^r}{\sqrt{\lambda}} \int_0^1 \frac{s^r}{\sqrt{1-s^2}} \left\{ 1 - \frac{1}{\alpha^2} (1 + o(1)) \frac{G(\alpha) - G(\alpha s)}{1-s^2} \right\} ds \\ &= \frac{2\alpha^r}{\sqrt{\lambda}} \left\{ A_r - \frac{1}{\alpha^2} (1 + o(1)) \int_0^1 \frac{s^r (G(\alpha) - G(\alpha s))}{(1-s^2)^{3/2}} ds \right\}. \end{aligned} \quad (2.7)$$

We put

$$D(\alpha) := \int_0^1 \frac{s^r (G(\alpha) - G(\alpha s))}{(1-s^2)^{3/2}} ds. \quad (2.8)$$

By combining [8, Lemma 2] and [10, Lemma 2.25], we have following equalities.

Lemma 2.1 ([8, Lemma 2], [10, Lemma 2.25]). *Assume that the function $f(r) \in C^2[0, 1]$, and $h(r) = \cos(\pi r/2)$. Then as $\mu \rightarrow \infty$*

$$\int_0^1 f(r) e^{i\mu h(r)} dr = e^{i(\mu - (\pi/4))} \sqrt{\frac{2}{\pi\mu}} f(0) + O\left(\frac{1}{\mu}\right). \quad (2.9)$$

In particular, by taking the imaginary part of (2.9),

$$\int_0^1 f(r) \sin(\mu h(r)) dr = \sqrt{\frac{2}{\pi\mu}} f(0) \sin\left(\mu - \frac{\pi}{4}\right) + O\left(\frac{1}{\mu}\right). \quad (2.10)$$

Lemma 2.2. Assume that $p = 0, q = 0$ or $p + 1 \geq 2q$. Then as $\alpha \rightarrow \infty$,

$$D(\alpha) = \sqrt{\frac{\pi}{2q}} \alpha^{p+1-q/2} \sin\left(\alpha^q - \frac{\pi}{4}\right) + O(\alpha^{-q}). \quad (2.11)$$

Proof. We put $s = \sin \theta$ in (2.8). Then by integration by parts, we obtain

$$\begin{aligned} D(\alpha) &= \int_0^{\pi/2} \frac{1}{\cos^2 \theta} \{\sin^r \theta (G(\alpha) - G(\alpha \sin \theta))\} d\theta \\ &= \int_0^{\pi/2} (\tan \theta)' \{\sin^r \theta (G(\alpha) - G(\alpha \sin \theta))\} d\theta \\ &= [\tan \theta \{\sin^r \theta (G(\alpha) - G(\alpha \sin \theta))\}]_0^{\pi/2} \\ &\quad - \int_0^{\pi/2} \tan \theta \{r \sin^{r-1} \theta \cos \theta (G(\alpha) - G(\alpha \sin \theta)) - \alpha \cos \theta \sin^r \theta g(\alpha \sin \theta)\} d\theta \\ &:= D_0(\alpha) - rD_1(\alpha) + \alpha D_2(\alpha), \end{aligned} \quad (2.12)$$

where

$$D_0(\alpha) := [\tan \theta \{\sin^r \theta (G(\alpha) - G(\alpha \sin \theta))\}]_0^{\pi/2}, \quad (2.13)$$

$$D_1(\alpha) := \int_0^{\pi/2} \sin^r \theta (G(\alpha) - G(\alpha \sin \theta)) d\theta, \quad (2.14)$$

$$D_2(\alpha) = \int_0^{\pi/2} \sin^{r+1} \theta g(\alpha \sin \theta) d\theta. \quad (2.15)$$

By l'Hôpital's rule, we obtain

$$\lim_{\theta \rightarrow \pi/2} \frac{\int_{\alpha \sin \theta}^{\alpha} y^p \sin(y^q) dy}{\cos \theta} = \lim_{\theta \rightarrow \pi/2} \frac{\alpha \cos \theta (\alpha \sin \theta)^p \sin((\alpha \sin \theta)^q)}{\sin \theta} = 0.$$

This implies that $D_0(\alpha) = 0$.

We put $S(\theta) := \int_0^{\theta} \sin^r x dx$, $\sin^q \theta = \sin x$. By this and (2.14), we obtain

$$\begin{aligned} D_1(\alpha) &= \int_0^{\pi/2} S'(\theta) (G(\alpha) - G(\alpha \sin \theta)) d\theta \\ &= [S(\theta) (G(\alpha) - G(\alpha \sin \theta))]_0^{\pi/2} + \alpha^{p+1} \int_0^{\pi/2} S(\theta) \cos \theta \sin^p \theta \sin(\alpha^q \sin^q \theta) d\theta \\ &= \alpha^{p+1} \int_0^{\pi/2} S(\theta) \cos \theta \sin^p \theta \sin(\alpha^q \sin^q \theta) d\theta \\ &= \frac{\alpha^{p+1}}{q} \int_0^{\pi/2} S(\sin^{-1}(\sin^{1/q} x)) \sin^{(p+1-q)/q} x \cos x \sin(\alpha^q \sin x) dx. \end{aligned} \quad (2.16)$$

By direct calculation, we obtain

$$\frac{d}{dx} S(\sin^{-1}(\sin^{1/q} x)) = \frac{1}{q} \sin^{(r+1-q)/q} x \sqrt{\frac{1 - \sin^2 x}{1 - \sin^{2/q} x}}. \quad (2.17)$$

Since $(r+1-q)/q \geq 1$, we see that $\sin^{(r+1-q)/q} x \in C^1[0, \pi/2]$. Furthermore, by direct calculation, we see that $\sqrt{\frac{1 - \sin^2 x}{1 - \sin^{2/q} x}} \in C^2[0, \pi/2]$. Consequently, $S(\sin^{-1}(\sin^{1/q} x)) \in C^2[0, \pi/2]$. Now we put $x = \frac{\pi}{2}(1-y)$ to obtain

$$\begin{aligned} D_1(\alpha) &= \frac{\pi}{2q} \alpha^{p+1} \int_0^1 S\left(\sin^{-1}\left(\cos^{1/q}\left(\frac{\pi}{2}y\right)\right)\right) \\ &\quad \times \cos^{(p+1-q)/q}\left(\frac{\pi}{2}y\right) \sin\left(\frac{\pi}{2}y\right) \sin\left(\alpha^q \cos\left(\frac{\pi}{2}y\right)\right) dy. \end{aligned} \quad (2.18)$$

Let $f(y) = S(\sin^{-1}(\cos^{1/q}(\frac{\pi}{2}y))) \cos^{(p+1-q)/q}(\frac{\pi}{2}y) \sin(\frac{\pi}{2}y)$ and $\mu = \alpha^q$. If $p = 0$ and $q = 1$, $p + 1 = 2q$ or $p + 1 \geq 3q$, then we directly apply Lemma 2.1 to (2.18) to obtain that $D_1(\alpha) = O(\alpha^{-q})$. If $2q < p + 1 < 3q$, then we are also able to apply Lemma 2.1 and obtain that $D_1(\alpha) = O(\alpha^{-q})$, although $\cos^{(p+1-q)/q}(\frac{\pi}{2}y) \in C^{1+\epsilon}[0, 1]$ with $0 < \epsilon = (p + 1 - 2q)/q < 1$. (cf. Appendix). Finally,

$$\begin{aligned} D_2(\alpha) &= \alpha^p \int_0^{\pi/2} \sin^{r+1+p} \theta \sin(\alpha^q \sin^q \theta) d\theta \\ &= \frac{\pi}{2q} \alpha^p \int_0^1 \cos^{(p+2+r-q)/q} \left(\frac{\pi}{2}y \right) \sqrt{\frac{1 - \cos^2(\frac{\pi}{2}y)}{1 - \cos^{2/q}(\frac{\pi}{2}r)}} \sin \left(\alpha^q \cos \left(\frac{\pi}{2}y \right) \right) dy. \end{aligned} \quad (2.19)$$

Let $f(y) = \cos^{(p+2+r-q)/q}(\frac{\pi}{2}y) \sqrt{\frac{1 - \cos^2(\frac{\pi}{2}y)}{1 - \cos^{2/q}(\frac{\pi}{2}r)}}$ and $\mu = \alpha^q$. Then it is easy to see that $f \in C^2[0, 1]$ and we are able to apply Lemma 2.1 to (2.19) and obtain

$$D_2(\alpha) = \sqrt{\frac{\pi}{2q}} \alpha^{p-q/2} \sin \left(\alpha^q - \frac{\pi}{4} \right) + O(\alpha^{-q}). \quad (2.20)$$

By this, (2.12), (2.18), we obtain that

$$D(\alpha) = \sqrt{\frac{\pi}{2q}} \alpha^{p+1-q/2} \sin \left(\alpha^q - \frac{\pi}{4} \right) + O(\alpha^{p+1-q}). \quad (2.21)$$

Thus the proof is complete. \square

Proof of Theorem 1.4. (i) By (2.7), Theorem 1.3, Lemma 2.2 and Taylor expansion, we obtain

$$\begin{aligned} \|u_\alpha\|_r^r &= \frac{2\alpha^r}{\sqrt{\lambda}} \left\{ A_r - \sqrt{\frac{\pi}{2q}} \alpha^{p-1-q/2} \sin \left(\alpha^q - \frac{\pi}{4} \right) + O(\alpha^{p-1-q}) \right\} \\ &= 2\alpha^r \left\{ A_r - \sqrt{\frac{\pi}{2q}} \alpha^{p-1-q/2} \sin \left(\alpha^q - \frac{\pi}{4} \right) + O(\alpha^{p-1-q}) \right\} \\ &\quad \times \left\{ \frac{\pi^2}{4} - \frac{\pi^{3/2}}{\sqrt{2q}} \alpha^{p-1-q/2} \sin \left(\alpha^q - \frac{\pi}{4} \right) + o(\alpha^{p-1-q/2}) \right\}^{-1/2} \\ &= \frac{4}{\pi} \alpha^r \left\{ A_r - \sqrt{\frac{\pi}{2q}} \alpha^{p-1-q/2} \sin \left(\alpha^q - \frac{\pi}{4} \right) + O(\alpha^{p-1-q}) \right\} \\ &\quad \times \left\{ 1 + \sqrt{\frac{2}{\pi q}} \alpha^{p-1-2/q} \sin \left(\alpha^q - \frac{\pi}{4} \right) + o(\alpha^{p-1-q/2}) \right\} \\ &= \frac{4}{\pi} \alpha^r \left\{ A_r + \left(A_r \sqrt{\frac{2}{q\pi}} - \sqrt{\frac{\pi}{2q}} \right) \alpha^{p-1-q/2} \sin \left(\alpha^q - \frac{\pi}{4} \right) + o(\alpha^{p-1-q/2}) \right\}. \end{aligned} \quad (2.22)$$

This implies Theorem 1.4 (i). The proof of Theorem 1.4 (ii) is a direct consequence of Theorem 1.3 and (2.22). Thus the proof is complete. \square

3 Appendix

The argument in this section, namely, (2.9) in Lemma 2.1 holds for $0 \leq p < 1$ and $0 < q \leq 1$, is taken from [15]. We put $m = 1/q$. For $0 \leq x \leq 1$, let

$$f(x) = f_1(x)f_2(x) := \cos^{(p+2-q)/q} \left(\frac{\pi}{2}x \right) \sqrt{\frac{1 - \cos^2(\frac{\pi}{2}x)}{1 - \cos^{2m}(\frac{\pi}{2}x)}}. \quad (3.1)$$

The essential point of the proof of (2.9) in this case is to show Lemma 2.24 in [10] (see also [10, Lemma 2.25]). Namely, as $\mu \rightarrow \infty$,

$$\Phi(\mu) := \int_0^1 f(x)e^{-i\mu x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{\mu}}e^{-i(\pi/4)}f(0) + O\left(\frac{1}{\mu}\right). \quad (3.2)$$

We put $w(x) = (f(x) - f(0))/x$. Then we have $f(x) = f(0) + xw(x)$. We know from [10, Lemma 2.24] that

$$\int_0^1 e^{-i\mu x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{\mu}}e^{-i\pi/4} + O\left(\frac{1}{\mu}\right). \quad (3.3)$$

Since $f(0) = \sqrt{q}$, by (3.3), we obtain

$$\begin{aligned} \Phi(\mu) &= f(0) \int_0^1 e^{-i\mu x^2} dx + \int_0^1 x e^{-i\mu x^2} w(x) dx \\ &= \frac{1}{2}\sqrt{\frac{\pi}{\mu}}e^{-i\pi/4} \sqrt{q} + O\left(\frac{1}{\mu}\right) + \int_0^1 x e^{-i\mu x^2} w(x) dx. \end{aligned} \quad (3.4)$$

We put

$$\Phi_1(\mu) := \int_0^1 x e^{-i\mu x^2} w(x) dx. \quad (3.5)$$

Now we prove that $w(x) \in C^1[0, 1]$, because if it is proved, then by integration by parts, we easily show that $\Phi_1(\mu) = O(1/\mu)$ and our conclusion (3.2) follows immediately from (3.4) and (3.5). To do this, there are several cases to consider. We note that, by direct calculation, we can show that if $q > 0$, namely, $m > 1$, then $f_2(x) \in C^2[0, 1]$.

Case 1. Assume that $p = 0$ and $q = 1$. Then we have $f(x) = \cos(\frac{\pi}{2}x)$ and $f \in C^2[0, 1]$.

Case 2. Assume that $0 < q < 1$ and $p + 2 \geq 3q$. Then $(p + 2 - q)/q \geq 2$ and $f_1(x) \in C^2[0, 1]$. Consequently, $f \in C^2[0, 1]$ in this case.

Case 3. Assume that $0 < p < 1$ and $q = 1$. Then $f(x) = \cos^{p+1}(\frac{\pi}{2}x) \notin C^2[0, 1]$. However, by direct calculation, we can show that $w(x) = (f(x) - f(0))/x \in C^1[0, 1]$. It is reasonable, because by Taylor expansion, for $0 < x \ll 1$, we have

$$w(x) = -\frac{(p+1)\pi^2}{8}x + O(x^3). \quad (3.6)$$

Case 4. Assume that $0 < q < 1$ and $p + 2 < 3q$. Then $1 < m < 3/2$ and $m(p + 2 - (1/m)) = mp + 2(m - 1) + 1 := \eta + 1 > 1$, $0 < \eta < 1$ and $f_1(x) = \cos^{\eta+1}x$. Since $f_2 \in C^2[0, 1]$, by the consequence of Case 3 above, we find that $w \in C^1[0, 1]$.

Thus the proof is complete. \square

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