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Infinitely many solutions for perturbed Kirchhoff type problems

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Abstract. In this paper, we discuss a superlinear Kirchhoff type problem where the non-linearity is not necessarily odd. By using variational and perturbative methods, we prove the existence of infinitely many solutions in the non-symmetric case.

Keywords: Kirchhoff type problem, infinitely many solutions, perturbative method.

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1 Introduction

In this paper, we are concerned with the problem

$$\begin{aligned} -\left(a+b\int_\Omega |\nabla u|^2 dx\right)\Delta u &= |u|^{p-1}u + f(x,u) \quad \text{in}\,\Omega, \\ u &= 0 \quad \text{on}\ \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ with smooth boundary, $a \geq 0, b > 0$. The function f is a perturbative term satisfying the following condition

(C) There are two nonnegative functions $\alpha \in L^\mu(\Omega), \beta \in L^\infty(\Omega)$ and constant $\gamma \geq 1$ such that

$$|f(x,u)| \leq \alpha(x) + \beta(x)|u|^{\gamma-1},$$

where $\mu > 2N/(N+2)$.

When $a \neq 0, b = 0$ in problem (1.1), it reduces to the classic semilinear elliptic problem and the existence of solutions for elliptic equations with zero Dirichlet boundary conditions has been widely studied by variational methods, for example, see [2,16,18] . Further suppose that $f \equiv 0$, and $1 < p < 2^* - 1$, here $2^* = 2N/(N-2)$ for $N \geq 3$, $2^* = +\infty$ if $N = 1,2$, it is well known that (1.1) has infinitely many distinct solutions $\{u_k\}$ associated with critical values $I(u_k)$ of the functional

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{p+1}\int_\Omega |u|^{p+1} dx$$

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such that $I(u_k) \to +\infty$ as $k \to \infty$. If $f \not\equiv 0$ and $f(x,u)$ is not odd in u, symmetry of the functional corresponding the equation is lost and the Symmetric Mountain Pass Theorem cannot be applied. A long standing question is whether the symmetry of the functional is necessary for the existence of infinitely many critical points. Since the 1980s, some mathematicians had been working on this problem for elliptic equations, see Bahri and Berestycki [4],Bahri and Lions [5], Bolle [6], Candela, Salvatore and Squassina [9], Rabinowitz [15], Struwe [17], Tanaka [19] and so on. Researchers gave various conditions guaranteeing the existence infinitely many solution when the symmetry of the problem is broken. Here we list a classical result about the following problem

$$\begin{aligned} -\Delta u &= |u|^{p-1}u + g(x) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

where $g \in L^2(\Omega)$.

Theorem 1.1. *If $1 < p < N/(N-2)$, then (1.2) has infinitely many solutions.*

Theorem 1.1 is a particular case of a more general one due to Bahri and Lions [5]. It is not known whether the bound $N/(N-2)$ is optimal. If $\Omega = B_R$ is the open ball of radius $R > 0$ and center 0 in $\mathbb{R}^N (N \geq 3)$, and g is a radial function, (1.2) has infinitely many radial solutions for any $1 < p < (N+2)/(N-2)$, see Theorem 1.2 of [8]. Whether the conclusion of Theorem 1.1 would still hold for all p up to the Sobolev exponent $2^* - 1 = (N+2)/(N-2)$ for the general function g when $N \geq 3$ is a open problem.

When $b \neq 0$, (1.1) is called nonlocal because of the presence of the term $(\int_\Omega |\nabla u|^2 dx)\Delta u$, which implies that the equation in (1.1) is no longer a point-wise identity. Kirchhoff type problem received great attention only after Lions [13] proposed an abstract functional analysis framework for the problem, see [1,3,11,14]. The nonlocal perturbation causes that the energy functional corresponding the equation has properties different than the case $b = 0$. There are some works showing that sometime the appearance of the term $(\int_\Omega |\nabla u|^2 dx)\Delta u$ is good in some sense, see [20].

The main purpose of the present paper is to show that (1.1) has infinitely many solutions when the exponent p is close to the Sobolev exponent if $N = 2, 3$. Using Morse indices, we obtain the growth estimate of critical level for the functional without perturbative term. Combining with the Bolle's Perturbation arguments, we prove the following result.

Theorem 1.2. *Let (C) hold. Then (1.1) has infinitely many solutions if one of the following conditions is satisfies*

(i) $N = 3$, $3 < p < 5$ *and* $1 \leq \gamma < (p+13)/4$,

(ii) $N = 2$, $3 < p < +\infty$ *and* $1 \leq \gamma < (p+5)/2$.

Corollary 1.3. *Assume that $N = 3$, $3 < p < 5$ and $g \in L^2(\Omega)$, then the equation*

$$\begin{aligned} -\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u &= |u|^{p-1}u + g(x) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1.3}$$

has infinitely many solutions.

This paper is organized as follows. In Section 2, we present Bolle's Perturbation method which is useful for proving multiplicity results for perturbed problems. In section 3 we apply this result to prove Theorem 1.2. Throughout the paper, the symbols $C_1, C_2, \ldots$ denote various positive constants whose exact values are not essential to the analysis of the problem.

2 Bolle's perturbation arguments

In order to apply the method introduced by Bolle [6] for dealing with problems with broken symmetry, we recall the main theorem as stated in [7]. The idea is to consider a continuous path of functional starting from a symmetric functional and to prove a preservation result for min-max critical levels in order to obtain critical points for a nonsymmetric functional.

Let H be a Hilbert space equipped with the norm $\|\cdot\|$. Assume that $H = H^- \oplus H^+$, where $\dim(H^-) < +\infty$, and let $(e_k)_{k\geq 1}$ be an orthonormal base of H^+. Consider

$$H_0 = H^-, \qquad H_{k+1} = H_k \oplus \mathbb{R}e_{k+1}, \qquad k \in \mathbb{N},$$

so $(H_k)_k$ is an increasing sequence of finite dimensional subspaces of H.

Let $J : [0,1] \times H \to \mathbb{R}$ be a C^1-functional and, taken any $\theta \in [0,1]$, set $J_\theta = J(\theta, \cdot) : H \to \mathbb{R}$ and $J'_\theta(v) = \partial J(\theta, v)/\partial v$. Assume that

(H1) J_θ satisfies the Palais–Smale condition, which means that every sequence $\{(\theta_n, u_n)\} \subset [0,1] \times H$ such that

$$J_{\theta_n}(u_n) \text{ is bounded and } \lim_{n\to\infty} I'_{\theta_n}(u_n) = 0$$

converges up to subsequences.

(H2) For all $d > 0$ there is a constant $C(d) > 0$ such that if $\forall(\theta, u) \in [0,1] \times H$, then

$$|J_\theta(u)| \leq d \Rightarrow \left|\frac{\partial J_\theta(u)}{\partial \theta}\right| \leq C(d)(\|J'_\theta(u)\| + 1)(\|u\| + 1).$$

(H3) There exist two continuous maps $\eta_1, \eta_2 : [0,1] \times \mathbb{R} \to \mathbb{R}$ Lipschitz continuous with respect to the second variable, such that $\eta_1(\theta, \cdot) \leq \eta_2(\theta, \cdot)$ and if $(\theta, u) \in [0,1] \times H$, then

$$J'_\theta(u) = 0 \Rightarrow \eta_1(\theta, J_\theta(u)) \leq \frac{\partial}{\partial \theta} J_\theta(u) \leq \eta_2(\theta, J_\theta(u)).$$

(H4) J_0 is even and for each finite dimensional subspace W of H it results

$$\lim_{u\in W: \|u\|\to\infty} \sup_{\theta\in[0,1]} J_\theta(u) = -\infty.$$

Define

$$\Gamma = \{\tau \in C(H,H) : \tau \text{ odd and there exists } R > 0 \text{ s.t. } \tau(u) = u \text{ if } \|u\| \geq R\},$$
$$c_k = \inf_{\tau\in\Gamma} \sup_{u\in H_k} J_0(\tau(u)).$$

For $i \in \{1,2\}$, let $\psi_i : [0,1] \times \mathbb{R} \to \mathbb{R}$ be the flow associated to η_i, i.e. the solution of problem

$$\begin{aligned} &\frac{\partial \psi_i}{\partial \theta}(\theta, s) = \eta_i(\theta, \psi_i(\theta, s)), \\ &\psi_i(0, s) = s. \end{aligned} \tag{2.1}$$

Note that $\psi_i(\theta, \cdot)$ is continuous, non-decreasing on $\mathbb{R}$ and $\psi_1(\theta, \cdot) \leq \psi_2(\theta, \cdot)$. Set

$$\bar\eta_1(s) = \sup_{\theta\in[0,1]} |\eta_1(\theta, s)|, \qquad \bar\eta_2(s) = \sup_{\theta\in[0,1]} |\eta_2(\theta, s)|.$$

The following abstract result is due to Bolle, Ghoussoub and Tehrani (for more details, see Theorem 2.2 in [7]).

Theorem 2.1. *Assume J_θ satisfies hypothesis $(H1)$–$(H4)$, then there exists $C > 0$ such that if $k \in \mathbb{N}$ then*

(i) either J_1 has a critical level $\bar{c}_k$ with $\psi_2(1, c_k) < \psi_1(1, c_{k+1}) \le \bar{c}_k$,

(ii) or $c_{k+1} - c_k \le C(\bar{\eta}_1(c_{k+1}) + \bar{\eta}_2(c_k) + 1)$.

Remark 2.2. In case (i), $\bar{c}_k \ge \psi_2(1, c_k) \ge c_k = \psi_2(0, c_k)$ if $\eta_2 \ge 0$ in $[0,1] \times \mathbb{R}$.

3 Proof of main result

Consider the Banach space $H = H_0^1(\Omega)$ with the norm $\|u\|^2 = \int_\Omega |\nabla u|^2 dx$ and define the functional $J : [0,1] \times H \to \mathbb{R}$ by

$$J(\theta, u) := J_\theta(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{p+1}\int_\Omega |u|^{p+1}dx - \theta \int_\Omega F(x,u)dx,$$

where $F(x,u) = \int_0^u f(x,r)dr$. It is clear that J_0 is an even functional and the solutions of problem (1.1) are the critical points of J_1. It is also easily shown that in any finite dimensional subspace of H, $\sup_{\theta \in [0,1]} J_\theta(u) \to -\infty$ as $\|u\| \to \infty$. Thus $(H4)$ is satisfied.

Using Young's inequality, we have

$$\int_\Omega \alpha(x)|u|dx \le \frac{1}{(p+1)^*}\left(\frac{1}{\varepsilon}\right)^{(p+1)^*} \int_\Omega \alpha(x)^{(p+1)^*}(x)dx + \frac{\varepsilon^{p+1}}{p+1}\int_\Omega |u|^{p+1}dx,$$
$$\int_\Omega \beta(x)|u|^\gamma dx \le \frac{1}{\left(\frac{p+1}{\gamma}\right)^*}\left(\frac{1}{\varepsilon}\right)^{(\frac{p+1}{\gamma})^*} \int_\Omega \beta(x)^{(\frac{p+1}{\gamma})^*}dx + \frac{\gamma\varepsilon^{\frac{p+1}{\gamma}}}{p+1}\int_\Omega |u|^{p+1}dx,$$

where $\varepsilon > 0$ and A^* is conjugate of A, which follow that for $\forall \varepsilon > 0$, there is $C(\varepsilon) > 0$ such that for all $u \in H$,

$$\int_\Omega |f(x,u)u|dx \le \varepsilon \int_\Omega |u|^{p+1}dx + C(\varepsilon), \tag{3.1}$$
$$\int_\Omega |F(x,u)|dx \le \varepsilon \int_\Omega |u|^{p+1}dx + C(\varepsilon). \tag{3.2}$$

Lemma 3.1. *The functional J_θ satisfies PS condition.*

Proof. Assume that there exist $(\theta_n, u_n) \in [0,1] \times H$ and $C > 0$ such that

$$|J_{\theta_n}(u_n)| < C, \qquad \|I'_{\theta_n}(u_n)\| \to 0.$$

Then,

$$a\|u_n\|^2 + b\|u_n\|^4 - \int_\Omega |u_n|^{p+1}dx - \theta_n \int_\Omega f(x,u_n)u_n dx = o(1)\|u_n\|.$$

For sufficiently large n,

$$a\|u_n\|^2 + b\|u_n\|^4 + \|u_n\| \ge \int_\Omega |u_n|^{p+1}dx - \int_\Omega |f(x,u_n)u_n|dx. \tag{3.3}$$

Using (C) and (3.1), we have

$$\int_\Omega |u_n|^{p+1}dx \le 2a\|u_n\|^2 + 2b\|u_n\|^4 + 2\|u_n\| + C_1 \le C_2\|u_n\|^4 + C_3 \tag{3.4}$$

for sufficiently large n and some $C_i > 0$, $i = 1, 2, 3$. Hence, for sufficiently large n

$$\begin{aligned}
C + 1 + \|u_n\| &\geq J_{\theta_n}(u_n) - \frac{1}{p+1} J'_{\theta_n}(u_n)u_n \\
&= \frac{(p-1)a}{2(p+1)}\|u_n\|^2 + \frac{(p-3)b}{4(p+1)}\|u_n\|^4 - \theta_n \int_\Omega \left(F(x,u_n) - \frac{1}{p+1} f(x,u_n)u_n \right) dx \\
&\geq \frac{(p-3)b}{4(p+1)}\|u_n\|^4 - \int_\Omega (|F(x,u_n)| + |f(x,u_n)u_n|)dx \\
&\geq \frac{(p-3)b}{4(p+1)}\|u_n\|^4 - 2\varepsilon \int_\Omega |u_n|^{p+1}dx - 2C(\varepsilon) \\
&\geq \left(\frac{(p-3)b}{4(p+1)} - 2\varepsilon C_2 \right) \|u_n\|^4 - 2\varepsilon C_3 - 2C(\varepsilon),
\end{aligned}$$

which implies that $\|u_n\|$ is bounded in H. There exist $\{n_j\} \subset \{n\}$, $u \in H$ and $\theta \in [0,1]$ such that

$$\begin{aligned}
u_j &\rightharpoonup u && \text{in } H, \\
u_j &\to u && \text{in } L^s(\Omega),\ s \in [1, 2^*), \\
u_j &\to u && \text{a.e. in } \Omega, \\
\theta_j &\to \theta && \text{in } \mathbb{R},
\end{aligned}$$

where $u_j := u_{n_j}, \theta_j := \theta_{n_j}$. From (C), we have

$$\begin{aligned}
\left| \int_\Omega (|u_j|^{p-1}u_j - |u|^{p-1}u)(u_j - u)dx \right| &\leq \left(\int_\Omega (|u_j|^p + |u|^p)^{\frac{2^*-\varsigma}{2^*-\varsigma-1}} dx \right)^{\frac{2^*-\varsigma-1}{2^*-\varsigma}} \left(\int_\Omega |u_j - u|^{2^*-\varsigma} dx \right)^{\frac{1}{2^*-\varsigma}} \\
&\to 0,
\end{aligned}$$

$$\int_\Omega f(x,u_j)(u_j - u)dx \to 0, \qquad \int_\Omega f(x,u)(u_j - u)dx \to 0$$

as $n \to \infty$, where $\varsigma = 5 - p$ if $N = 3$, $2^* - \varsigma = 2$ if $N = 2$. Hence,

$$\begin{aligned}
(a + b\|u_j\|^2)\|u_j - u\|^2 = (J'_{\theta_j}(u_j) - J'_\theta(u))(u_j - u) &+ \int_\Omega (|u_j|^{p-1}u_j - |u|^{p-1}u)(u_j - u)dx \\
&+ \theta_j \int_\Omega f(x,u_j)(u_j - u)dx + \theta \int_\Omega f(x,u)(u_j - u)dx \to 0,
\end{aligned}$$

which follows that $\|u_j - u\| \to 0$ or $\|u_j\| \to 0$ (when $a = 0$). If $\|u_j\| \to 0$, then $u = 0$, that is, $\|u_j - u\| \to 0$. Hence, $(\theta_j, u_j) \to (\theta, u)$ in $[0,1] \times H$. The proof is completed. □

Lemma 3.2. *For all $d > 0$ there is a constant $C(d) > 0$ such that*

$$\left| \frac{\partial J_\theta(u)}{\partial \theta} \right| \leq C(d)(\|J'_\theta(u)\| + 1)(\|u\| + 1)$$

if $|J_\theta(u)| \leq d$.

Proof. Since $J'_\theta(u)u = a\|u\|^2 + b\|u\|^4 - \int_\Omega |u|^{p+1}dx - \theta \int_\Omega f(x,u)udx$, we have

$$a\|u\|^2 + b\|u\|^4 + |J'_\theta(u)u| \geq \int_\Omega |u|^{p+1}dx - \int_\Omega |f(x,u)u|dx.$$

Using (3.1), we obtain that

$$\int_\Omega |u|^{p+1}dx \leq 2a\|u\|^2 + 2b\|u\|^4 + 2|J'_\theta(u)u| + C_4 \leq C_5\|u\|^4 + C_6 + 2|J'_\theta(u)u|$$

for some $C_{i+3} > 0, i = 1,2,3$. From $|J_\theta(u)| \le d$, we have

$$\begin{aligned} d + |J_\theta'(u)u| &\ge J_\theta(u) - \frac{1}{p+1}J_\theta'(u)u \\ &\ge \frac{(p-1)a}{2(p+1)}\|u\|^2 + \frac{(p-3)b}{4(p+1)}\|u\|^4 - \int_\Omega (|F(x,u)| + |f(x,u)u|)dx \\ &\ge \frac{(p-3)b}{4(p+1)}\|u\|^4 - 2\varepsilon\int_\Omega |u|^{p+1}dx - 2C(\varepsilon) \\ &\ge \left(\frac{(p-3)b}{4(p+1)} - 2\varepsilon C_5\right)\|u\|^4 - 2\varepsilon C_6 - 4\varepsilon|J_\theta'(u)u| - 2C(\varepsilon). \end{aligned}$$

There exist $C_1(d) > 0, C_2(d) > 0$ such that $\|u\|^4 \le C_1(d)|J_\theta'(u)u| + C_2(d)$. Hence,

$$\int_\Omega |u|^{p+1}dx \le C_3(d)|J_\theta'(u)u| + C_4(d)$$

for some $C_3(d) > 0$ and $C_4(d) > 0$. And

$$\begin{aligned} \left|\frac{\partial J_\theta(u)}{\partial\theta}\right| &\le \int_\Omega |F(x,u)|dx \le \varepsilon\int_\Omega |u|^{p+1}dx + C(\varepsilon) \\ &\le C_3(d)\varepsilon|J_\theta'(u)u| + C_4(d)\varepsilon + C(\varepsilon) \\ &\le C(d)(\|J_\theta'(u)\| + 1)(\|u\| + 1), \end{aligned}$$

where $C(d) = C_3(d)\varepsilon + C_4(d)\varepsilon + C(\varepsilon)$. The proof is completed. □

Lemma 3.3. *If $J_\theta'(u) = 0$, there exists a constant $C^* > 0$ such that*

$$\left|\frac{\partial J_\theta(u)}{\partial\theta}\right| \le C^*\left(J_\theta^2(u) + 1\right)^{\frac{\gamma}{2(p+1)}}.$$

Proof. Since $J_\theta'(u) = 0$, we have

$$a\|u\|^2 + b\|u\|^4 = \int_\Omega |u|^{p+1}dx + \theta\int_\Omega f(x,u)udx,$$

$$J_\theta(u) = \frac{(p-1)a}{2(p+1)}\|u\|^2 + \frac{(p-3)b}{4(p+1)}\|u\|^4 - \theta\int_\Omega\left(F(x,u) - \frac{1}{p+1}f(x,u)u\right)dx.$$

From (3.1) and (3.2), there exist $C_7 > 1, C_8 > 1$ such that

$$\int_\Omega |u|^{p+1}dx \le C_7\|u\|^4 + C_8,$$
$$\|u\|^4 \le C_7|J_\theta(u)| + C_8.$$

Hence,

$$\begin{aligned} \left|\frac{\partial J_\theta(u)}{\partial\theta}\right| &\le \int_\Omega |F(x,u)|dx \le \int_\Omega \alpha(x)|u|dx + \frac{1}{\gamma}\int_\Omega \beta(x)|u|^\gamma dx \\ &\le C_9\left(\int_\Omega |u|^{p+1}dx\right)^{\frac{1}{p+1}} + C_{10}\left(\int_\Omega |u|^{p+1}dx\right)^{\frac{\gamma}{p+1}} \\ &\le (C_9 + C_{10})\left(\int_\Omega |u|^{p+1}dx\right)^{\frac{\gamma}{p+1}} + C_9 \\ &\le (C_9 + C_{10})\left(C_7^2|J_\theta(u)| + C_7C_8 + C_8\right)^{\frac{\gamma}{p+1}} + C_9 \\ &\le C^*\left(J_\theta^2(u) + 1\right)^{\frac{\gamma}{2p+2}}, \end{aligned}$$

where $C^* = 2(C_9 + C_{10})(C_7^2 + C_7C_8 + C_8 + 1)$. The proof is completed. □

Now let H_k be the subspace of H spanned by the first k eigenfunctions of Δ. In order to estimate critical levels c_k of J_0, we need the following classical result.

Lemma 3.4 ([12, 19])**.** *Let Ω be a bounded smooth domain in $\mathbb{R}^N (N \geq 2)$, let $q > 1$ with $q = \frac{N}{2}$ if $N \geq 3$, and let $V \in L^q(\Omega)$. Denote by $m(V)$ the number of non-positive eigenvalues of the following eigenvalue problem*

$$\begin{aligned} -\Delta u - Vu &= \lambda u, \quad \text{in } \Omega \\ u &= 0, \qquad \text{on } \partial\Omega. \end{aligned} \tag{3.5}$$

Then there is a constant $C_q > 0$ such that $m(V) \leq C_q \int_\Omega |V|^q dx$.

Lemma 3.5.

(1) If $N = 3$, there exists $\hat{C} > 0$ such that

$$c_k \geq \hat{C} k^{\frac{4}{3} \cdot \frac{p+1}{p-3}}. \tag{3.6}$$

(2) If $N = 2$, for any $1 < \epsilon < (p+1)/(p-1)$, there exists $C_\epsilon > 0$ only dependent of ϵ such that

$$c_k \geq C_\epsilon k^{\frac{2}{\epsilon} \cdot \frac{p+1}{p-3}}. \tag{3.7}$$

Proof. We prove the lemma by using Morse indices. One identifies a cohomotopic family $\mathfrak{F}$ of dimension k (see Definition 5.1 in [10]) in such a way that if D_k denotes the ball in H_k of radius R_k and if $\tau \in \Gamma_k$, then $\tau(D_k) \in \mathfrak{F}$. It follows from Theorem 5.1 in [10] that there exist $v_k \in H$ such that

$$J_0(v_k) \leq c_k, \quad J_0'(v_k) = 0, \quad \text{index}_0 J_0''(v_k) \geq k,$$

where

$$\text{index}_0 J_0''(v) = \max\{\dim W : W \subset H \text{ is a subspace such that } J_0''(v)(h,h) \leq 0 \text{ for } h \in W\}.$$

Noting that

$$\begin{aligned} J_0''(v)(h,h) &= a\int_\Omega |\nabla h|^2 dx + 2b\left(\int_\Omega \nabla v \nabla h dx\right)^2 + b\int_\Omega |\nabla v|^2 dx \int_\Omega |\nabla h|^2 dx - p\int_\Omega |v|^{p-1} h^2 dx \\ &\geq \left(a + b\int_\Omega |\nabla v|^2 dx\right)\int_\Omega |\nabla h|^2 dx - p\int_\Omega |v|^{p-1}h^2 dx \\ &= \left\langle \left(-\left(a + b\int_\Omega |\nabla v|^2 dx\right)\Delta - p|v|^{p-1}\right)h, h\right\rangle \\ &= \left(a + b\int_\Omega |\nabla v|^2 dx\right)\left\langle\left(-\Delta - \frac{p|v|^{p-1}}{a + b\int_\Omega |\nabla v|^2 dx}\right)h, h\right\rangle, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality product between $H^{-1}(\Omega)$ and H, one have

$$\left\langle\left(-\Delta - \frac{p|v_k|^{p-1}}{a + b\int_\Omega |\nabla v_k|^2 dx}\right)h, h\right\rangle \leq 0$$

if $J_0''(v_k)(h,h) \leq 0$, which follows that

$$-\Delta - \frac{p|v_k|^{p-1}}{a + b\int_\Omega |\nabla v_k|^2 dx}$$

possesses at least k non-positive eigenvalues. In addition,

$$a\|v_k\|^2 + b\|v_k\|^4 = \int_\Omega |v_k|^{p+1}dx.$$

Set

$$V = \frac{p|v_k|^{p-1}}{a + b\int_\Omega |\nabla v_k|^2 dx}, \qquad \epsilon = \begin{cases} \frac{3}{2}, & \text{if } N = 3, \\ 1 < \epsilon < \frac{p+1}{p-1}, & \text{if } N = 2. \end{cases}$$

It is easy to check that $(p-1)\epsilon < p+1 < 2^*$ and $V \in L^\epsilon(\Omega)$. Applying Lemma 3.4 to V, we have

$$\begin{aligned}
k &\le \tilde{C}p^\epsilon \frac{\int_\Omega |v_k|^{(p-1)\epsilon}dx}{\left(a + b\int_\Omega |\nabla v_k|^2 dx\right)^\epsilon} \le \tilde{C}p^\epsilon \frac{\left(\int_\Omega dx\right)^{1-\frac{(p-1)\epsilon}{(p+1)}} \left(\int_\Omega |v_k|^{p+1}dx\right)^{\frac{(p-1)\epsilon}{(p+1)}}}{\left(a + b\int_\Omega |\nabla v_k|^2 dx\right)^\epsilon} \\
&= \tilde{C}p^\epsilon |\Omega|^{1-\frac{(p-1)\epsilon}{p+1}} \frac{\left(\int_\Omega |v_k|^{p+1}dx\right)^{\frac{(p-1)\epsilon}{(p+1)}}}{\left(\frac{\int_\Omega |v_k|^{p+1}dx}{\|v_k\|^2}\right)^\epsilon} \\
&= \tilde{C}p^\epsilon \|v_k\|^{2\epsilon} |\Omega|^{1-\frac{(p-1)\epsilon}{p+1}} \left(\int_\Omega |v_k|^{p+1}dx\right)^{-\frac{2\epsilon}{p+1}}.
\end{aligned}$$

Hence, there are $\rho_1 > 0, \rho_2 > 0$ such that

$$\|v_k\| \ge \rho_1 k^{\frac{1}{2\epsilon}} \left(\int_\Omega |v_k|^{p+1}dx\right)^{\frac{1}{p+1}}, \tag{3.8}$$

$$\int_\Omega |v_k|^{p+1}dx \ge \rho_2 k^{\frac{(p+1)}{\epsilon(p-1)}} \left(a + b\|v_k\|^2\right)^{\frac{p+1}{p-1}}. \tag{3.9}$$

From (3.8) and (3.9), one have

$$\begin{aligned}
\int_\Omega |v_k|^{p+1}dx &\ge \rho_2 b^{\frac{p+1}{p-1}} k^{\frac{(p+1)}{\epsilon(p-1)}} \|v_k\|^{\frac{2(p+1)}{p-1}} \\
&\ge \rho_2 b^{\frac{p+1}{p-1}} k^{\frac{(p+1)}{\epsilon(p-1)}} \left(\rho_1 k^{\frac{1}{2\epsilon}} \left(\int_\Omega |v_k|^{p+1}dx\right)^{\frac{1}{p+1}}\right)^{\frac{2(p+1)}{p-1}} \\
&\ge \rho_2 b^{\frac{p+1}{p-1}} \rho_1^{\frac{2(p+1)}{p-1}} k^{\frac{2(p+1)}{\epsilon(p-1)}} \left(\int_\Omega |v_k|^{p+1}dx\right)^{\frac{2}{p-1}}, \\
\int_\Omega |v_k|^{p+1}dx &\ge \rho_2^{\frac{p-1}{p-3}} b^{\frac{p+1}{p-3}} \rho_1^{\frac{2(p+1)}{p-3}} k^{\frac{2(p+1)}{\epsilon(p-3)}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
c_k \ge J_\theta(v_k) &= \frac{a}{2}\|v_k\|^2 + \frac{b}{4}\|v_k\|^4 - \frac{1}{p+1}\int_\Omega |v_k|^{p+1}dx \\
&= \frac{a}{4}\|v_k\|^2 + \left(\frac{1}{4} - \frac{1}{p+1}\right)\int_\Omega |v_k|^{p+1}dx \\
&\ge \left(\frac{1}{4} - \frac{1}{p+1}\right) \rho_2^{\frac{p-1}{p-3}} b^{\frac{p+1}{p-3}} \rho_1^{\frac{2(p+1)}{p-3}} k^{\frac{2(p+1)}{\epsilon(p-3)}} =: C_\epsilon k^{\frac{2(p+1)}{\epsilon(p-3)}}.
\end{aligned}$$

The proof is completed. □

Remark 3.6. Similar to [15], by applying the Borsuk–Ulam theorem, one can obtain a new estimate of c_k with

$$c_k \geq Mk^{\left(\frac{1}{N}-\frac{1}{2}+\frac{1}{p+1}\right)\cdot\frac{4(p+1)}{p-3}} \tag{3.10}$$

for some $M > 0$. Obviously,

$$\frac{4}{3}\cdot\frac{p+1}{p-3} > \left(\frac{1}{N}-\frac{1}{2}+\frac{1}{p+1}\right)\cdot\frac{4(p+1)}{p-3} \quad \text{if } N=3,$$
$$\frac{2}{\epsilon}\cdot\frac{p+1}{p-3} > \left(\frac{1}{N}-\frac{1}{2}+\frac{1}{p+1}\right)\cdot\frac{4(p+1)}{p-3} \quad \text{if } N=2,\ 1<\epsilon<\frac{p+1}{p-1}.$$

Proof of Theorem 1.2. Note that $(H1)$–$(H4)$ in Section 2 hold with

$$\eta_2(\theta,s) = C^*\left(s^2+1\right)^{\frac{\gamma}{2(p+1)}}, \qquad \eta_1(\theta,s) = -C^*\left(s^2+1\right)^{\frac{\gamma}{2(p+1)}}.$$

If we assume that alternative (ii) occurs for k large, by the form of η_i it follows that

$$c_{k+1}-c_k \leq \mathbf{C}\left(c_{k+1}^{\frac{\gamma}{p+1}} + c_k^{\frac{\gamma}{p+1}} + 1\right) \tag{3.11}$$

for some $\mathbf{C} > 0$. Therefore, since $\{c_k\}$ is a nondecreasing sequence, from (3.11) we can find a constant $\mathbf{C_1} > 0$ and integer k_0 such that

$$c_k \leq \mathbf{C_1} k^{\frac{p+1}{p+1-\gamma}} \quad \text{for all } k \geq k_0. \tag{3.12}$$

Noting

$$\frac{4(p+1)}{3(p-3)} > \frac{p+1}{p+1-\gamma} \quad \text{if } p>3,\ p+13>4\gamma; \tag{3.13}$$

$$\frac{2(p+1)}{\epsilon(p-3)} > \frac{p+1}{p+1-\gamma} \quad \text{if } p>3,\ p+5>2\gamma,\ 1<\epsilon<\min\left\{\frac{2p+2-2\gamma}{p-3},\frac{p+1}{p-1}\right\}, \tag{3.14}$$

we can obtain a contradiction. Hence, alternative (i) of Theorem 2.1 occurs for finitely many integers $k \in \mathbb{N}$. Thus, in correspondence of these integers, there are critical levels $\bar{c}_k$ of J_1 such that $\bar{c}_k \geq c_k$. Since $c_k \to +\infty$ as $k \to \infty$, it follows that J_1 has finitely many critical points. □

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