



Existence of bounded weak solutions of the Robin problem for a quasi-linear elliptic equation with $p(x)$ -Laplacian.

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Received 3 February 2019, appeared 9 March 2019

Communicated by Maria Alessandra Ragusa

Abstract. We prove the existence of bounded weak solutions to the Robin problem for an elliptic quasi-linear second-order equation with the variable $p(x)$ -Laplacian.

Keywords: $p(x)$ -Laplacian, angular and conical points, the nonlinear Robin boundary condition.

2010 Mathematics Subject Classification: 35J20, 35J25, 35J91.

1 Introduction

The aim of our article is the existence of bounded weak solutions to the Robin problem for an elliptic quasi-linear second-order equation with the variable $p(x)$ -Laplacian in the Lipschitz bounded n -dimensional domain. Boundary value problems for elliptic second order equations with a non-standard growth in function spaces with variable exponents have been an active investigations in recent years. We refer to [16] for an overview. Differential equations with variable exponents-growth conditions arise from the nonlinear elasticity theory, electrorheological fluids, etc. There are many essential differences between the variable exponent problems and the constant exponent problems. In the variable exponent problems, many singular phenomena occurred and many special questions were raised. V. Zhikov [26] has gave examples of the Lavrentiev phenomenon for the variational problems with variable exponent.

Most of the works devoted to the quasi-linear elliptic second-order equations with the variable $p(x)$ -Laplacian refers to the Dirichlet problem in smooth bounded domains (see [16]). In [1, 2, 8, 9, 18] the Robin problem for such equations has been considered, but in smooth domains only. What is more, in these works the lower order terms depend only on (x, u) and do not depend on $|\nabla u|$. A problem with a lower order term that does not depend on $|\nabla u|$ in a non-smooth domain has been recently studied in [7]. Our recent works [4, 5] are devoted to the Robin problem in a cone for such equations with a singular $p(x)$ -power gradient lower order term.

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The Robin boundary conditions appear in the solving Sturm–Liouville problems which are used in many contexts of science and engineering: for example, in electromagnetic problems, in heat transfer problems and for convection-diffusion equations (Fick’s law of diffusion).

The Robin problem plays a major role in the study of reflected shocks in transonic flow. Important applications of this problem is the capillary problem.

We shall investigate the existence of bounded weak solutions of the Robin problem:

$$\begin{cases} -\Delta_{p(x)}u + b(x, u, \nabla u) = f(x), & x \in G, \\ |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \bar{n}} + \frac{\gamma}{|x|^{p(x)-1}} u |u|^{p(x)-2} = 0, & x \in \partial G, \end{cases} \quad (RQL)$$

where $G \in C^{0,1}$ is a bounded domain in \mathbb{R}^n with the boundary ∂G , containing a **conical point in the origin** \mathcal{O} , $\gamma = \text{const} > 0$ and

$$\Delta_{p(x)}u \equiv \text{div} \left(|\nabla u|^{p(x)-2} \nabla u \right). \quad (1.1)$$

We shall work under the following **assumptions**:

- (i) $1 < p_- \leq p(x) \leq p_+ < n$, $\forall x \in \bar{G}$;
- (ii) $p \in C^{0,1}(\bar{G})$;
- (iii) $b : G \times \mathbb{R}^{n+1} \Rightarrow \mathbb{R}$ is a Carathéodory function ($b \in \text{CAR}$) satisfying for almost all $x \in G$ and for all $(u, \xi) \in \mathbb{R}^{n+1}$, the following inequalities:

$$(iii)_a \quad |b(x, u, \xi)| \leq b_1 \left(b_0(x) + |u(x)|^{q_0(x)} + |\xi|^{q_1(x)} \right), \quad \text{where } b_1 = \text{const} \geq 0,$$

$$b_0 \in L_{q^*(x)}(G), \quad \frac{1}{q^*(x)} + \frac{1}{p^*(x)} = 1, \quad p^*(x) = \frac{np(x)}{n-p(x)};$$

$$q_0(x) < p^*(x) - 1, \quad q_1(x) < p(x) - 1 + \frac{p(x)}{n};$$

$$(iii)_b \quad ub(x, u, \xi) \geq |u|^{p(x)} \quad \text{for } |u| > 1;$$

$$(iv) \quad f \in L_{p'(x)}(G), \quad \frac{1}{p'(x)} + \frac{1}{p(x)} = 1.$$

We shall use the space $M(G)$: it is the set of all measurable and bounded almost everywhere in \bar{G} functions $u(x)$ with the norm

$$\|u\| = \text{vrai max}_{x \in \bar{G}} |u(x)| = \inf_{\text{meas } E=0} \left\{ \sup_{x \in \bar{G} \setminus E} |u(x)| \right\}.$$

The convergence in $M(G)$ is the uniform convergence almost everywhere.

Definition 1.1. The function u is called a bounded weak solution of problem (RQL) provided that $u \in V_{p(x)}(G) \equiv W^{1,p(x)}(G) \cap M(G)$ and satisfies the integral identity

$$\begin{aligned} \int_G \left\langle |\nabla u(x)|^{p(x)-2} \nabla u(x) \nabla \eta(x) + b(x, u(x), \nabla u(x)) \eta(x) \right\rangle dx \\ + \gamma \int_{\partial G} |x|^{1-p(x)} u(x) |u(x)|^{p(x)-2} \eta(x) dS = \int_G f(x) \eta(x) dx \quad (II) \end{aligned}$$

for all $\eta \in V_{p(x)}(G)$.

The main result is the following statement.

Theorem 1.2. *Let the assumptions (i)–(iv) be satisfied. Then problem (RQL) has at least one bounded weak solution.*

2 Preliminaries

At first, we recall some theories on variable exponent Sobolev space $W^{1,p(x)}(G)$ (we refer to [3, 10, 14, 17, 19–24]). Let G be an open subset of \mathbb{R}^n and let $p : G \rightarrow \mathbb{R}$ be a measurable function satisfying condition (i). The variable exponent Lebesgue space $L_{p(x)}(G)$ is defined by

$$L_{p(x)}(G) = \left\{ u : G \rightarrow \mathbb{R} \text{ is measurable, } A_{p(\cdot)}(u) := \int_G |u(x)|^{p(x)} dx < \infty \right\}$$

with the norm $\|u\|_{p(x)} = \inf \left\{ \lambda > 0 : A_{p(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1 \right\}$.

Proposition 2.1. *The following inequalities hold (see e.g. [3, (15)], [10, Lemma 3.2.5]):*

$$\min \left(|u|_{p(\cdot)}^{p_-}, |u|_{p(\cdot)}^{p_+} \right) \leq A_{p(\cdot)}(u) \leq \max \left(|u|_{p(\cdot)}^{p_-}, |u|_{p(\cdot)}^{p_+} \right); \quad (2.1)$$

$$\min \left(A_{p(\cdot)}^{\frac{1}{p_-}}(u), A_{p(\cdot)}^{\frac{1}{p_+}}(u) \right) \leq \|u\|_{p(\cdot)} \leq \max \left(A_{p(\cdot)}^{\frac{1}{p_-}}(u), A_{p(\cdot)}^{\frac{1}{p_+}}(u) \right). \quad (2.2)$$

Proposition 2.2 (Generalized Hölder inequality (see e.g. [3, (16)], [10, Lemma 2.6.5]). *The inequality*

$$\int_G |f(x)g(x)| dx \leq 2 \|f\|_{p(x)} \|g\|_{p'(x)}$$

holds for every $f \in L_{p(x)}(G)$ and $g \in L_{p'(x)}(G)$, where $\frac{1}{p'(x)} + \frac{1}{p(x)} = 1$.

The variable exponent Sobolev space $W^{1,p(x)}(G)$ is defined by

$$W^{1,p(x)}(G) = \left\{ u \in L_{p(x)}(G) : |\nabla u| \in L_{p(x)}(G) \right\}$$

with the norm $\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}$.

The spaces $L_{p(x)}(G), W^{1,p(x)}(G)$ are separable, uniformly convex and reflexive Banach spaces (see e.g. [10, Theorems 3.2.7, 3.4.7, 3.4.9, 8.1.6, Corollary 3.4.5], [17, Theorem 2.5, Corollary 2.7, Corollary 2.12, Theorem 3.1], [14, Theorems 1.10 and 2.1]).

We need some properties on spaces $W^{1,p(x)}(G)$.

Proposition 2.3 (See [15, Theorem 1.1], [10, Corollary 8.3.2]). *Let $p \in C^{0,1}(\overline{G})$ and $q : \overline{G} \rightarrow \mathbb{R}$ is measurable. Assume that*

$$p(x) \leq q(x) \leq \frac{np(x)}{n-p(x)} = p^*(x), \quad \text{a.e. } x \in \overline{G}.$$

Then there is a **continuous embedding** $W^{1,p(x)}(G) \rightarrow L_{q(x)}(G)$.

Proposition 2.4 (See [14, Theorem 2.3], [10, Corollary 8.4.4]). *Let $p, q \in C(\overline{G})$ and $p, q \in L_+^\infty(G) = \{t \in L_\infty(G) : \text{ess inf}_G t \geq 1\}$. Assume that*

$$p(x) < n, \quad q(x) < \frac{np(x)}{n-p(x)} = p^*(x), \quad \forall x \in \overline{G}.$$

Then there is a **continuous and compact embedding** $W^{1,p(x)}(G) \rightarrow L_{q(x)}(G)$.

Proposition 2.5 (See [12, Theorem 2.1]). *Let $G \subset \mathbb{R}^n$ be an open bounded domain with Lipschitz boundary. Suppose that $p \in W^{1,s}(G)$ with $1 \leq p_- \leq p_+ < n < s$ and $q : \partial G \rightarrow \mathbb{R}$ is measurable. Let $t(x) = \frac{(n-1)p(x)}{n-p(x)}$ for $x \in \partial G$. Then there is a boundary trace embedding $W^{1,p(x)}(G) \rightarrow L_{q(x)}(\partial G)$ which is **continuous** for $q(x) = t(x)$ and **compact** for $1 \leq q(x) < t(x)$, $x \in \partial G$.*

Theorem 2.6 (Leray–Lions (see [11, Theorem 5.3.23])). *Let X be a reflexive real Banach space. Let $T : X \rightarrow X^*$ be an operator satisfying the conditions*

(i) *T is bounded;*

(ii) *T is demicontinuous;*

(iii) *T is coercive.*

Moreover, let there exist a bounded mapping $\Phi : X \times X \rightarrow X^*$ such that

(iv) $\Phi(u, u) = T(u)$ for every $u \in X$;

(v) for all $u, w, h \in X$ and any sequence $\{t_n\}_{n=1}^{\infty}$ of real numbers such that $t_n \rightarrow 0$ we have

$$\Phi(u + t_n h, w) \rightarrow \Phi(u, w);$$

(vi) for all $u, w \in X$ we have

$$\langle \Phi(u, u) - \Phi(w, w), u - w \rangle \geq 0$$

(the so-called condition of monotonicity in the principal part);

(vii) if $u_n \rightarrow u$ and

$$\lim_{n \rightarrow \infty} \langle \Phi(u_n, u_n) - \Phi(u, u), u_n - u \rangle = 0,$$

then we have

$$\Phi(w, u_n) \rightarrow \Phi(w, u) \quad \forall w \in X;$$

(viii) if $w \in X$, $u_n \rightarrow u$, $\Phi(w, u_n) \rightarrow z$, then

$$\lim_{n \rightarrow \infty} \langle \Phi(w, u_n), u_n \rangle = \langle z, u \rangle.$$

Then the equation $T(u) = f^*$ has at least one solution $u \in X$ for every $f^* \in X^*$.

3 Proof of the existence theorem

Proof. We define nonlinear operators $J, B, \Gamma : V_{p(x)}(G) \rightarrow V_{p(x)}^*(G)$ and an element $f^* \in V_{p(x)}^*(G)$ by

$$\langle J(u), \eta \rangle = \int_G |\nabla u(x)|^{p(x)-2} \nabla u(x) \nabla \eta(x) dx,$$

$$\langle B(u), \eta \rangle = \int_G b(x, u(x), \nabla u(x)) \eta(x) dx,$$

$$\langle \Gamma(u), \eta \rangle = \int_{\partial G} |x|^{1-p(x)} u(x) |u(x)|^{p(x)-2} \eta(x) dS,$$

$$\langle f^*, \eta \rangle = \int_G f(x) \eta(x) dx$$

for all $u, \eta \in V_{p(x)}(G)$. By the definition of $V_{p(x)}(G)$, it is obvious that J and f^* are well defined. Now, we shall verify that B, Γ also are well defined. We denote $M_0 = \|u(x)\|$ (here

norm is in $M(G)$, see [4]). At first, we estimate $|\langle \Gamma(u), \eta \rangle|$. For this it is sufficient to assume $1 < |u(x)| \leq M_0$. Then we get $\|u(x)\|^{p(x)-1} \leq M_0^{p_+-1}$ and therefore

$$\begin{aligned}
 |\langle \Gamma(u), \eta \rangle| &= \left| \int_{\partial G} |x|^{1-p(x)} u(x) |u(x)|^{p(x)-2} \eta(x) dS \right| \\
 &\leq \left| \int_{\{|x|<d\} \cap \partial G} |x|^{1-p(x)} |u(x)|^{p(x)-1} \eta(x) dS \right| \\
 &\quad + \left| \int_{\partial G \setminus \{|x|<d\} \cap \partial G} |x|^{1-p(x)} |u(x)|^{p(x)-1} \eta(x) dS \right| \\
 &\leq \sup_G |\eta(x)| \cdot M_0^{p_+-1} \left\{ \text{meas } \partial\Omega \int_0^d r^{n-p_+-1} dr + d^{1-p_+} \text{meas } \partial G \right\} \\
 &= \sup_G |\eta(x)| \cdot M_0^{p_+-1} \left(\text{meas } \partial\Omega \cdot \frac{d^{n-p_+}}{n-p_+} + d^{1-p_+} \text{meas } \partial G \right),
 \end{aligned} \tag{3.1}$$

where Ω is a domain on the unit sphere with smooth boundary $\partial\Omega$, obtained by the intersection of the cone with the unit sphere; (we can choose d so: $0 < d \ll 1$).

Further, according to (iii)_a, it is clear that

$$\begin{aligned}
 |\langle B(u), \eta \rangle| &\leq \int_G |b(x, u(x), \nabla u(x))| \cdot |\eta(x)| dx \\
 &\leq b_1 \sup_G |\eta(x)| \int_G \left(|b_0(x)| + |u(x)|^{q_0(x)} + |\nabla u(x)|^{q_1(x)} \right) dx.
 \end{aligned}$$

Next, we derive using the Hölder inequality

$$\int_G |b_0(x)| dx \leq 2 |b_0(x)|_{q^*(x)} \cdot |1|_{(q^*(x))'} \leq \text{const}(n, p_+, p_-, \text{meas } G) \cdot |b_0(x)|_{q^*(x)}.$$

Further, it is clear that

$$q_1(x) < p(x), \quad q_0(x) < \frac{n(p_+ - 1) + p_+}{n - p_+}.$$

Therefore

$$\begin{aligned}
 \int_G |u(x)|^{q_0(x)} dx &= \int_{G \cap \{|u(x)| \leq 1\}} |u(x)|^{q_0(x)} dx + \int_{G \cap \{1 < |u(x)| \leq M_0\}} |u(x)|^{q_0(x)} dx \\
 &\leq \left(1 + M_0^{\frac{n(p_+-1)+p_+}{n-p_+}} \right) \cdot \text{meas } G.
 \end{aligned}$$

Again using the Hölder inequality

$$\begin{aligned}
 \int_G |\nabla u(x)|^{q_1(x)} dx &\leq 2 |\nabla u(x)|_{p(x)} \cdot |1|_{\frac{p(x)}{p(x)-q_1(x)}} \\
 &\leq \text{const}(n, p_+, p_-, \text{meas } G) \cdot |\nabla u(x)|_{p(x)}.
 \end{aligned}$$

Thus, it is proved that

$$|\langle B(u), \eta \rangle| < \infty. \tag{3.2}$$

Lemma 3.1. J, B, Γ are bounded and continuous operators.

Proof. The boundedness and the continuity of J is proved in [17] (see Corollary 4.4). The estimates (3.1), (3.2) mean the boundedness of B, Γ .

Now, we consider the so-called Nemytski operator $H(u)(x) = b(x, u, \nabla u) \in CAR(G)$. By assumption (iii)_a, we get that operator $H(u)$ maps the space $V_{p(x)}(G)$ into $L_{q^*(x)}(G)$ and this map is continuous and bounded (see [17, Theorems 4.1–4.3]). Moreover, the operator $B : V_{p(x)}(G) \rightarrow V_{p(x)}^*(G)$ defined above as well as is continuous and bounded (see [17, Corollary 4.4]).

Next, we study the operator Γ . Let $\{u_n\}_{n=1}^\infty \subset V_{p(x)}(G)$ be any sequence and let for $u, u_n \in M(G) : \|u_n - u\| \rightarrow 0$. By the property of $M(G)$, we get that $u_n(x) \rightarrow u(x)$ uniformly almost everywhere in ∂G . Moreover, since $u_n \in W^{1,p(x)}(G)$ and, by the Proposition 2.5, the boundary trace embedding $W^{1,p(x)}(G) \rightarrow L_{q(x)}(\partial G)$, $1 \leq q(x) < \frac{(n-1)p(x)}{n-p(x)}$ is compact, we have $|u_n(x)|^{p(x)-1} \leq M_0^{p_+-1}$, $x \in \partial G$. Therefore, we can pass to the limit under the symbol of integral over ∂G and we obtain

$$\lim_{n \rightarrow \infty} |\langle \Gamma(u_n) - \Gamma(u), \eta \rangle| = 0,$$

i.e. the operator Γ is continuous. \square

Lemma 3.2. *The operator J is monotone on the space $V_{p(x)}(G)$, i.e. for any $u, \eta \in V_{p(x)}(G)$ one has*

$$\langle J(u) - J(\eta), u - \eta \rangle \geq 0. \quad (3.3)$$

Moreover,

$$\langle J(u) - J(\eta), u - \eta \rangle \geq \frac{1}{2^{p_+}} \min \left\{ |\nabla(u - \eta)|_{p(x)}^{p_-}; |\nabla(u - \eta)|_{p(x)}^{p_+} \right\} \quad \text{if } p(x) \geq 2; \quad (3.4)$$

$$\langle J(u) - J(\eta), u - \eta \rangle \geq \frac{(p_- - 1) \min \left\{ |\nabla(u - \eta)|_{p(x)}^{p_-}; |\nabla(u - \eta)|_{p(x)}^{p_+} \right\}}{2 \max \left\{ \left(\int_G (|\nabla u(x)|^{p(x)} + |\nabla \eta(x)|^{p(x)}) dx \right)^{\frac{2-p_-}{2}}; \left(\int_G (|\nabla u(x)|^{p(x)} + |\nabla \eta(x)|^{p(x)}) dx \right)^{\frac{2-p_+}{2}} \right\}}$$

if $1 < p(x) < 2$. (3.5)

Proof. By direct calculation, we have

$$\begin{aligned} & \langle J(u) - J(\eta), u - \eta \rangle \\ &= \int_G \left(|\nabla u(x)|^{p(x)-2} \nabla u(x) - |\nabla \eta(x)|^{p(x)-2} \nabla \eta(x) \right) (\nabla u(x) - \nabla \eta(x)) dx. \end{aligned} \quad (3.6)$$

Now, we use the known inequalities (see e.g. proof of Theorem 3.1 (i) [13], inequality (4.8) [2]) for any $\xi, \eta \in \mathbb{R}^n$:

$$\begin{cases} (|\xi|^{p-2}\xi - |\eta|^{p-2}\eta) (\xi - \eta) \geq (p-1)(|\xi|^p + |\eta|^p)^{\frac{p-2}{p}} |\xi - \eta|^2 & \text{if } 1 < p < 2; \\ (|\xi|^{p-2}\xi - |\eta|^{p-2}\eta) (\xi - \eta) \geq \frac{1}{2^p} |\xi - \eta|^p & \text{if } p \geq 2. \end{cases} \quad (3.7)$$

Therefore, for $p(x) \geq 2$ we obtain

$$\begin{aligned} \langle J(u) - J(\eta), u - \eta \rangle &= \int_G \left(|\nabla u(x)|^{p(x)-2} \nabla u(x) - |\nabla \eta(x)|^{p(x)-2} \nabla \eta(x) \right) (\nabla u(x) - \nabla \eta(x)) dx \\ &\geq \frac{1}{2^{p_+}} \int_G |\nabla u - \nabla \eta|^{p(x)} dx \geq \frac{1}{2^{p_+}} \min \left\{ |\nabla(u - \eta)|_{p(x)}^{p_-}; |\nabla(u - \eta)|_{p(x)}^{p_+} \right\}, \end{aligned}$$

by the inequality (2.1).

Now we consider the case $1 < p(x) < 2$. For this case let $s(x) = \frac{2}{p(x)}$, $s'(x) = \frac{2}{2-p(x)}$. Then we have

$$s_- = \frac{2}{p_+} \leq s(x) \leq \frac{2}{p_-} = s_+; \quad s'_- = \frac{2}{2-p_-} \leq s'(x) \leq \frac{2}{2-p_+} = s'_+; \implies \quad (3.8)$$

$$\frac{1}{s_+} = \frac{p_-}{2}, \quad \frac{1}{s_-} = \frac{p_+}{2}; \quad \frac{1}{s'_+} = \frac{2-p_+}{2}, \quad \frac{1}{s'_-} = \frac{2-p_-}{2}.$$

By (3.7) for $1 < p(x) < 2$, we obtain

$$\begin{aligned} \langle J(u) - J(\eta), u - \eta \rangle &= \int_G \left(|\nabla u(x)|^{p(x)-2} \nabla u(x) - |\nabla \eta(x)|^{p(x)-2} \nabla \eta(x) \right) (\nabla u(x) - \nabla \eta(x)) dx \\ &\geq (p_- - 1) \int_G \frac{|\nabla u(x) - \nabla \eta(x)|^2}{(|\nabla u(x)|^{p(x)} + |\nabla \eta(x)|^{p(x)})^{\frac{2-p(x)}{p(x)}}} dx. \end{aligned}$$

Next we consider the integral

$$\begin{aligned} &\int_G |\nabla u(x) - \nabla \eta(x)|^{p(x)} dx \\ &= \int_G \frac{|\nabla u(x) - \nabla \eta(x)|^{p(x)}}{(|\nabla u(x)|^{p(x)} + |\nabla \eta(x)|^{p(x)})^{\frac{2-p(x)}{2}}} \left(|\nabla u(x)|^{p(x)} + |\nabla \eta(x)|^{p(x)} \right)^{\frac{2-p(x)}{2}} dx \\ &\leq 2 \left| \frac{|\nabla u(x) - \nabla \eta(x)|^{p(x)}}{(|\nabla u(x)|^{p(x)} + |\nabla \eta(x)|^{p(x)})^{\frac{2-p(x)}{2}}} \right|_{s(x)} \cdot \left| \left(|\nabla u(x)|^{p(x)} + |\nabla \eta(x)|^{p(x)} \right)^{\frac{2-p(x)}{2}} \right|_{s'(x)} \\ &\leq 2 \max \left\{ \left(\int_G \frac{|\nabla u(x) - \nabla \eta(x)|^2}{(|\nabla u(x)|^{p(x)} + |\nabla \eta(x)|^{p(x)})^{\frac{2-p(x)}{p(x)}}} dx \right)^{\frac{1}{s_-}} ; \left(\int_G \frac{|\nabla u(x) - \nabla \eta(x)|^2}{(|\nabla u(x)|^{p(x)} + |\nabla \eta(x)|^{p(x)})^{\frac{2-p(x)}{p(x)}}} dx \right)^{\frac{1}{s_+}} \right\} \\ &\quad \times \max \left\{ \left(\int_G (|\nabla u(x)|^{p(x)} + |\nabla \eta(x)|^{p(x)}) dx \right)^{\frac{1}{s'_-}} ; \left(\int_G (|\nabla u(x)|^{p(x)} + |\nabla \eta(x)|^{p(x)}) dx \right)^{\frac{1}{s'_+}} \right\}. \end{aligned}$$

Hence, by (3.8), it follows that

$$\begin{aligned} &\max \left\{ \left(\int_G \frac{|\nabla u(x) - \nabla \eta(x)|^2}{(|\nabla u(x)|^{p(x)} + |\nabla \eta(x)|^{p(x)})^{\frac{2-p(x)}{p(x)}}} dx \right)^{\frac{p_+}{2}} ; \left(\int_G \frac{|\nabla u(x) - \nabla \eta(x)|^2}{(|\nabla u(x)|^{p(x)} + |\nabla \eta(x)|^{p(x)})^{\frac{2-p(x)}{p(x)}}} dx \right)^{\frac{p_-}{2}} \right\} \\ &\geq \frac{\int_G |\nabla u(x) - \nabla \eta(x)|^{p(x)} dx}{2 \max \left\{ \left(\int_G (|\nabla u(x)|^{p(x)} + |\nabla \eta(x)|^{p(x)}) dx \right)^{\frac{2-p_-}{2}} ; \left(\int_G (|\nabla u(x)|^{p(x)} + |\nabla \eta(x)|^{p(x)}) dx \right)^{\frac{2-p_+}{2}} \right\}} \\ &\geq \frac{\min \left\{ |\nabla(u-\eta)|_{p(x)}^{p_-}; |\nabla(u-\eta)|_{p(x)}^{p_+} \right\}}{2 \max \left\{ \left(\int_G (|\nabla u(x)|^{p(x)} + |\nabla \eta(x)|^{p(x)}) dx \right)^{\frac{2-p_-}{2}} ; \left(\int_G (|\nabla u(x)|^{p(x)} + |\nabla \eta(x)|^{p(x)}) dx \right)^{\frac{2-p_+}{2}} \right\}}, \end{aligned}$$

by the inequality (2.1). □

Lemma 3.3. *If $u_n \rightharpoonup u$ in $V_{p(x)}(G)$ (weak convergence) and*

$$|\nabla u_n|_{p(x)} \rightarrow |\nabla u|_{p(x)}, \quad (3.9)$$

then $B(u_n) \rightarrow B(u)$ in $(V_{p(x)}(G))^$.*

Proof. By the assumption (i), the space $W^{1,p(x)}$ is a Banach space, which is separable and reflexive (see Theorem 3.1 [17]). Therefore, from the weak convergence $u_n \rightharpoonup u$ follows the boundedness of the set $\{|u_n|_{1,p(x)}\}$. In addition, by proposition 2.4, there is the operator of the compact embedding that maps this set to the compact set in $L_{p(x)}(G)$, i.e. this operator is compact operator. But then there is subsequence $\{u_{n_k}\} \rightarrow u$ in $L_{p(x)}(G)$. Together with (3.9) we have that $\{u_{n_k}\} \rightarrow u$ in $V_{p(x)}(G)$. By Lemma 3.1 operator B is continuous operator. Therefore, we can perform the passage to the limit under the integral symbol and thus $B(u_n) \rightarrow B(u)$ in $(V_{p(x)}(G))^*$. \square

Set $T := J + B + \Gamma$. Then the operator equation

$$T(u) = f^* \quad (3.10)$$

is equivalent to validity of the integral identity (II). This fact means that the solutions of (3.10) correspond one-to-one to the weak solutions of (RQL). Now, we shall verify the assumptions (i)–(viii) of the Leray–Lions Theorem 2.6 to prove that there is a solution of (3.10).

Assumptions (i)–(ii) follow directly from Lemma 3.1. The coercivity of T (assumption (iii)) is a direct consequence of (iii)_b and $\gamma > 0$:

$$\begin{aligned} \langle T(u), u \rangle &= \int_G \langle |\nabla u(x)|^{p(x)} + b(x, u(x), \nabla u(x)) u(x) \rangle dx + \gamma \int_{\partial G} |x|^{1-p(x)} |u(x)|^{p(x)} dS \\ &\geq \int_G \left(|\nabla u(x)|^{p(x)} + |u(x)|^{p(x)} \right) \quad \text{for } |u| > 1. \end{aligned}$$

Now, we use the inequality (2.1)

$$\int_G \left(|\nabla u(x)|^{p(x)} + |u(x)|^{p(x)} \right) \geq \min \left(|u|_{1,p(x)}^{p_-}, |u|_{1,p(x)}^{p_+} \right).$$

Then we obtain

$$\lim_{|u|_{1,p(x)} \rightarrow \infty} \frac{\langle T(u), u \rangle}{|u|_{1,p(x)}} \geq \lim_{|u|_{1,p(x)} \rightarrow \infty} |u|_{1,p(x)}^{p_{\pm}-1} = \infty.$$

Let us define an operator $\Phi : V_{p(x)}(G) \times V_{p(x)}(G) \rightarrow (V_{p(x)}(G))^*$ by

$$\langle \Phi(u, w), \eta \rangle := \langle J(u), \eta \rangle + \langle B(w), \eta \rangle + \langle \Gamma(w), \eta \rangle \quad \text{for all } u, w, \eta \in V_{p(x)}(G).$$

The assumption (iv) is obvious.

Next, let $u, w, h \in V_{p(x)}(G)$ and $t_n \rightarrow 0$. Then, by continuity of operator J , we have

$$\Phi(u + t_n h, w) = J(u + t_n h) + B(w) + \Gamma(w) \rightarrow J(u) + B(w) + \Gamma(w) = \Phi(u, w).$$

Thus, the assumption (v) is satisfied.

The assumption (vi) satisfies by Lemma 3.2, because of

$$\Phi(u, u) - \Phi(w, u) = J(u) - J(w).$$

Now, we shall verify the assumption (vii). Let $u_n \rightharpoonup u$ in $V_{p(x)}(G)$ and

$$\begin{aligned} \lim_{n \rightarrow \infty} \langle \Phi(u_n, u_n) - \Phi(u, u_n), u_n - u \rangle &= 0 \implies \\ \lim_{n \rightarrow \infty} \langle J(u_n) - J(u), u_n - u \rangle &= 0. \end{aligned} \quad (3.11)$$

From (3.11) and (3.4)–(3.5) it follows that $|\nabla u_n|_{p(x)} \rightarrow |\nabla u|_{p(x)}$, i.e. (3.9) is satisfied. Moreover, $u_n \rightarrow u$ in M . The last facts and that $W^{1,p(x)}$ is a uniformly convex Banach space together with the weak convergence imply

$$u_n \rightarrow u \quad \text{in } V_{p(x)}(G)$$

(see e.g. Proposition 2.1.22 (iv)). Further, by Lemmas 3.1 and 3.3, we have

$$\Phi(w, u_n) = J(w) + B(u_n) + \Gamma(u_n) \rightarrow J(w) + B(u) + \Gamma(u) = \Phi(w, u) \quad \text{for arbitrary } w \in V_{p(x)}(G).$$

Finally, we verify the assumption (viii). Let $w \in V_{p(x)}(G)$, $u_n \rightarrow u$ in $V_{p(x)}(G)$. Then, in virtue of $B(u_n) \rightarrow B(u)$ and $\Gamma(u_n) \rightarrow \Gamma(u)$ in $(V_{p(x)}(G))^*$ (see Lemmas 3.1, 3.3), we obtain

$$\begin{aligned} \langle \Phi(w, u_n), u_n \rangle &= \langle J(w) + B(u_n) + \Gamma(u_n), u_n \rangle \\ &\rightarrow \langle J(w), u \rangle + \langle B(u), u \rangle + \langle \Gamma(u), u \rangle \\ &= \langle \Phi(w, u), u \rangle. \end{aligned}$$

Hence we have: $u_n \rightarrow u$ in $V_{p(x)}(G)$ implies that $\Phi(w, u_n) \rightarrow J(w) + B(u) + \Gamma(u)$.

Thus, all assumptions of the Leray–Lions Theorem are satisfied. \square

References

- [1] M. ALLAoui, Robin problems involving the $p(x)$ -Laplacian, *Appl. Math. Comput.* **332**(2018), 457–468. <https://doi.org/10.1016/j.amc.2018.03.052>; MR3788704
- [2] S. ANTONtSEV, L. CONSIGLIERI, Elliptic boundary value problems with nonstandard growth conditions, *Nonlinear Anal.* **71**(2009), 891–902. <https://doi.org/10.1016/j.na.2008.10.109>; MR2527510
- [3] S. ANTONtSEV, S. SHMAREV, Elliptic equations and systems with nonstandard growth conditions: existence, uniqueness and localization properties of solutions, *Nonlinear Anal.* **65**(2006), 728–761. <https://doi.org/10.1016/j.na.2005.09.035>
- [4] M. BORSUK, L_∞ -estimate for the Robin problem of a singular variable p -Laplacian equation in a conical domain, *Electron. J. Differential Equations* **2018**, No. 49, 1–9. MR3781164
- [5] M. BORSUK, The Robin problem for singular $p(x)$ -Laplacian equation in a cone, *Electron. J. Qual. Theory Differ. Equ.* **2018**, No. 93, 1–14. <https://doi.org/10.14232/ejqtde.2018.1.93>; MR3884524
- [6] M. BORSUK, S. JANKOWSKI, The Robin problem for singular p -Laplacian equation in a cone, *Complex Var. Elliptic Equ.* **63**(2018), No. 3, 333–345. <https://doi.org/10.1080/17476933.2017.1307837>; MR3764765
- [7] M. M. BOUREANU, A. VÉLEZ-SANTIAGO, Fine regularity for elliptic and parabolic anisotropic Robin problems with variable exponents, *J. Differential Equations*, in press, 2019. <https://doi.org/10.1016/j.jde.2018.12.026>
- [8] SH.-G. DENG, Positive solutions for Robin problem involving the $p(x)$ -Laplacian, *J. Math. Anal. Appl.* **360**(2009), 548–560. <https://doi.org/10.1016/j.jmaa.2009.06.032>; MR2561253

- [9] SH.-G. DENG, Q. WANG, SH. CHENG, On the $p(x)$ -Laplacian Robin eigenvalue problem, *Appl. Math. Comput.* **217**(2011), 5643–5649. <https://doi.org/10.1016/j.amc.2010.12.042>; MR2770184
- [10] L. DIENING, P. HARJULEHTO, P. HÄSTÖ, M. RŮŽIČKA, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, Vol. 2017, Springer, Heidelberg, 2011. <https://doi.org/10.1007/978-3-642-18363-8>; MR2790542
- [11] P. DRÁBEK, J. MILOTA, *Methods of nonlinear analysis. Applications to differential equations*, Birkhäuser, Basel, 2007. <https://doi.org/10.1007/978-3-0348-0387-8>; MR2323436
- [12] X. FAN, Boundary trace embedding theorems for variable exponent Sobolev spaces, *J. Math. Anal. Appl.* **339**(2008), 1395–1412. <https://doi.org/10.1016/j.jmaa.2007.08.003>
- [13] X.-L. FAN, Q.-H. ZHANG, Existence of solutions for $p(x)$ -Laplacian Dirichlet problem, *Nonlinear Anal.* **52**(2003), 1843–1852. [https://doi.org/10.1016/S0362-546X\(02\)00150-5](https://doi.org/10.1016/S0362-546X(02)00150-5); MR1954585
- [14] X. FAN, D. ZHAO, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, *J. Math. Anal. Appl.* **263**(2001), 424–446. <https://doi.org/10.1006/j.jmaa.2000.7617>; MR1866056
- [15] X. FAN, J. SHEN, D. ZHAO, Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$, *J. Math. Anal. Appl.* **262**(2001), 749–760. <https://doi.org/10.1006/j.jmaa.2001.7618>; MR1859337
- [16] P. HARJULEHTO, P. HÄSTÖ, ÚT LÊ, M. NUORITO, Overview of differential equations with non-standard growth, *Nonlinear Anal.* **72**(2010), 4551–4574. <https://doi.org/10.1016/j.na.2010.02.033>; MR2639204
- [17] O. KOVÁČIK, J. RÁKOSNÍK, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, *Czech. Math. J.* **41**(1991), No. 4, 592–618. MR1134951
- [18] M. MIHĂILESCU, C. VARGA, Multiplicity results for some elliptic problems with nonlinear boundary conditions involving variable exponents, *Comput. Math. Appl.* **62**(2011), 3464–3471. <https://doi.org/10.1016/j.camwa.2011.08.062>; MR2844897
- [19] M. A. RAGUSA, A. TACHIKAWA, H. TAKABAYASHI, Partial regularity of $p(x)$ -harmonic maps, *Trans. Amer. Math. Soc.* **365**(2013), No. 6, 3329–3353. <https://doi.org/10.1090/S0002-9947-2012-05780-1>; MR3034468
- [20] M. A. RAGUSA, A. TACHIKAWA, Boundary regularity of minimizers of $p(x)$ -energy functionals, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **33**(2016), No. 2, 451–476. <https://doi.org/10.1016/j.anihpc.2014.11.003>; MR3465382
- [21] M. A. RAGUSA, A. TACHIKAWA, New challenges on the regularity of minimizers of functionals, *J. Convex Anal.* **25**(2018), No. 2, 675–690. MR3783779
- [22] M. A. RAGUSA, A. TACHIKAWA, On the regularity in some variational problems, *AIP Conference Proceedings* **1798**(2017), 020128, 6 pp. <https://doi.org/10.1063/1.4972720>

- [23] A. SCAPELLATO, Homogeneous Herz spaces with variable exponents and regularity results, *Electron. J. Qual. Theory Differ. Equ.* **2018**, No. 82, 1–11. <https://doi.org/10.14232/ejqtde.2018.1.82>; MR3863876
- [24] A. SCAPELLATO, Regularity of solutions to elliptic equations on Herz spaces with variable exponents, *Bound. Value Probl.* **2019**, Paper No. 2, 9 pp. <https://doi.org/10.1186/s13661-018-1116-6>; MR3895830
- [25] V. V. ZHIKOV, On Lavrentiev's phenomenon, *Russian J. Math. Phys.* **13**(1994), No. 2, 249–269. MR1350506
- [26] V. V. ZHIKOV, On some variational problems, *Russian J. Math. Phys.* **5**(1997), No. 1, 105–116. MR1486765