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A two-point boundary value problem for third order asymptotically linear systems

Armands Gritsans ✉ and **Felix Sadyrbaev**

Institute of Life Sciences and Technology, Daugavpils University,
Parades street 1, Daugavpils, LV–5400, Latvia

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Abstract. We consider a third order system $x''' = f(x)$ with the two-point boundary conditions $x(0) = \mathbf{0}$, $x'(0) = \mathbf{0}$, $x(1) = \mathbf{0}$, where $f(\mathbf{0}) = \mathbf{0}$ and the vector field $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ is asymptotically linear with the derivative at infinity $f'(\infty)$. We introduce an asymptotically linear vector field ϕ such that its singular points (zeros) are in a one-to-one correspondence with the solutions of the boundary value problem. Using the vector field rotation theory, we prove that under the non-resonance conditions for the linearized problems at zero and infinity the indices of ϕ at zero and infinity can be expressed in the terms of the eigenvalues of the matrices $f'(\mathbf{0})$ and $f'(\infty)$, respectively. This proof constitutes an essential part of our article. If these indices are different, then standard arguments of the vector field rotation theory ensure the existence of at least one nontrivial solution to the boundary value problem. At the end of the article we consider the consequences for the scalar case.

Keywords: boundary value problem, spectrum, asymptotically linear vector field, index of isolated singular point.

2010 Mathematics Subject Classification: 34B15.

1 Introduction

Third order ordinary differential equations play an important role in modeling various processes in physics, engineering and technology, for example, in modeling compressible flows, thin viscous films, three-layer beams, electric circuits, and many others. The reader may consult, for instance, [4,8,9,13,25] and references therein, for more information about applications of third order equations.

The present article regards the existence results for a system of n third order ordinary differential equations

$$x''' = f(x), \tag{1.1}$$

✉ Corresponding author. Email: arminge@inbox.lv

satisfying the two-point boundary conditions

$$\boldsymbol{x}(0) = \mathbf{0}, \quad \boldsymbol{x}'(0) = \mathbf{0}, \quad \boldsymbol{x}(1) = \mathbf{0}. \tag{1.2}$$

The vector field $\boldsymbol{f}\colon \mathbb{R}^n \to \mathbb{R}^n$ is supposed to present a linear behavior near zero and infinity. We prove the existence of at least one nontrivial solution to the problem (1.1), (1.2) when appropriate indices associated with the linearized problems at zero and infinity are different. The same approach was used in the authors papers [10,11] considering the Dirichlet boundary value problem for second order systems with asymptotically linear and asymptotically asymmetric behavior at infinity, respectively. The idea of comparison the behavior of solutions near zero and infinity was developed in [1,24].

The literature on boundary value problems for third order nonlinear systems is not extensive, see, for instance, [5,15,16] and references therein. In the works [6,8,17,19,22,23,27] boundary values problems for m-th order systems are considered.

First, in Section 2, we consider some properties of asymptotically linear vector fields. We assume that a vector field $\boldsymbol{f}\colon \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following conditions.

(A1) $\boldsymbol{f} \in C^1(\mathbb{R}^n, \mathbb{R}^n)$.

(A2) $\boldsymbol{f}(\mathbf{0}) = (\mathbf{0})$, where $\mathbf{0} = (\underbrace{0, 0, \dots, 0}_{n})^T \in \mathbb{R}^n$.

(A3) $\boldsymbol{f}$ is asymptotically linear.

Then we introduce a vector field $\boldsymbol{\phi}\colon \mathbb{R}^n \to \mathbb{R}^n$ associated with the problem (1.1), (1.2). The singular points (zeros) of $\boldsymbol{\phi}$ are in a one-to-one correspondence with the solutions of the boundary value problem (1.1), (1.2).

In Section 3, we will prove a number of auxiliary propositions which will be used in the next sections.

In Section 4, we explore a linear vector field $\boldsymbol{\psi} : \mathbb{R}^n \to \mathbb{R}^n$ associated with the linear boundary value problem

$$\boldsymbol{p}''' = A\,\boldsymbol{p}, \tag{1.3}$$

$$\boldsymbol{p}(0) = \mathbf{0}, \quad \boldsymbol{p}'(0) = \mathbf{0}, \quad \boldsymbol{p}(1) = \mathbf{0}, \tag{1.4}$$

where A is a $n \times n$ matrix with real entries. If this problem is non-resonant, that is, the problem (1.3), (1.4) has only the trivial solution, then Theorem 4.2 states that the index of $\boldsymbol{\psi}$ at zero can be expressed in the terms of the eigenvalues of A. The proof of Theorem 4.2 constitutes an essential part of our article.

In Section 5, we explore the vector field $\boldsymbol{\phi}$ near zero. We consider a vector field $\boldsymbol{\phi}_0\colon \mathbb{R}^n \to \mathbb{R}^n$ associated with the linearized at zero problem (5.4). The assumptions (A1) to (A3) coupled with the condition that (5.4) is non-resonant ensure that $\operatorname{ind}(\mathbf{0}, \boldsymbol{\phi}) = \operatorname{ind}(\mathbf{0}, \boldsymbol{\phi}_0)$.

In Section 6, we prove that the vector field $\boldsymbol{\phi}$ is asymptotically linear. We consider the vector field $\boldsymbol{\phi}_\infty\colon \mathbb{R}^n \to \mathbb{R}^n$ associated with the linearized at infinity problem (6.3). The assumptions (A1) and (A3) combined with the condition that (6.3) is non-resonant ensure that $\operatorname{ind}(\infty, \boldsymbol{\phi}) = \operatorname{ind}(\mathbf{0}, \boldsymbol{\phi}_\infty)$.

In Section 7, we prove the main theorem 7.1 of this paper. The assumptions (A1) to (A3) coupled with the condition that the problems (5.4) and (6.3) are non-resonant provide that zero and infinity are isolated singular points of $\boldsymbol{\phi}$. The standard arguments of the vector field rotation theory (the Brouwer degree theory) ensure the existence of at least one nontrivial solution to the problem (1.1), (1.2) whenever $\mathrm{ind}(\mathbf{0}, \boldsymbol{\phi}) \neq \mathrm{ind}(\infty, \boldsymbol{\phi})$.

In Section 8, we consider the example illustrating the main theorem and providing calculations of indices of $\boldsymbol{\phi}$ at zero and infinity based on Theorem 4.2.

In Section 9, we analyze the main theorem in the scalar case.

The concluding remarks in Section 10 finish our paper.

2 Vector field $\boldsymbol{\phi}$ associated with the boundary value problem

Definition 2.1 ([20])**.** A vector field $\boldsymbol{f} \in C(\mathbb{R}^n, \mathbb{R}^n)$ is called asymptotically linear if there exists a real $n \times n$ matrix $\boldsymbol{f}'(\infty)$ such that

$$\lim_{\|\boldsymbol{x}\| \to \infty} \frac{\|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}'(\infty)\, \boldsymbol{x}\|}{\|\boldsymbol{x}\|} = 0, \tag{2.1}$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$. The matrix $\boldsymbol{f}'(\infty)$ is called the derivative of $\boldsymbol{f}$ at infinity.

If a vector field $\boldsymbol{f} \in C(\mathbb{R}^n, \mathbb{R}^n)$ is asymptotically linear, then, see [2], its derivative $\boldsymbol{f}'(\infty)$ at infinity is uniquely determined by $\boldsymbol{f}$.

Definition 2.2 ([31])**.** A vector field $\boldsymbol{f} \in C(\mathbb{R}^n, \mathbb{R}^n)$ is called linearly bounded if there exist non-negative constants a and b such that

$$\|\boldsymbol{f}(\boldsymbol{x})\| \le a + b\,\|\boldsymbol{x}\|, \quad \forall \boldsymbol{x} \in \mathbb{R}^n. \tag{2.2}$$

Proposition 2.3. *Consider a vector field $\boldsymbol{f} \in C(\mathbb{R}^n, \mathbb{R}^n)$.*

(a) If $\boldsymbol{f}$ is asymptotically linear and

$$\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}'(\infty)\, \boldsymbol{x}, \quad \forall \boldsymbol{x} \in \mathbb{R}^n, \tag{2.3}$$

then for every $\varepsilon > 0$ there exists $M(\varepsilon) > 0$ such that

$$\|\boldsymbol{g}(\boldsymbol{x})\| \le M(\varepsilon) + \varepsilon \|\boldsymbol{x}\|, \quad \forall \boldsymbol{x} \in \mathbb{R}^n. \tag{2.4}$$

(b) Suppose that there exist a real $n \times n$ matrix B and a vector field $\boldsymbol{g}\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\boldsymbol{f}$ has the form

$$\boldsymbol{f}(\boldsymbol{x}) = B\boldsymbol{x} + \boldsymbol{g}(\boldsymbol{x}), \quad \forall \boldsymbol{x} \in \mathbb{R}^n. \tag{2.5}$$

If for every $\varepsilon > 0$ there exists $M(\varepsilon) > 0$ such that (2.4) is fulfilled, then $\boldsymbol{f}$ is asymptotically linear and $B = \boldsymbol{f}'(\infty)$.

(c) If $\boldsymbol{f}$ is asymptotically linear, then $\boldsymbol{f}$ is linearly bounded.

Proof. (a) The statement is valid due to the reference [20].

(b) Consider $\varepsilon > 0$. Then there exists $M(\varepsilon) > 0$ such that (2.4) fulfills. It follows from (2.4) and (2.5) that

$$\lim_{\|\boldsymbol{x}\| \to \infty} \frac{\|\boldsymbol{f}(\boldsymbol{x}) - B\, \boldsymbol{x}\|}{\|\boldsymbol{x}\|} \le \varepsilon.$$

Since $\varepsilon > 0$ can be arbitrary, then (2.1) is fulfilled with $f'(\infty) = B$, by the uniqueness of the derivative of f at infinity.

(c) If f is asymptotically linear with the derivative at infinity $f'(\infty)$, then it follows from (2.3) that

$$\|f(x)\| \leq \|f'(\infty)\,x\| + \|g(x)\| \leq \|f'(\infty)\|\|x\| + \|g(x)\|, \quad \forall x \in \mathbb{R}^n, \tag{2.6}$$

where

$$\|f'(\infty)\| = \max_{\|x\|=1} \|f'(\infty)\,x\|$$

is the induced matrix norm on the Euclidean space $\mathbb{R}^n$. Consider $\varepsilon > 0$. Then, in accordance with (a), there exists $M(\varepsilon) > 0$ such that (2.4) is valid. It follows from (2.4) and (2.6) that (2.2) is fulfilled with $a = M(\varepsilon) > 0$ and $b = \|f'(\infty)\| + \varepsilon > 0$. □

Corollary 2.4. *Consider a vector field $f \in C(\mathbb{R}^n, \mathbb{R}^n)$.*

(a) If f is bounded on $\mathbb{R}^n$, then f is asymptotically linear with $f'(\infty) = O_n$, where O_n is the $n \times n$ zero matrix.

(b) If f is quasi-linear, that is, f has the form (2.5), where g is bounded on $\mathbb{R}^n$, then f is asymptotically linear with $f'(\infty) = B$.

From now on, we assume that the conditions (A1) to (A3) are fulfilled.

Let us rewrite the system (1.1) in the equivalent form $w' = F(w)$, where

$$F(w) = \big(y, z,\ f(x)\big)^T,\ w = (x,\ y,\ z)^T \in \mathbb{R}^N,\ y = x',\ z = y' = x'',\ N = 3n.$$

Denote by $\|\cdot\|_N$ the Euclidean norm on $\mathbb{R}^N$.

Proposition 2.5. *Suppose that the conditions (A1) to (A3) hold. The vector field $F\colon \mathbb{R}^N \to \mathbb{R}^N$ has the following properties.*

(a) $F \in C^1(\mathbb{R}^N, \mathbb{R}^N)$.

(b) $F(o) = o$, where $o = (\mathbf{0}, \mathbf{0}, \mathbf{0})^T \in \mathbb{R}^N$, besides

$$F'(o) = \begin{pmatrix} O_n & I_n & O_n \\ O_n & O_n & I_n \\ f'(\mathbf{0}) & O_n & O_n \end{pmatrix}, \tag{2.7}$$

where I_n is the $n \times n$ unit matrix.

(c) F is asymptotically linear and its derivative at infinity is

$$F'(\infty) = \begin{pmatrix} O_n & I_n & O_n \\ O_n & O_n & I_n \\ f'(\infty) & O_n & O_n \end{pmatrix}. \tag{2.8}$$

(d) F is linearly bounded.

Proof. The statements (a) and (b) are direct consequences of (A1) and (A2).

(c) Since $\boldsymbol{f}$ is asymptotically linear, then it follows from (a) of Proposition 2.3 that for every $\varepsilon > 0$ there exists $M(\varepsilon) > 0$ such that (2.4) fulfills. For each $\boldsymbol{w} = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})^T \in \mathbb{R}^N$ we have

$$\|\boldsymbol{F}(\boldsymbol{w}) - \boldsymbol{F}'(\infty)\boldsymbol{w}\|_N = \|\boldsymbol{g}(\boldsymbol{x})\| \leq M(\varepsilon) + \varepsilon\|\boldsymbol{x}\| \leq M(\varepsilon) + \varepsilon\|\boldsymbol{w}\|_N,$$

where $\boldsymbol{F}'(\infty)$ and $\boldsymbol{g}$ are given by (2.8) and (2.3), respectively. In accordance with (b) of Proposition 2.3, the vector field $\boldsymbol{F}$ is asymptotically linear with the derivative at infinity $\boldsymbol{F}'(\infty)$.

(d) It follows from (c) of Proposition 2.3 that $\boldsymbol{f}$ is linearly bounded, that is, there exist nonnegative constants a and b such that (2.2) is valid. Consider an arbitrary $\boldsymbol{w} = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})^T \in \mathbb{R}^N$. Since

$$\big(\|\boldsymbol{y}\| - \|\boldsymbol{z}\|\big)^2 \geq 0, \quad \big(\|\boldsymbol{x}\| - b\|\boldsymbol{y}\|\big)^2 \geq 0, \quad \big(\|\boldsymbol{x}\| - b\|\boldsymbol{z}\|\big)^2 \geq 0,$$

then

$$2\|\boldsymbol{y}\|\,\|\boldsymbol{z}\| \leq \|\boldsymbol{y}\|^2 + \|\boldsymbol{z}\|^2, \quad 2b\,\|\boldsymbol{x}\|\,\|\boldsymbol{y}\| \leq \|\boldsymbol{x}\|^2 + b^2\|\boldsymbol{y}\|^2, \quad 2b\,\|\boldsymbol{x}\|\,\|\boldsymbol{z}\| \leq \|\boldsymbol{x}\|^2 + b^2\|\boldsymbol{z}\|^2.$$

Taking into account the last inequalities, we obtain

$$\big(\|\boldsymbol{y}\| + \|\boldsymbol{z}\| + b\,\|\boldsymbol{x}\|\big)^2 \leq (2 + b^2)\left(\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 + \|\boldsymbol{z}\|^2\right) = (2 + b^2)\,\|\boldsymbol{w}\|_N^2. \tag{2.9}$$

In view of (2.2) and (2.9), we have

$$\begin{aligned}\|\boldsymbol{F}(\boldsymbol{w})\|_N &= \sqrt{\|\boldsymbol{y}\|^2 + \|\boldsymbol{z}\|^2 + \|\boldsymbol{f}(\boldsymbol{x})\|^2} \leq \|\boldsymbol{y}\| + \|\boldsymbol{z}\| + \|\boldsymbol{f}(\boldsymbol{x})\| \\ &\leq a + \big(\|\boldsymbol{y}\| + \|\boldsymbol{z}\| + b\,\|\boldsymbol{x}\|\big) \leq a + \sqrt{2 + b^2}\,\|\boldsymbol{w}\|_N.\end{aligned}$$

Consequently, $\boldsymbol{F}$ is linearly bounded. □

It follows from Proposition 2.5 that $\boldsymbol{F} \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ and $\boldsymbol{F}$ is linearly bounded. Therefore, see [3,31], the flow $\boldsymbol{\Phi}^t(\boldsymbol{\xi}) = \boldsymbol{w}(t;\boldsymbol{\xi})$ of $\boldsymbol{F}$ is complete and $\boldsymbol{\Phi}^t \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ for every $t \in \mathbb{R}$, where $\boldsymbol{w}(t;\boldsymbol{\xi}) = \big(\boldsymbol{x}(t;\boldsymbol{\xi}), \boldsymbol{x}'(t;\boldsymbol{\xi}), \boldsymbol{x}''(t;\boldsymbol{\xi})\big)^T$, is the solution of the Cauchy problem

$$\boldsymbol{w}' = \boldsymbol{F}(\boldsymbol{w}), \quad \boldsymbol{w}(0) = \boldsymbol{\xi}.$$

Consider $\boldsymbol{\xi} = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})^T \in \mathbb{R}^N$. If $\boldsymbol{\alpha} = \boldsymbol{\beta} = \boldsymbol{0}$, then

$$\boldsymbol{x}(t;\boldsymbol{\gamma}) := \boldsymbol{x}(t;\boldsymbol{\xi}) \tag{2.10}$$

solves the Cauchy problem (1.1),

$$\boldsymbol{x}(0) = \boldsymbol{0}, \;\; \boldsymbol{x}'(0) = \boldsymbol{0}, \;\; \boldsymbol{x}''(0) = \boldsymbol{\gamma}. \tag{2.11}$$

Consider a vector field $\boldsymbol{\phi}\colon \mathbb{R}^n \to \mathbb{R}^n$,

$$\boldsymbol{\phi}(\boldsymbol{\gamma}) = \boldsymbol{x}(1;\boldsymbol{\gamma}), \quad \forall \boldsymbol{\gamma} \in \mathbb{R}^n.$$

Since $\boldsymbol{\phi}$ is the first component of the restriction $\boldsymbol{\Phi}^1\Big|_{\substack{\boldsymbol{\alpha}=\boldsymbol{0}\\ \boldsymbol{\beta}=\boldsymbol{0}}}$, then $\boldsymbol{\phi} \in C^1(\mathbb{R}^n, \mathbb{R}^n)$.

Definition 2.6. A point $\boldsymbol{\gamma} \in \mathbb{R}^n$ is called a singular point of the vector field $\boldsymbol{\phi}$ if $\boldsymbol{\phi}(\boldsymbol{\gamma}) = \boldsymbol{0}$.

The singular points of $\boldsymbol{\phi}$ are in a one-to-one correspondence with the solutions of the boundary value problem (1.1), (1.2), since $\boldsymbol{\phi}(\boldsymbol{\gamma}) = \boldsymbol{0}$ if and only if $\boldsymbol{x}(t;\boldsymbol{\gamma})$ solves (1.1), (1.2). It follows from (A2) that $\boldsymbol{\gamma} = \boldsymbol{0}$ is a singular point of $\boldsymbol{\phi}$ and it corresponds to the trivial solution of (1.1), (1.2). Each singular point $\boldsymbol{\gamma} \neq \boldsymbol{0}$ of $\boldsymbol{\phi}$ generates a nontrivial solution of (1.1), (1.2).

In this paper, we will prove that under the conditions, formulated in the main theorem 7.1, the vector field $\boldsymbol{\phi}$ has a singular point $\boldsymbol{\gamma} \neq \boldsymbol{0}$. For this we will use the vector field rotation theory. The reader may consult, for instance, [21,30], for definitions of isolated singular points of vector fields and their indices.

3 Auxiliary results

Consider the function

$$h(z) = e^z + \varepsilon_1 e^{\varepsilon_1 z} + \varepsilon_2 e^{\varepsilon_2 z} = e^z - 2e^{-\frac{z}{2}} \sin\left(\frac{\sqrt{3}}{2}z + \frac{\pi}{6}\right), \quad \forall z \in \mathbb{C}, \tag{3.1}$$

where $\varepsilon_{1,2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ are cube roots of unity and e^z is the complex exponential function. The function h is analytic on $\mathbb{C}$ and

$$h(\varepsilon_1 z) = \varepsilon_2 h(z), \quad h(\varepsilon_2 z) = \varepsilon_1 h(z), \quad \forall z \in \mathbb{C}. \tag{3.2}$$

If τ is a zero of h, then it follows from (3.2) that $\varepsilon_1\tau$ and $\varepsilon_2\tau$ are zeros of h, too.

The next lemma plays an important role in our considerations.

Lemma 3.1. *The function h has no complex zeros outside of the lines* $\operatorname{Im} z = 0$ *and* $\operatorname{Im} z = \pm\sqrt{3}\operatorname{Re} z$.

For the proof of this lemma we will prove a number of auxiliary propositions, some of which are of independent interest and will be used in the next sections.

3.1 The change of the argument along a path

Definition 3.2. Let α and β be real numbers with $\alpha < \beta$. A continuous mapping $w\colon [\alpha, \beta] \to \mathbb{C}$ is called a path from $w(\alpha)$ to $w(\beta)$ and its image $[w] = w([\alpha, \beta])$ is called the trace of w. A path $w\colon [\alpha, \beta] \to \mathbb{C}$ is called a loop if $w(\alpha) = w(\beta)$. A subset $L \subset \mathbb{C}$ is called a Jordan curve if there exists a loop $w\colon [\alpha, \beta] \to \mathbb{C}$ such that $L = [w]$ and the mapping w, restricted to $[\alpha, \beta)$, is injective.

For each path $w\colon [\alpha, \beta] \to \mathbb{C} \setminus \{0\}$ there exist, see [29], continuous functions $\rho\colon [\alpha, \beta] \to \mathbb{R}_+$ and $\varphi\colon [\alpha, \beta] \to \mathbb{R}$ such that

$$w(t) = \rho(t)\, e^{i\,\varphi(t)}, \quad \forall t \in [\alpha, \beta]. \tag{3.3}$$

Definition 3.3 ([29])**.** Every continuous function φ satisfying (3.3) is called a continuous branch of the argument along the path w.

If φ and ψ are two continuous branches of the argument along the path w, then, see [29], there exists $k \in \mathbb{Z}$ such that $\psi(t) = \varphi(t) + 2\pi k$ for every $t \in [\alpha, \beta]$. Therefore, if φ is a continuous branch of the argument along the path $w\colon [\alpha, \beta] \to \mathbb{C} \setminus \{0\}$, then the difference $\Delta_w \arg = \varphi(\beta) - \varphi(\alpha)$ does not depend on the choice of the branch φ along w.

Definition 3.4. The difference $\Delta_w \arg$ is called the change of the argument along the path w.

If $w\colon [\alpha, \beta] \to \mathbb{C} \setminus \{0\}$ is a loop, then, see [29], the change of the argument along w is an integer multiple of 2π, and, as already mentioned above, it does not depend on the choice of the branch φ along w.

Definition 3.5. The integer $\operatorname{wind} w = \frac{\Delta_w \arg}{2\pi}$ is called the winding number of w.

The reader may consult, for instance, [29], for more information about winding numbers.

Proposition 3.6. *If* $p, q\colon [\alpha,\beta] \to \mathbb{C}$ *are paths such that*

$$\big|p(t) - q(t)\big| < \big|q(t)\big|, \quad \forall t \in [\alpha,\beta], \tag{3.4}$$
$$p(\alpha)\, q(\beta) = p(\beta)\, q(\alpha), \tag{3.5}$$

then

$$[p], [q] \subset \mathbb{C} \setminus \{0\}, \tag{3.6}$$
$$\Delta_p \arg = \Delta_q \arg. \tag{3.7}$$

Proof. It follows from (3.4) that (3.6) fulfills. Taking into account (3.5), the path $\omega = \frac{p}{q}\colon [\alpha,\beta] \to \mathbb{C} \setminus \{0\}$ is a loop. The inequality (3.4) yields

$$\big|\omega(t) - 1\big| < 1, \quad \forall t \in [\alpha,\beta]. \tag{3.8}$$

Consider the constant loop $\kappa(t) \equiv 1$ and a mapping $H\colon [0,1] \times [\alpha,\beta] \to \mathbb{C}$,

$$H(s,t) = (1-s)\,\kappa(t) + s\,\omega(t), \quad \forall s \in [0,1], \quad \forall t \in [\alpha,\beta].$$

It follows from (3.6) and (3.8) that $H(s,t) \neq 0$ for every $s \in [0,1]$ and $t \in [\alpha,\beta]$. The mapping $H(s,\cdot)\colon [\alpha,\beta] \to \mathbb{C} \setminus \{0\}$ is a loop for every $s \in [0,1]$. Therefore, the loop ω is homotopic through loops to the constant loop κ in the region $\mathbb{C} \setminus \{0\}$. In view of the reference [26], $\operatorname{wind} w = 0$. Hence, for every continuous branch φ of the argument along the loop ω we have $\Delta_\omega \arg = \varphi(\beta) - \varphi(\alpha) = 0$. Taking into account (3.6), there exist continuous branches φ_p and φ_q of the argument along the paths p and q, respectively. Since $\omega = \frac{p}{q}$, then $\varphi = \varphi_p - \varphi_q$ is a continuous branch of the argument along the loop ω. We have

$$\begin{aligned} 0 = \Delta_\omega \arg = \varphi(\beta) - \varphi(\alpha) &= \big(\varphi_p(\beta) - \varphi_q(\beta)\big) - \big(\varphi_p(\alpha) - \varphi_q(\alpha)\big) \\ &= \big(\varphi_p(\beta) - \varphi_p(\alpha)\big) - \big(\varphi_q(\beta) - \varphi_q(\alpha)\big) = \Delta_p \arg - \Delta_q \arg. \end{aligned}$$

Consequently, (3.7) fulfills. □

Remark 3.7. If $p, q\colon [\alpha,\beta] \to \mathbb{C}$ are loops, then Lemma 3.6 actually is the Rouché's Theorem for loops, since (3.5) fulfills and (3.7) yields $\operatorname{wind} p = \operatorname{wind} q$.

3.2 The number of zeros of h in the interior of the triangle T_k

Consider the function

$$g(z) = \varepsilon_1 e^{\varepsilon_1 z} + \varepsilon_2 e^{\varepsilon_2 z} = -2e^{-\frac{z}{2}} \sin\left(\frac{\sqrt{3}}{2} z + \frac{\pi}{6}\right), \quad \forall z \in \mathbb{C}. \tag{3.9}$$

If

$$\xi_k = -\frac{2\pi}{\sqrt{3}} \frac{2+3k}{3} \quad (k \in \mathbb{N}_0), \tag{3.10}$$

then $\sin\big(\frac{\sqrt{3}}{2}\xi_k + \frac{\pi}{6}\big) = (-1)^{k+1}$ $(k \in \mathbb{N}_0)$. For every $k \in \mathbb{N}_0$ consider a mapping $w_k\colon [-1,5] \to \mathbb{C}$,

$$w_k(t) = \begin{cases} w_{1,k}(t) = \xi_k + i\sqrt{3}\xi_k\, t, & \text{if } t \in [-1,1], \\ w_{2,k}(t) = \varepsilon_1 w_{1,k}(t-2) = -\frac{\xi_k}{2}(-5+3t) + i\,\frac{\sqrt{3}\xi_k}{2}\,(3-t), & \text{if } t \in [1,3], \\ w_{3,k}(t) = \varepsilon_1 w_{2,k}(t-2) = \varepsilon_2 w_{1,k}(t-4) & \\ \qquad = \frac{\xi_k}{2}(-13+3t) - i\,\frac{\sqrt{3}\xi_k}{2}\,(-3+t), & \text{if } t \in [3,5]. \end{cases} \tag{3.11}$$

The mapping w_k ($k \in \mathbb{N}_0$) is a loop and its trace $[w_k] = T_k \subset \mathbb{C} \setminus \{0\}$ is a positively oriented Jordan curve. The set T_k is a triangle with the vertices

$$z_{1,k} = w_{1,k}(-1) = \xi_k - i\sqrt{3}\xi_k, \tag{3.12}$$
$$z_{2,k} = w_{2,k}(1) = \varepsilon_1 z_{1,k} = \xi_k + i\sqrt{3}\xi_k, \tag{3.13}$$
$$z_{3,k} = w_{3,k}(3) = \varepsilon_1 z_{2,k} = \varepsilon_2 z_{1,k} = -2\xi_k,$$

and it consists of the line segments

$$L[z_{1,k}; z_{2,k}], \quad L[z_{2,k}; z_{3,k}] = \varepsilon_1 L[z_{1,k}; z_{2,k}], \quad L[z_{3,k}; z_{1,k}] = \varepsilon_2 L[z_{1,k}; z_{2,k}] \tag{3.14}$$

with the parametrizations $w_{1,k}$, $w_{2,k}$, $w_{3,k}$, respectively.

Proposition 3.8. *For every $k \in \mathbb{N}_0$ the number of zeros of h in the interior of the Jordan curve T_k, counted with multiplicity, is equal to $2 + 3k$.*

Proof. Suppose that $k \in \mathbb{N}_0$. Consider the loop $p_k = h \circ w_k \colon [-1, 5] \to \mathbb{C}$,

$$p_k(t) = \begin{cases} lcl p_{1,k}(t) = (h \circ w_{1,k})(t), & \text{if } t \in [-1, 1], \\ p_{2,k}(t) = (h \circ w_{2,k})(t) = \varepsilon_2 p_{1,k}(t-2), & \text{if } t \in [1, 3], \\ p_{3,k}(t) = (h \circ w_{3,k})(t) = \varepsilon_1 p_{1,k}(t-4), & \text{if } t \in [3, 5], \end{cases} \tag{3.15}$$

and the path $q_{1,k} = g \circ w_{1,k} \colon [-1, 1] \to \mathbb{C}$.

Let us prove that

$$\left| p_{1,k}(t) - q_{1,k}(t) \right| < \left| q_{1,k}(t) \right|, \quad \forall t \in [-1, 1]. \tag{3.16}$$

For every $t \in [-1, 1]$ we have

$$\left| p_{1,k}(t) - q_{1,k}(t) \right| = \left| e^{w_{1,k}(t)} \right| = e^{\xi_k} < 1. \tag{3.17}$$

Since

$$q_{1,k}(t) = g\big(w_{1,k}(t)\big) = -2e^{-\frac{w_{1,k}(t)}{2}} \sin\left(\frac{\sqrt{3}}{2} w_{1,k}(t) + \frac{\pi}{6} \right) = 2(-1)^k \cosh\left(\frac{3}{2}\xi_k t \right) e^{-\frac{\xi_k}{2}} e^{-i\frac{\sqrt{3}\xi_k t}{2}}, \tag{3.18}$$

then for every $t \in [-1, 1]$ we obtain

$$\left| q_{1,k}(t) \right| = 2e^{-\frac{\xi_k}{2}} \cosh\left(\frac{3}{2}\xi_k t \right) > 2. \tag{3.19}$$

In view of (3.17) and (3.19) we deduce that (3.16) fulfills.

From (3.1), (3.9), (3.10) and (3.11) it follows that

$$\frac{p_{1,k}(-1)}{q_{1,k}(-1)} = 1 + \frac{1}{1 + e^{-3\xi_k}} = \frac{p_{1,k}(1)}{q_{1,k}(1)}.$$

Therefore,

$$p_{1,k}(-1) q_{1,k}(1) = p_{1,k}(1) q_{1,k}(-1). \tag{3.20}$$

On account of (3.16) and (3.20), we infer from Proposition 3.6 that

$$[p_{1,k}], [q_{1,k}] \subset \mathbb{C} \setminus \{0\}, \tag{3.21}$$
$$\Delta_{p_{1,k}} \arg = \Delta_{q_{1,k}} \arg. \tag{3.22}$$

It follows from (3.18) that

$$q_{1,k}(t) = \nu(t)e^{i\theta(t)}, \quad \forall t \in [-1,1],$$

where

$$\nu(t) = 2\cosh\left(\frac{3}{2}\xi_k t\right) e^{-\frac{\xi_k}{2}} > 0, \quad \theta(t) = \begin{cases} -\frac{\sqrt{3}\xi_k t}{2}, & \text{if } k = 0,2,4,\dots, \\ \pi - \frac{\sqrt{3}\xi_k t}{2}, & \text{if } k = 1,3,5,\dots, \end{cases}$$

are continuous functions on the interval $[-1,1]$. Therefore, θ is a continuous branch of the argument along the path $q_{1,k}$ and

$$\Delta_{q_{1,k}} \arg = \theta(1) - \theta(-1) = \frac{2+3k}{3} 2\pi.$$

In view of (3.22),

$$\Delta_{p_{1,k}} \arg = \frac{2+3k}{3} 2\pi. \tag{3.23}$$

Taking into account (3.21), there exist continuous functions

$$r\colon [-1,1] \to \mathbb{R}_+, \quad a\colon [-1,1] \to \mathbb{R}$$

such that

$$p_{1,k}(t) = h\big(w_{1,k}(t)\big) = r(t)e^{ia(t)}, \quad \forall t \in [-1,1]. \tag{3.24}$$

Since (3.23) is valid for every continuous branch of the argument along the path $p_{1,k}$, then

$$a(1) - a(-1) = \frac{2+3k}{3} 2\pi. \tag{3.25}$$

It follows from (3.2), (3.12), (3.13) and (3.24) that

$$\begin{aligned} r(1)e^{ia(1)} &= p_{1,k}(1) = h\big(w_{1,k}(1)\big) = h\big(\varepsilon_1 w_{1,k}(-1)\big) \\ &= \varepsilon_2 h\big(w_{1,k}(-1)\big) = \varepsilon_2 p_{1,k}(-1) = \varepsilon_2 r(-1)e^{ia(-1)}. \end{aligned}$$

Hence,

$$r(1) = r(-1). \tag{3.26}$$

Consider functions $\rho_k, \varphi_k\colon [-1,5] \to \mathbb{R}$,

$$\rho_k(t) = \begin{cases} \rho_{1,k}(t) = r(t), & \text{if } \ t \in [-1,1], \\ \rho_{2,k}(t) = r(t-2), & \text{if } \ t \in [1,3], \\ \rho_{3,k}(t) = r(t-4), & \text{if } \ t \in [3,5], \end{cases} \tag{3.27}$$

$$\varphi_k(t) = \begin{cases} \varphi_{1,k}(t) = a(t), & \text{if } \ t \in [-1,1], \\ \varphi_{2,k}(t) = \frac{4\pi}{3} + a(t-2) + 2\pi k, & \text{if } \ t \in [1,3], \\ \varphi_{3,k}(t) = \frac{2\pi}{3} + a(t-4) + 2\pi(1+2k), & \text{if } \ t \in [3,5]. \end{cases} \tag{3.28}$$

Since $r(t) > 0$ for every $t \in [-1,1]$, then $\rho_k(t) > 0$ for every $t \in [-1,5]$. It follows from (3.2), (3.11), (3.15), (3.27) and (3.28), that

$$\rho_k(t)e^{i\,\varphi_k(t)} = p_k(t), \quad \forall t \in [1,5].$$

On account of (3.25) and (3.26), the functions ρ_k and φ_k are continuous on the interval $[-1,5]$. Consequently, φ_k is a continuous branch of the argument along the loop p_k and the change of the argument along the loop p_k is

$$\Delta_{p_k} \arg = \varphi_k(5) - \varphi_k(-1) = \varphi_{3,k}(5) - \varphi_{1,k}(-1) = (2+3k)2\pi.$$

The winding number of the function h along the Jordan curve T_k is

$$\operatorname{wind}_{T_k} h := \operatorname{wind} p_k = \frac{\Delta_{p_k} \arg}{2\pi} = 2 + 3k.$$

It follows from the argument principle for analytic functions, see [29], that the number of zeros of h in the interior of T_k, counted with multiplicity, is equal to $2 + 3k$. □

3.3 The scalar eigenvalue problem

Proposition 3.9.

(a) The number $r_0 = 0$ is a double zero of h. Every zero $\tau \in \mathbb{C} \setminus \{0\}$ of h, if any, is simple.

(b) The function h is positive on the interval $(0, +\infty)$.

(c) The function h on the real axis has a countable number of zeros r_k $(k \in \mathbb{N}_0)$ which can be ordered as

$$\cdots < \xi_{k+1} < r_{k+1} < \xi_k < r_k < \cdots < \xi_2 < r_2 < \xi_1 < r_1 < \xi_0 < r_0 = 0, \tag{3.29}$$

where ξ_k $(k \in \mathbb{N}_0)$ are given by (3.10).

(d) The function h is positive on (r_{k+1}, r_k) if $k \in \mathbb{N}_0$ is even and negative if $k \in \mathbb{N}_0$ is odd.

Proof. (a) The number $r_0 = 0$ is a double zero of h, since

$$h(0) = 1 + \varepsilon_1 + \varepsilon_2 = 0, \quad h'(0) = 1 + \varepsilon_1^2 + \varepsilon_2^2 = 0, \quad h''(0) = 1 + \varepsilon_1^3 + \varepsilon_2^3 = 3 \neq 0. \tag{3.30}$$

Consider $\tau \in \mathbb{C} \setminus \{0\}$ such that $h(\tau) = 0$. Let us prove that $h'(\tau) \neq 0$. Suppose, on the contrary, that $h'(\tau) = 0$. Since $h(z) = e^z + g(z)$ for every $z \in \mathbb{C}$, then $e^\tau + g(\tau) = 0$ and $e^\tau + g'(\tau) = 0$, where the function g is given by (3.9). Therefore,

$$\varepsilon_2 e^{\varepsilon_1 \tau} + \varepsilon_1 e^{\varepsilon_2 \tau} = g'(\tau) = g(\tau) = \varepsilon_1 e^{\varepsilon_1 \tau} + \varepsilon_2 e^{\varepsilon_2 \tau}.$$

We deduce that $e^{(\varepsilon_1 - \varepsilon_2)\tau} = 1$ and thus there exists $m \in \mathbb{Z}$ such that $\tau = \frac{2\pi m}{\sqrt{3}}$. If follows from

$$0 = h(\tau) = e^{-\frac{\pi m}{\sqrt{3}}} \left(e^{\sqrt{3}\pi m} - (-1)^m \right)$$

that $e^{\sqrt{3}\pi m} = 1$. Hence, $m = 0$ and $\tau = 0$. The contradiction obtained proves the statement.

(b) Let us consider the function h on the real axis. Taking into account (3.1), we can write

$$h(t) = 2e^{-\frac{t}{2}} q(t), \quad \forall t \in \mathbb{R}, \tag{3.31}$$

where

$$q(t) = q_1(t) - q_2(t), \quad q_1(t) = \frac{1}{2} e^{\frac{3t}{2}}, \quad q_2(t) = \sin\left(\frac{\sqrt{3}}{2} t + \frac{\pi}{6} \right), \quad \forall t \in \mathbb{R}.$$

It follows from (3.31) that the functions h and q have the same zeros.

1) Since $q_2(\xi_k) = (-1)^{k+1}$ $(k \in \mathbb{Z})$, then ξ_k $(k = 0, \pm 2, \pm 4, \ldots)$ are local minima of q_2 and ξ_k $(k = \pm 1, \pm 3, \pm 5, \ldots)$ are local maxima of q_2. The points $\eta_k = -\frac{2\pi}{\sqrt{3}} \frac{1+6k}{6}$ $(k \in \mathbb{Z})$ are zeros of q_2 and

$$\cdots < \xi_{k+1} < \eta_{k+1} < \xi_k < \eta_k < \cdots < \eta_2 < \xi_1 < \eta_1 < \xi_0 < \eta_0 < 0 < \xi_{-1} < \eta_{-1} < \cdots$$

The function q_1 is positive and strictly increasing on $\mathbb{R}$, besides, $\lim_{t\to-\infty} q_1(t) = 0$ and $q_1(0) = \frac{1}{2}$. Therefore, for each $k \in \mathbb{N}_0$ the points ξ_k and η_k are not zeros of the function q, since

$$q(\eta_k) = q_1(\eta_k) > 0 \quad (k \in \mathbb{N}_0),$$
$$q(\xi_k) = q_1(\xi_k) + (-1)^k \begin{cases} < 0, & \text{if } k = 1,3,5,\dots, \\ > 0, & \text{if } k = 0,2,4,\dots \end{cases}$$

2) Since the function q_1 is positive on (η_1, η_0) and the function q_2 is negative on this interval, then the function q is positive on (η_1, η_0).

3) The function q_1 is strictly convex on $\mathbb{R}$ and $q_1(0) = \frac{1}{2}$, $q_1'(0) = \frac{3}{4}$. Therefore, for $t \in \mathbb{R}$ the graph of q_1 is strictly above the tangent line $y = \frac{1}{2} + \frac{3}{4}t$ to the graph of q_1 at $\left(0, \frac{1}{2}\right)$ except at the point of tangency. The function q_2 is strictly concave on the interval (η_0, ξ_{-1}) and $q_2(0) = \frac{1}{2}$, $q_2'(0) = \frac{3}{4}$. Therefore, for $t \in (\eta_0, \xi_{-1})$ the graph of q_2 is strictly below the tangent line $y = \frac{1}{2} + \frac{3}{4}t$ to the graph of q_2 at $\left(0, \frac{1}{2}\right)$ except at the point of tangency. Consequently, the function q is positive on (η_0, ξ_{-1}) except at the point $r_0 = 0$, where $q(r_0) = 0$.

4) Since $q_1(\xi_{-1}) = \frac{1}{2} e^{\frac{\pi}{\sqrt{3}}} = 3.0668$ and the function q_1 is strictly increasing on $\mathbb{R}$, then for every $t \in [\xi_{-1}, +\infty)$ we have $q_1(t) \geq q_1(\xi_{-1}) > 1 \geq q_2(t)$. Therefore, the function q is positive on $[\xi_{-1}, +\infty)$.

It follows from 1) to 4) that $h(t) > 0$ for every $t \in [\eta_1, +\infty)$ except at the point $r_0 = 0$, where $h(r_0) = 0$. In particular, $h(t) > 0$ for every $t > 0$.

(c) To prove the statement, it is sufficient, taking into account (b), to prove that for every $k \in \mathbb{N}_0$ the function q has a unique zero r_{k+1} in the interval (ξ_{k+1}, ξ_k).

Since the function q_1 is positive on $\mathbb{R}$ and the function q_2 is nonnegative on $C_k = [\eta_{k+1}, \eta_k]$ $(k = 1,3,5,\dots)$, then

$$q''(t) = \frac{9}{8} q_1(t) + \frac{3}{4} q_2(t) > 0, \quad \forall t \in C_k \quad (k = 1,3,5,\dots).$$

Therefore, q is a strictly convex function on the interval C_k $(k = 1,3,5,\dots)$, and its restriction to every subinterval of C_k $(k = 1,3,5,\dots)$ is a strictly convex function as well.

Suppose that $k = 0,2,4,\dots$ The function q has no zeros in the subinterval $I_k = (\eta_{k+1}, \xi_k) \subset (\xi_{k+1}, \xi_k)$, since the function q_1 is positive on I_k and the function q_2 is negative on I_k. The function q has a unique zero in the subinterval $I_{k+1} = (\xi_{k+1}, \eta_{k+1}) \subset (\xi_{k+1}, \xi_k)$, since $q(\xi_{k+1}) < 0$, $q(\eta_{k+1}) > 0$ and the function q is strictly convex and continuous on $\bar{I}_{k+1} = [\xi_{k+1}, \eta_{k+1}] \subset C_{k+1}$. We have $(\xi_{k+1}, \xi_k) = I_{k+1} \cup \{\eta_{k+1}\} \cup I_k$ and $q(\eta_{k+1}) > 0$. Consequently, the function q has a unique zero r_{k+1} in the interval (ξ_{k+1}, ξ_k).

Suppose that $k = 1,3,5,\dots$ The function q has a unique zero in the subinterval $J_{k-1} = (\eta_{k+1}, \xi_k) \subset (\xi_{k+1}, \xi_k)$, since $q(\eta_{k+1}) > 0$, $q(\xi_k) < 0$ and the function q is strictly convex and continuous on $\bar{J}_{k-1} = [\eta_{k+1}, \xi_k] \subset C_k$. The function q has no zeros in the subinterval $J_k = (\xi_{k+1}, \eta_{k+1}) \subset (\xi_{k+1}, \xi_k)$, since the function q_1 is positive on J_k and the function q_2 is negative on J_k. We have $(\xi_{k+1}, \xi_k) = J_{k+1} \cup \{\eta_{k+1}\} \cup J_k$ and $q(\eta_{k+1}) > 0$. Consequently, the function q has a unique zero r_{k+1} in the interval (ξ_{k+1}, ξ_k).

(d) It follows from the proof of (b) and (c) that the statement is valid. □

Consider the scalar boundary value problem

$$x''' = \lambda x, \quad x(0) = 0, \quad x'(0) = 0, \quad x(1) = 0. \tag{3.32}$$

Definition 3.10. A number λ is called an eigenvalue of (3.32) if there exists a nontrivial solution of (3.32). The set σ of all eigenvalues of (3.32) is called the spectrum of (3.32).

Proposition 3.11. *The set $\sigma = \{r_j^3 : \ j \in \mathbb{N}\}$ is the spectrum of (3.32), where r_j $(j \in \mathbb{N})$ are negative zeros of the function h.*

Proof. If $y(t;\lambda)$ solves the Cauchy problem

$$y''' = \lambda y, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1, \tag{3.33}$$

then $x(t;\lambda;\gamma) = \gamma\, y(t;\lambda)$ is the solution of the Cauchy problem

$$x''' = \lambda x, \quad x(0) = 0, \quad x'(0) = 0, \quad x''(0) = \gamma.$$

The boundary value problem (3.32) has a nontrivial solution $x(t;\lambda;\gamma)$ if and only if $\gamma \neq 0$ and $y(1;\lambda) = 0$.

1) If $\lambda = 0$, then $y(t;\lambda) = \frac{t^2}{2}$ solves (3.33). Since

$$y(1;\lambda) = \frac{1}{2} > 0, \tag{3.34}$$

then $\lambda = 0$ does not belong to the spectrum σ of (3.32).

2) If $\lambda > 0$, then $\lambda = r^3$, where $r = \sqrt[3]{\lambda} > 0$. The function $y(t;\lambda) = \frac{1}{3r^2} h(rt)$ solves (3.33). Taking into account (b) of Proposition 3.9,

$$y(1;\lambda) = \frac{1}{3r^2} h(r) > 0. \tag{3.35}$$

Therefore, positive λ do not belong to the spectrum σ of (3.32).

3) If $\lambda < 0$, then $\lambda = -r^3$, where $r = \sqrt[3]{|\lambda|} > 0$. The function $y(t;\lambda) = \frac{1}{3r^2} h(-rt)$ solves (3.33). Taking into account (c) of Proposition 3.9, $y(1;\lambda) = \frac{1}{3r^2} h(-r) = 0$ if and only if there exists $j \in \mathbb{N}$ such that $-r = r_j$. Therefore, a negative λ belongs to the spectrum σ of (3.32) if and only if there exists $j \in \mathbb{N}$ such that $\lambda = -r^3 = r_j^3$.

If follows from 1) to 3) that the statement is valid. □

Remark 3.12. The numerical values of r_j $(j = 1, 2, 3, 4, 5)$ are

$$r_5 = -18.7426, \ r_4 = -15.115, \ r_3 = -11.4874, \ r_2 = -7.85979, \ r_1 = -4.23321.$$

The numerical values of the first five eigenvalues of the spectrum σ are

$$r_5^3 = -6583.99, \ r_4^3 = -3453.22, \ r_3^3 = -1515.88, \ r_2^3 = -485.549, \ r_1^3 = -75.8593.$$

3.4 The proof of Lemma 3.1

Proof. In accordance with (c) of Proposition 3.9, the numbers r_j $(j \in \mathbb{N})$ are negative zeros of h. Taking into account (3.2), we deduce that $\varepsilon_1 r_j$, $\varepsilon_2 r_j$ $(j \in \mathbb{N})$ are zeros of h as well. It follows from (a) of Proposition 3.9 that $r_0 = 0$ is a double zero of h and r_j, $\varepsilon_1 r_j$, $\varepsilon_2 r_j$ $(j \in \mathbb{N})$ are simple zeros of h. All these zeros

$$r_0, r_j, \varepsilon_1 r_j, \ \varepsilon_2 r_j \ \ (j \in \mathbb{N}) \tag{3.36}$$

of h are located on the lines $\operatorname{Im} z = 0$ and $\operatorname{Im} z = \pm\sqrt{3}\operatorname{Re} z$.

Since $L[z_{1,k}; z_{2,k}]$ $(k \in \mathbb{N}_0)$ is the vertical line segment with the end points $z_{1,k} = \xi_k - i\sqrt{3}\xi_k$ and $z_{2,k} = \xi_k + i\sqrt{3}\xi_k$, then it follows from (3.29) that the interior of the triangle T_k $(k \in \mathbb{N}_0)$ with the edges (3.14) contains $2 + 3k$ zeros (3.36) of h, counted with multiplicity. More detailed,

- the interior of T_0 contains the double zero $r_0 = 0$ of h,
- the interior of T_k ($k \in \mathbb{N}$) contains the double zero $r_0 = 0$ of h and $3k$ simple zeros $r_j, \varepsilon_1 r_j, \varepsilon_2 r_j$ ($1 \le j \le k$) of h.

It follows from Proposition 3.8 that for every $k \in \mathbb{N}_0$ the number of zeros of h in the interior of T_k, counted with multiplicity, is equal to $2 + 3k$ and thus the numbers (3.36) are exactly the zeros of h. If $z \in \mathbb{C}$ is not located on the lines $\operatorname{Im} z = 0$ and $\operatorname{Im} z = \pm\sqrt{3}\operatorname{Re} z$, then z is not a zero of h and thus $h(z) \neq 0$. □

Remark 3.13. Actually, we have proved more than Lemma 3.1 claims, namely that the numbers (3.36) form the set of all zeros of h.

4 Vector field $\boldsymbol{\psi}$ associated with the linear boundary value problem

Definition 4.1. The vectorial boundary value problem (1.3), (1.4) is called non-resonant if the problem (1.3), (1.4) has only the trivial solution.

Suppose that P solves the $n \times n$ matrix Cauchy problem

$$P''' = A\,P, \quad P(0) = O_n, \quad P'(0) = O_n, \quad P''(0) = I_n. \tag{4.1}$$

If $\boldsymbol{p}(t;\gamma)$ is the solution of the vectorial Cauchy problem (1.3),

$$\boldsymbol{p}(0) = \mathbf{0}, \quad \boldsymbol{p}'(0) = \mathbf{0}, \quad \boldsymbol{p}''(0) = \gamma, \tag{4.2}$$

then $\boldsymbol{p}(t;\gamma) = P(t)\,\gamma$ for every $t \in \mathbb{R}$ and $\gamma \in \mathbb{R}^n$. Let us introduce a linear vector field $\boldsymbol{\psi}\colon \mathbb{R}^n \to \mathbb{R}^n$,

$$\boldsymbol{\psi}(\gamma) = \boldsymbol{p}(1;\gamma) = P(1)\,\gamma, \quad \forall \gamma \in \mathbb{R}^n. \tag{4.3}$$

Consequently, $\boldsymbol{\psi}'(\gamma) = \boldsymbol{\psi}'(\mathbf{0}) = P(1)$ for every $\gamma \in \mathbb{R}^n$.

Theorem 4.2.

(A) The following statements are equivalent.

(1) The boundary value problem (1.3), (1.4) *is non-resonant.*

(2) $\det P(1) \neq 0$.

(3) $\gamma = \mathbf{0}$ *is a unique singular point of the vector field* $\boldsymbol{\psi}$.

(4) No eigenvalue of the matrix A belongs to the spectrum σ of (3.32).

(B) Suppose that one of the statements (1)–(4) *holds. Then*

$$\operatorname{ind}(\mathbf{0}, \boldsymbol{\psi}) = \operatorname{sgn}\det P(1). \tag{4.4}$$

If the matrix A does not have negative eigenvalues with odd algebraic multiplicities, then $\operatorname{ind}(\mathbf{0}, \boldsymbol{\psi}) = 1$. *If the matrix A has m ($1 \le m \le n$) different negative eigenvalues λ_s ($1 \le s \le m$) with odd algebraic multiplicities, then*

$$\operatorname{ind}(\mathbf{0}, \boldsymbol{\psi}) = \prod_{s=1}^{m} \operatorname{sgn} h\left(-\sqrt[3]{|\lambda_s|}\right) = (-1)^{j_1 + \cdots + j_m}, \tag{4.5}$$

where

$$r_{j_s+1} < -\sqrt[3]{|\lambda_s|} < r_{j_s} \quad (1 \le s \le m) \tag{4.6}$$

and r_j ($j \in \mathbb{N}_0$) are real zeros of h ordered as in (3.29).

Proof. (A) Since the nonzero singular points of the linear vector field $\boldsymbol{\psi}$ are in one-to-one correspondence with the nontrivial solutions of the boundary value problem (1.3), (1.4), then equivalencies (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) hold by (4.3).

Let us prove (2) $\Leftrightarrow$ (4).

Since the matrix A has real elements, then, see [14], there exists nonsingular matrix M with real elements such that $J = M^{-1}AM$, where J is the real Jordan form of A. If $\boldsymbol{p} = M\boldsymbol{q}$ and $\boldsymbol{\gamma} = M\boldsymbol{\eta}$, then the Cauchy problem (1.3), (4.2) transforms to the Cauchy problem

$$\boldsymbol{q}''' = J\boldsymbol{q}, \quad \boldsymbol{q}(0) = \mathbf{0}, \quad \boldsymbol{q}'(0) = \mathbf{0}, \quad \boldsymbol{q}''(0) = \boldsymbol{\eta}. \tag{4.7}$$

If $\boldsymbol{q}(t;\boldsymbol{\eta})$ is the solution of (4.7) and Q solves the $n \times n$ matrix Cauchy problem

$$Q''' = J\,Q, \quad Q(0) = O_n, \quad Q'(0) = O_n, \quad Q''(0) = I_n, \tag{4.8}$$

then $\boldsymbol{q}(t;\boldsymbol{\eta}) = Q(t)\,\boldsymbol{\eta}$ for every $t \in \mathbb{R}$ and $\boldsymbol{\eta} \in \mathbb{R}^n$. It follows from $\boldsymbol{q}(1;\boldsymbol{\eta}) = M^{-1}\,\boldsymbol{p}(1;\boldsymbol{\gamma})$ that $Q(1)\boldsymbol{\eta} = M^{-1}P(1)M\boldsymbol{\eta}$ for every $\boldsymbol{\eta} \in \mathbb{R}^n$. Therefore, $Q(1) = M^{-1}P(1)M$. The matrices $Q(1)$ and $P(1)$ are similar and have the same eigenvalues, counted with multiplicity. Consequently,

$$\det Q(1) = \det P(1). \tag{4.9}$$

Next we will analyze $\det Q(1)$.

The blocks of the real Jordan form J of A are of two types. A real eigenvalue λ of A generates blocks

$$J_k(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix} \tag{4.10}$$

of the size k. A pair of complex conjugate eigenvalues $\lambda = a + ib$ and $\overline{\lambda} = a - ib$, where $b \neq 0$, generates blocks

$$J_k(\lambda) = \begin{pmatrix} C_2(\lambda) & I_2 & O_2 & \cdots & O_2 & O_2 \\ O_2 & C_2(\lambda) & I_2 & \cdots & O_2 & O_2 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ O_2 & O_2 & O_2 & \cdots & C_2(\lambda) & I_2 \\ O_2 & O_2 & O_2 & \cdots & O_2 & C_2(\lambda) \end{pmatrix} \tag{4.11}$$

of the size $k = 2m$, where

$$C_2(\lambda) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad O_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Suppose that Q_k solves the $k \times k$ matrix Cauchy problem

$$Q_k''' = J_k(\lambda)\,Q_k, \quad Q_k(0) = O_k, \quad Q_k'(0) = O_k, \quad Q_k''(0) = I_k. \tag{4.12}$$

If $\lambda = r^3 \operatorname{sgn}\lambda$, where $r = \sqrt[3]{|\lambda|} \geq 0$, is a real eigenvalue of A, then corresponding to λ Jordan blocks have the form (4.10). Taking into account [28], we obtain

$$Q_k(t) = \sum_{j=0}^{\infty} \frac{t^{3j+2}}{(3j+2)!}\,[J_k(\lambda)]^j, \quad \forall t \in \mathbb{R}, \tag{4.13}$$

where $[J_k(\lambda)]^j$ ($j \in \mathbb{N}_0$) is an upper triangular $k \times k$ matrix with the same element λ^j on the main diagonal. Therefore, the matrix $Q_k(t)$ is an upper triangular matrix also with the same element

$$y(t;\lambda) = \sum_{j=0}^{\infty} \frac{t^{3j+2}}{(3j+2)!}\,\lambda^j, \quad \forall t \in \mathbb{R},$$

on the main diagonal. It follows from (4.12) that $y(t;\lambda)$ solves the Cauchy problem (3.33) and $\det Q_k(1) = \big[y(1;\lambda)\big]^k$.

(a) If $\lambda = 0$, then it follows from (3.34) that $\det Q_k(1) = \frac{1}{2^k} > 0$.

(b) If $\lambda > 0$, then it follows from (3.35) that $\det Q_k(1) > 0$.

(c) If $\lambda < 0$, then $\lambda = -r^3$. It follows from Proposition 3.11 that

$$\det Q_k(1) = \left[\frac{h(-r)}{3r^2}\right]^k \neq 0 \Leftrightarrow -r \neq r_j \ (j \in \mathbb{N}) \Leftrightarrow \lambda \notin \sigma.$$

(d) If $\lambda = a + ib$ and $\overline{\lambda} = a - ib$, where $b \neq 0$, is a pair of complex conjugate eigenvalues of A, then corresponding to λ and $\overline{\lambda}$ Jordan blocks $J_k(\lambda)$ have the form (4.11). The power $\big[J_k(t)\big]^j$ ($k = 2m;\ j \in \mathbb{N}_0$) is an $m \times m$ upper triangular block matrix of 2×2 blocks with the same block $\big[C_2(\lambda)\big]^j$ on the main diagonal. Therefore, the matrix $Q_k(t)$, given by (4.13), is an $m \times m$ upper triangular block matrix of 2×2 blocks also with the same block

$$D_2(t) = \begin{pmatrix} u_2(t) & -v_2(t) \\ v_2(t) & u_2(t) \end{pmatrix} = \sum_{j=0}^{\infty} \frac{t^{3j+2}}{(3j+2)!}\,\big[C_2(\lambda)\big]^j$$

on the main diagonal, where

$$u_2(t) = \sum_{j=0}^{\infty} \frac{t^{3j+2}}{(3j+2)!}\,|\lambda|^j \cos\big(j \arg\lambda\big), \quad v_2(t) = \sum_{j=0}^{\infty} \frac{t^{3j+2}}{(3j+2)!}\,|\lambda|^j \sin\big(j \arg\lambda\big),$$

and

$$\det Q_k(1) = \big[\det D_2(1)\big]^m = \big[u_2^2(1) + u_2^2(1)\big]^m \geq 0. \tag{4.14}$$

It follows from (4.12) that $D_2(t)$ solves the 2×2 matrix Cauchy problem

$$D_2'''(t) = C_2(\lambda)\,D_2(t), \quad D_2(0) = O_2, \quad D_2'(0) = O_2, \quad D_2''(0) = I_2.$$

Let us introduce a complex-valued function $w_2(t) = u_2(t) + i\,v_2(t)$ of a real variable t. Then we can rewrite the last Cauchy problem in the complex form

$$w_2'''(t) = \lambda\,w_2(t),\ w_2(0) = 0,\ w_2'(0) = 0,\ w_2''(0) = 1. \tag{4.15}$$

The function

$$w_2(t;\lambda) = \frac{1}{3\mu^2}\left(e^{\mu t} + \varepsilon_1 e^{\varepsilon_1 \mu t} + \varepsilon_2 e^{\varepsilon_2 \mu t}\right) = \frac{1}{3\mu^2} h(\mu t), \quad t \in \mathbb{R}, \tag{4.16}$$

solves (4.15), where μ is a fixed cube root of $\lambda \in \mathbb{C} \setminus \mathbb{R}$. The solution (4.16) of (4.15) does not depend on the particular choice of the cube root μ, since, in view of (3.2),

$$\frac{1}{3\mu_1^2} h(\mu_1 t) = \frac{1}{3\mu_2^2} h(\mu_2 t) = w_2(t;\lambda),$$

where $\mu_1 = \varepsilon_1\mu$ and $\mu_2 = \varepsilon_2\mu$ are the two other cube roots of λ. Notice that

$$\det D_2(1) = u_2^2(1) + v_2^2(1) = \left|w_2(1;\lambda)\right|^2. \tag{4.17}$$

It follows from (4.14) and (4.17) that $\det Q_k(1) = \left|w_2(1;\lambda)\right|^{2m} \geq 0$. Hence,

$$\det Q_k(1) > 0 \Leftrightarrow h(\mu) \neq 0. \tag{4.18}$$

Since the cube root μ of $\lambda \in \mathbb{C} \setminus \mathbb{R}$ is not located on the lines $\operatorname{Im} z = 0$ and $\operatorname{Im} z = \pm\sqrt{3}\operatorname{Re} z$, then it follows from Lemma 3.1 that $h(\mu) \neq 0$. Taking into account (4.18), $\det Q_k(1) > 0$.

Since $\det Q(1)$ is equal to the product of $\det Q_k(1)$ for all Jordan blocks $J_k(\lambda)$ in the real Jordan form J of the matrix A, then if follows from (4.9) and (a)-(d) that $\det P(1) = \det Q(1) \neq 0$ if and only if the eigenvalues of the matrix A do not belong to the spectrum σ of (3.32). Hence, the equivalence (2) $\Leftrightarrow$ (4) is valid.

(B) Suppose that one of the statements (1)–(4) holds. Then $\det \boldsymbol{\psi}'(\mathbf{0}) = \det P(1) \neq 0$. It follows from [21] that (4.4) fulfills.

The sign of $\det P(1) = \det Q(1) \neq 0$ is equal to the product of signs $\det Q_k(1)$ for the Jordan blocks $J_k(\lambda)$ of the real Jordan form J of A. Taking into account (a), (b) and (d), we deduce that $\operatorname{sgn}\det Q_k(1) = 1$ for the Jordan blocks $J_k(\lambda)$ corresponding to nonnegative and complex conjugate eigenvalues λ of A. Consider a negative eigenvalue $\lambda = -r^3$ of A, where $r = \sqrt[3]{|\lambda|}$, with the algebraic multiplicity α and the geometric multiplicity γ, $1 \leq \gamma \leq \alpha \leq n$. Since the statement (4) holds, then $h(-r) \neq 0$. The matrix $Q(1)$ has the γ blocks $Q_{k_1}(1), \ldots, Q_{k_\gamma}(1)$, where $k_1 + \cdots + k_\gamma = \alpha$, corresponding to the Jordan blocks $J_{k_1}(\lambda), \ldots, J_{k_\gamma}(\lambda)$ and

$$\det Q_{k_1}(1) \cdot \ldots \cdot \det Q_{k_\gamma}(1) = \left[\frac{h(-r)}{3r^2}\right]^{k_1} \cdot \ldots \cdot \left[\frac{h(-r)}{3r^2}\right]^{k_\gamma} = \left[\frac{h(-r)}{3r^2}\right]^{\alpha} \neq 0.$$

Consequently,

$$\operatorname{sgn}\prod_{j=1}^{\gamma}\det Q_{k_j}(1) = \begin{cases} +1, & \text{if } \alpha \text{ is even,} \\ \operatorname{sgn} h\left(-\sqrt[3]{|\lambda|}\right), & \text{if } \alpha \text{ is odd.} \end{cases}$$

If the matrix A does not have negative eigenvalues with odd algebraic multiplicities, then $\operatorname{ind}(\mathbf{0}, \boldsymbol{\psi}) = 1$. If the matrix A has m $(1 \leq m \leq n)$ different negative eigenvalues λ_s $(1 \leq s \leq m)$ with odd algebraic multiplicities, then

$$\operatorname{ind}(\mathbf{0}, \boldsymbol{\psi}) = \prod_{s=1}^{m}\operatorname{sgn} h\left(-\sqrt[3]{|\lambda_s|}\right). \tag{4.19}$$

It follows from Proposition 3.9 that for each $s \in \{1, \ldots, m\}$ there exists a unique $j_s \in \mathbb{N}_0$ such that (4.6) fulfills and

$$\operatorname{sgn} h\left(-\sqrt[3]{|\lambda_s|}\right) = (-1)^{j_s} \quad (1 \leq s \leq m). \tag{4.20}$$

From (4.19) and (4.20) it follows that (4.5) is valid. $\square$

5 Vector field $\boldsymbol{\phi}$ near zero

Suppose that U solves the $n \times n$ matrix Cauchy problem

$$U''' = \boldsymbol{f}'(\mathbf{0})\,U, \quad U(0) = O_n, \quad U'(0) = O_n, \quad U''(0) = I_n. \tag{5.1}$$

If $\boldsymbol{u}(t;\gamma)$ is the solution of the Cauchy problem

$$\boldsymbol{u}''' = \boldsymbol{f}'(\mathbf{0})\,\boldsymbol{u}, \quad \boldsymbol{u}(0) = \mathbf{0}, \quad \boldsymbol{u}'(0) = \mathbf{0}, \quad \boldsymbol{u}''(0) = \gamma, \tag{5.2}$$

then $\boldsymbol{u}(t;\gamma) = U(t)\,\gamma$ for every $t \in \mathbb{R}$ and $\gamma \in \mathbb{R}^n$. Consider a linear vector field $\boldsymbol{\phi}_0 \colon \mathbb{R}^n \to \mathbb{R}^n$,

$$\boldsymbol{\phi}_0(\gamma) = \boldsymbol{u}(1;\gamma) = U(1)\,\gamma, \quad \forall \gamma \in \mathbb{R}^n. \tag{5.3}$$

Theorem 5.1. *Suppose that the conditions (A1) to (A3) are fulfilled. If the linear boundary value problem*

$$\boldsymbol{u}''' = \boldsymbol{f}'(\mathbf{0})\,\boldsymbol{u}, \quad \boldsymbol{u}(0) = \mathbf{0}, \quad \boldsymbol{u}'(0) = \mathbf{0}, \quad \boldsymbol{u}(1) = \mathbf{0} \tag{5.4}$$

is non-resonant, then $\gamma = \mathbf{0}$ is an isolated singular point of the vector field $\boldsymbol{\phi}$ and

$$\operatorname{ind}(\mathbf{0}, \boldsymbol{\phi}) = \operatorname{sgn}\det U(1) = \operatorname{ind}(\mathbf{0}, \boldsymbol{\phi}_0).$$

Proof. It follows from Proposition 2.5 that $\boldsymbol{F} \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ and $\boldsymbol{w}(t;\boldsymbol{o}) = \boldsymbol{o}$ for every $t \in \mathbb{R}$ is a solution of the system $\boldsymbol{w}' = \boldsymbol{F}(\boldsymbol{w})$. Besides, as mentioned above, for every $t \in \mathbb{R}$ the flow

$$\boldsymbol{\Phi}^t(\xi) = \boldsymbol{w}(t;\xi) = \big(\boldsymbol{x}(t;\xi), \boldsymbol{x}'(t;\xi), \boldsymbol{x}''(t;\xi)\big)^T$$

of $\boldsymbol{F}$ is of class C^1, where $\xi = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma)^T$. In accordance with [18, Theorem 8.43], the $N \times N$ matrix

$$\boldsymbol{Z}(t) = \frac{\partial \boldsymbol{w}(t;\boldsymbol{o})}{\partial \xi} = \begin{pmatrix} \dfrac{\partial \boldsymbol{x}(t;\boldsymbol{o})}{\partial \boldsymbol{\alpha}} & \dfrac{\partial \boldsymbol{x}(t;\boldsymbol{o})}{\partial \boldsymbol{\beta}} & \dfrac{\partial \boldsymbol{x}(t;\boldsymbol{o})}{\partial \gamma} \\ \dfrac{\partial \boldsymbol{x}'(t;\boldsymbol{o})}{\partial \boldsymbol{\alpha}} & \dfrac{\partial \boldsymbol{x}'(t;\boldsymbol{o})}{\partial \boldsymbol{\beta}} & \dfrac{\partial \boldsymbol{x}'(t;\boldsymbol{o})}{\partial \gamma} \\ \dfrac{\partial \boldsymbol{x}''(t;\boldsymbol{o})}{\partial \boldsymbol{\alpha}} & \dfrac{\partial \boldsymbol{x}''(t;\boldsymbol{o})}{\partial \boldsymbol{\beta}} & \dfrac{\partial \boldsymbol{x}''(t;\boldsymbol{o})}{\partial \gamma} \end{pmatrix}$$

solves the variational equation

$$\boldsymbol{Z}' = \boldsymbol{F}'(\boldsymbol{o})\,\boldsymbol{Z} \tag{5.5}$$

of $\boldsymbol{w}' = \boldsymbol{F}(\boldsymbol{w})$ along the solution $\boldsymbol{w}(t;\boldsymbol{o}) = \boldsymbol{o}$ and satisfies the initial condition

$$\boldsymbol{Z}(0) = I_N. \tag{5.6}$$

On account of (2.7), (5.5) and (5.6),

$$\frac{\partial \boldsymbol{x}'''(t;\boldsymbol{o})}{\partial \gamma} = \boldsymbol{f}'(\mathbf{0})\,\frac{\partial \boldsymbol{x}(t;\boldsymbol{o})}{\partial \gamma}, \quad \frac{\partial \boldsymbol{x}(0;\boldsymbol{o})}{\partial \gamma} = O_n, \quad \frac{\partial \boldsymbol{x}'(0;\boldsymbol{o})}{\partial \gamma} = O_n, \quad \frac{\partial \boldsymbol{x}''(0;\boldsymbol{o})}{\partial \gamma} = I_n. \tag{5.7}$$

If $\xi = (\mathbf{0}, \mathbf{0}, \gamma)^T \in \mathbb{R}^N$, then uniqueness of solutions for the $n \times n$ matrix Cauchy problems (5.1) and (5.7), in view of the notation (2.10), imply that

$$U(t) = \frac{\partial \boldsymbol{x}(t;\mathbf{0})}{\partial \gamma}, \quad \forall t \in \mathbb{R}.$$

Hence,

$$U(1) = \frac{\partial \boldsymbol{x}(1;\mathbf{0})}{\partial \gamma} = \boldsymbol{\phi}'(\mathbf{0}). \tag{5.8}$$

Since the problem (5.4) is non-resonant, then it follows from (5.8) and Theorem 4.2 that

$$\det \boldsymbol{\phi}'(\mathbf{0}) = \det U(1) \neq 0. \tag{5.9}$$

Therefore, see [21, Theorem 6.3], the point $\gamma = \mathbf{0}$ is an isolated singular point of the vector field $\boldsymbol{\phi}$ and, taking into account (5.9) and Theorem 4.2, we have

$$\operatorname{ind}(\mathbf{0}, \boldsymbol{\phi}) = \operatorname{sgn}\det \boldsymbol{\phi}'(\mathbf{0}) = \operatorname{sgn}\det U(1) = \operatorname{ind}(0, \boldsymbol{\phi}_0). \qquad \square$$

6 Vector field $\boldsymbol{\phi}$ near infinity

Suppose that V solves the $n \times n$ matrix Cauchy problem

$$V''' = \boldsymbol{f}'(\infty)\, V, \quad V(0) = O_n, \quad V'(0) = O_n, \quad V''(0) = I_n. \tag{6.1}$$

If $\boldsymbol{v}(t;\gamma)$ is the solution of the Cauchy problem

$$\boldsymbol{v}''' = \boldsymbol{f}'(\infty)\, \boldsymbol{v}, \quad \boldsymbol{v}(0) = \boldsymbol{0}, \quad \boldsymbol{v}'(0) = \boldsymbol{0}, \quad \boldsymbol{v}''(0) = \boldsymbol{\gamma}, \tag{6.2}$$

then $\boldsymbol{v}(t;\gamma) = V(t)\,\gamma$ for every $t \in \mathbb{R}$ and $\gamma \in \mathbb{R}^n$. Consider a linear vector field $\boldsymbol{\phi}_\infty \colon \mathbb{R}^n \to \mathbb{R}^n$,

$$\boldsymbol{\phi}_\infty(\gamma) = \boldsymbol{v}(1;\gamma) = V(1)\,\gamma, \quad \forall \gamma \in \mathbb{R}^n.$$

Theorem 6.1. *Suppose that the conditions (A1) and (A3) are fulfilled. If the linear boundary value problem*

$$\boldsymbol{v}''' = \boldsymbol{f}'(\infty)\, \boldsymbol{v}, \quad \boldsymbol{v}(0) = \boldsymbol{0}, \quad \boldsymbol{v}'(0) = \boldsymbol{0}, \quad \boldsymbol{v}(1) = \boldsymbol{0} \tag{6.3}$$

is non-resonant, then $\gamma = \infty$ is an isolated singular point of the vector field $\boldsymbol{\phi}$ and

$$\operatorname{ind}(\infty, \boldsymbol{\phi}) = \operatorname{sgn}\det V(1) = \operatorname{ind}(\boldsymbol{0}, \boldsymbol{\phi}_\infty).$$

Proof. It follows from Proposition 2.5 that the vector field $\boldsymbol{F} \colon \mathbb{R}^N \to \mathbb{R}^N$ is asymptotically linear and $\boldsymbol{F}'(\infty)$, given by (2.8), is its derivative at infinity. In accordance with [20, Theorem 2.2], the flow $\boldsymbol{\Phi}^t$ $(t \in \mathbb{R})$ of the vector field $\boldsymbol{F}$ is asymptotically linear and its derivative at infinity is the matrix exponential $W(t) = e^{t\,\boldsymbol{F}'(\infty)}$,

$$\lim_{\|\xi\|_N \to \infty} \frac{\left\|\boldsymbol{\Phi}^t(\xi) - W(t)\,\xi\right\|_N}{\|\xi\|_N} = 0. \tag{6.4}$$

The matrix exponential $W(t)$ solves the $N \times N$ matrix Cauchy problem

$$W' = \boldsymbol{F}'(\infty)\, W, \quad W(0) = I_N. \tag{6.5}$$

If we represent the $N \times N$ matrix W as the 3×3 block matrix

$$W = \begin{pmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{pmatrix},$$

where W_{ij} $(i, j = 1, 2, 3)$ are $n \times n$ matrices, then it follows from (6.5) and (2.8) that W has the form

$$W = \begin{pmatrix} W_{11} & W_{12} & W_{13} \\ W'_{11} & W'_{12} & W'_{13} \\ W''_{11} & W''_{12} & W''_{13} \end{pmatrix},$$

besides, W_{13} solves the $n \times n$ matrix Cauchy problem

$$W'''_{13} = \boldsymbol{f}'(\infty)\, W_{13}, \quad W_{13}(0) = O_n, \quad W'_{13}(0) = O_n, \quad W''_{13}(0) = I_n. \tag{6.6}$$

Uniqueness of solutions for the $n \times n$ matrix Cauchy problems (6.1) and (6.6) implies that $W_{13}(t) = V(t)$ for every $t \in \mathbb{R}$.

It follows from (6.4), taking into account (a) of Proposition 2.3, that for every $\varepsilon > 0$ there exists $M(\varepsilon) > 0$ such that

$$\left\| \boldsymbol{\Phi}^1(\boldsymbol{\xi}) - \boldsymbol{W}(1)\,\boldsymbol{\xi} \right\|_N \leq M(\varepsilon) + \varepsilon\, \|\boldsymbol{\xi}\|_N, \quad \forall \boldsymbol{\xi} \in \mathbb{R}^N. \tag{6.7}$$

Consider an arbitrary $\boldsymbol{\xi} = (\mathbf{0}, \mathbf{0}, \boldsymbol{\gamma})^T \in \mathbb{R}^N$. In view of the notation (2.10), the time one flow is

$$\boldsymbol{\Phi}^1(\boldsymbol{\xi}) = \boldsymbol{w}(1;\boldsymbol{\xi}) = \begin{pmatrix} \boldsymbol{x}(1;\boldsymbol{\gamma}) \\ \boldsymbol{x}'(1;\boldsymbol{\gamma}) \\ \boldsymbol{x}''(1;\boldsymbol{\gamma}) \end{pmatrix}.$$

Since $\|\boldsymbol{\xi}\|_N = \|\boldsymbol{\gamma}\|$ and

$$\boldsymbol{W}(1)\,\boldsymbol{\xi} = \begin{pmatrix} W_{11}(1) & W_{12}(1) & W_{13}(1) \\ W'_{11}(1) & W'_{12}(1) & W'_{13}(1) \\ W''_{11}(1) & W''_{12}(1) & W''_{13}(1) \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \boldsymbol{\gamma} \end{pmatrix} = \begin{pmatrix} V(1)\boldsymbol{\gamma} \\ V'(1)\boldsymbol{\gamma} \\ V''(1)\boldsymbol{\gamma} \end{pmatrix},$$

then it follows from (6.7) that

$$\|\boldsymbol{\phi}(\boldsymbol{\gamma}) - V(1)\boldsymbol{\gamma}\| = \|\boldsymbol{x}(1;\boldsymbol{\gamma}) - V(1)\boldsymbol{\gamma}\| \leq \left\| \begin{pmatrix} \boldsymbol{x}(1;\boldsymbol{\gamma}) \\ \boldsymbol{x}'(1;\boldsymbol{\gamma}) \\ \boldsymbol{x}''(1;\boldsymbol{\gamma}) \end{pmatrix} - \begin{pmatrix} V(1)\boldsymbol{\gamma} \\ V'(1)\boldsymbol{\gamma} \\ V''(1)\boldsymbol{\gamma} \end{pmatrix} \right\|_N \leq M(\varepsilon) + \varepsilon\, \|\boldsymbol{\gamma}\|$$

for every $\boldsymbol{\gamma} \in \mathbb{R}^n$. Therefore, in view of (b) of Proposition 2.3, the vector field $\boldsymbol{\phi}$ is asymptotically linear and its derivative at infinity is

$$\boldsymbol{\phi}'(\infty) = V(1). \tag{6.8}$$

Since the problem (6.3) is non-resonant, then it follows from (6.8) and Theorem 4.2 that

$$\det \boldsymbol{\phi}'(\infty) = \det V(1) \neq 0. \tag{6.9}$$

Therefore, see [30, Theorem 7], the point $\boldsymbol{\gamma} = \infty$ is an isolated singular point of the vector field $\boldsymbol{\phi}$ and, taking into account (6.9) and Theorem 4.2, we have

$$\operatorname{ind}(\infty, \boldsymbol{\phi}) = \operatorname{sgn}\det \boldsymbol{\phi}'(\infty) = \operatorname{sgn}\det V(1) = \operatorname{ind}(0, \boldsymbol{\phi}_\infty).$$ □

7 The main theorem

Theorem 7.1. *Suppose that the conditions (A1) to (A3) are fulfilled. If the linear boundary value problems (5.4) and (6.3) are non-resonant and* $\operatorname{ind}(\mathbf{0}, \boldsymbol{\phi}) \neq \operatorname{ind}(\infty, \boldsymbol{\phi})$*, then the nonlinear boundary value problem (1.1), (1.2) has a nontrivial solution.*

Proof. It follows from Theorems 5.1 and 6.1 that $\boldsymbol{\gamma} = \mathbf{0}$ and $\boldsymbol{\gamma} = \infty$ are isolated singular points of the vector field $\boldsymbol{\phi}$. Therefore, there exist positive r and R such that $r < R$ and the vector field $\boldsymbol{\phi}$ has no singular points on the sets

$$\overline{B_r(\mathbf{0})} \setminus \{\mathbf{0}\} = \{\boldsymbol{\gamma} \in \mathbb{R}^n : \ 0 < \|\boldsymbol{\gamma}\| \leq r\}, \quad \overline{B_R(\infty)} = \{\boldsymbol{\gamma} \in \mathbb{R}^n : \ \|\boldsymbol{\gamma}\| \geq R\}.$$

The rotations $\Gamma\big(\boldsymbol{\phi}, B_r(\mathbf{0})\big)$ and $\Gamma\big(\boldsymbol{\phi}, B_R(\mathbf{0})\big)$ on the spheres $\partial B_r(\mathbf{0})$ and $\partial B_R(\mathbf{0})$, respectively, are different, since

$$\Gamma\big(\boldsymbol{\phi}, B_r(\mathbf{0})\big) = \operatorname{ind}(\mathbf{0}, \boldsymbol{\phi}) \neq \operatorname{ind}(\infty, \boldsymbol{\phi}) = \Gamma\big(\boldsymbol{\phi}, B_R(\mathbf{0})\big).$$

It follows from [30] that the set

$$\mathbb{R}^n \setminus \left(\overline{B_r(\mathbf{0})} \cup \overline{B_R(\infty)}\right) = \{\boldsymbol{\gamma} \in \mathbb{R}^n : \ r < \|\boldsymbol{\gamma}\| < R\}$$

contains a singular point $\boldsymbol{\gamma}_* \neq \mathbf{0}$ of $\boldsymbol{\phi}$. Therefore, $\boldsymbol{x}(t;\boldsymbol{\gamma}_*)$ is a nontrivial solution of the boundary value problem (1.1), (1.2). □

8 Example

Consider the system

$$\begin{cases} x_1''' = x_1 + x_2 - 101\sin(x_1 + x_2), \\ x_2''' = -x_1 + x_2 - 49\arctan(x_1 - x_2) \end{cases} \tag{8.1}$$

together with the boundary conditions

$$x_1(0) = x_2(0) = x_1'(0) = x_2'(0) = 0, \quad x_1(1) = x_2(1) = 0. \tag{8.2}$$

Consider a vector field $\boldsymbol{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$,

$$\boldsymbol{f}(\boldsymbol{x}) = B\boldsymbol{x} + \boldsymbol{g}(\boldsymbol{x}), \quad \forall \boldsymbol{x} \in \mathbb{R}^2,$$

where

$$B = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

and

$$\boldsymbol{g}(\boldsymbol{x}) = \big(-101\sin(x_1 + x_2), -49\arctan(x_1 - x_2) \big)^T, \quad \forall \boldsymbol{x} = (x_1, x_2)^T \in \mathbb{R}^2.$$

The conditions $(A1)$ and $(A2)$ are fulfilled. Since $\boldsymbol{g}$ is bounded, then it follows from Corollary 2.4 that $\boldsymbol{f}$ is asymptotically linear and $\boldsymbol{f}'(\infty) = B$. Consequently, $(A3)$ fulfills as well.

The matrix $\boldsymbol{f}'(\infty)$ has the complex conjugate eigenvalues $1 \pm i$. Therefore, no eigenvalue of the matrix $\boldsymbol{f}'(\infty)$ belongs to the spectrum σ. It follows from Theorems 4.2 and 6.1 that $\operatorname{ind}(\infty, \boldsymbol{\phi}) = 1$.

The matrix

$$\boldsymbol{f}'(\boldsymbol{0}) = \begin{pmatrix} -100 & -100 \\ -50 & 50 \end{pmatrix}$$

has the eigenvalues

$$\lambda_1 = 25\left(-1 - \sqrt{17}\right) = -128.078, \quad \lambda_2 = 25\left(-1 + \sqrt{17}\right) = 78.0776,$$

which, by Proposition 3.9 and Remark 3.12, do not belong to the spectrum σ. Taking into account Theorems 4.2 and 5.1,

$$\operatorname{ind}(\boldsymbol{0}, \boldsymbol{\phi}) = \operatorname{sgn} h\left(-\sqrt[3]{|\lambda_1|}\right) = \operatorname{sgn}(-16.0161) = -1.$$

Since $\operatorname{ind}(\boldsymbol{0}, \boldsymbol{\phi}) \neq \operatorname{ind}(\infty, \boldsymbol{\phi})$, then it follows from the main theorem 7.1 that the problem (8.1), (8.2) has a nontrivial solution.

9 The main theorem in the scalar case

Theorem 9.1. *Consider a function $f \in C^1(\mathbb{R}, \mathbb{R})$ such that $f(0) = 0$ and f is asymptotically linear with the derivative at infinity $f'(\infty)$. Suppose that $f'(0)$ and $f'(\infty)$ do not belong to the spectrum σ of (3.32). The boundary value problem*

$$x''' = f(x), \quad x(0) = 0, \quad x'(0) = 0, \quad x(1) = 0 \tag{9.1}$$

has at least one nontrivial solution if one of the following conditions holds:

(B1) $f'(0) \geq 0$ *and* $f'(\infty) < 0$, $h\left(-\sqrt[3]{|f'(\infty)|}\right) < 0$;

(B2) $f'(0) < 0$, $h\left(-\sqrt[3]{|f'(0)|}\right) < 0$ *and* $f'(\infty) \geq 0$;

(B3) $f'(0) < 0$, $f'(\infty) < 0$ *and* $h\left(-\sqrt[3]{|f'(0)|}\right) h\left(-\sqrt[3]{|f'(\infty)|}\right) < 0$.

Proof. The statement is valid in view of Theorems 4.2 and 7.1. □

The last theorem can be reformulated in the terms of conjugate points of the linear equation $y''' = \lambda y$. The next definition adapts the similar definition in [12] for the case under consideration.

Suppose that $\lambda < 0$. The function $y(t;\lambda) = \frac{1}{3r^2} h(-rt)$ solves the Cauchy problem (3.33), where $r = \sqrt[3]{|\lambda|} > 0$. Hence, $t = 0$ is a double zero of $y(t;\lambda)$. In accordance with Proposition 3.9, the points r_j ($j \in \mathbb{N}$) are negative simple zeros of h. Therefore, the points $a_j = -\frac{r_j}{r}$ ($j \in \mathbb{N}$) are positive simple zeros of $y(t;\lambda)$ and, in view of (3.29), $0 < a_1 < a_2 < \cdots < a_j < \cdots$

Definition 9.2. Let λ be a negative number. The points

$$a_j = -\frac{r_j}{\sqrt[3]{|\lambda|}} \quad (j \in \mathbb{N})$$

are called the conjugate points of the linear equation $y''' = \lambda y$ with respect to $t = 0$ (or simply conjugate points).

Suppose that λ is negative and it does not belong to the spectrum σ of (3.32). The interval $(0, 1)$ contains $k \in \mathbb{N}_0$ conjugate points of $y''' = \lambda y$ if and only if $-\sqrt[3]{|\lambda|} \in (r_{k+1}, r_k)$. Taking into account (4.20),

- $h\left(-\sqrt[3]{|\lambda|}\right) > 0$ if and only if the interval $(0, 1)$ contains an even number of the conjugate points of $y''' = \lambda y$;
- $h\left(-\sqrt[3]{|\lambda|}\right) < 0$ if and only if the interval $(0, 1)$ contains an odd number of the conjugate points of $y''' = \lambda y$;

The conditions (B1) to (B3) in Theorem 9.1 can be equivalently reformulated in the terms of the conjugate points of $y''' = f'(0)y$ and $z''' = f'(\infty)z$.

(B1′) $f'(0) \geq 0$, $f'(\infty) < 0$ and the interval $(0, 1)$ contains an odd number of the conjugate points of $z''' = f'(\infty)z$;

(B2′) $f'(0) < 0$, $f'(\infty) \geq 0$ and the interval $(0, 1)$ contains an odd number of the conjugate points of $y''' = f'(0)y$;

(B3′) $f'(0) < 0$, $f'(\infty) < 0$ and the interval $(0, 1)$ contains either an odd number of the conjugate points of $y''' = f'(0)y$ and an even number of the conjugate points of $z''' = f'(\infty)z$ or an even number of the conjugate points of $y''' = f'(0)y$ and an odd number of the conjugate points of $z''' = f'(\infty)z$.

Corollary 9.3. *Consider a function* $f \in C^1(\mathbb{R}, \mathbb{R})$ *such that* $f(0) = 0$ *and* f *is bounded on* $\mathbb{R}$. *Suppose that* $f'(0)$ *is negative and it does not belong to the spectrum* σ *of* (3.32). *If the interval* $(0, 1)$ *contains an odd number of the conjugate points of* $y''' = f'(0)y$, *then the boundary value problem* (9.1) *has at least one nontrivial solution.*

Proof. Since the function f is bounded on $\mathbb{R}$, then it follows from Corollary 2.4 that f is asymptotically linear with $f'(\infty) = 0$. Taking into account Theorem 9.1 and the equivalence of (B2) and (B2′), the statement fulfills. □

More can be said about solvability and the number of solutions for the boundary value problem (9.1) if the nonlinearity f has additional properties, for example, if $f'(0) < 0$, f is bounded on $\mathbb{R}$ and f' is non-positive on $\mathbb{R}$, see [7].

10 Concluding remarks

This article considers a two-point boundary value problem for third order systems with a linear behavior near zero and infinity. We introduce an auxiliary vector field which is linear at zero and at infinity and which reflects the behavior of solutions to the boundary value problem. We prove that the auxiliary vector field under the non-resonance conditions has well-defined indices at zero and infinity. The main difficulties in the proof are related with the case of complex eigenvalues of corresponding matrices. By overcoming these difficulties, the standard arguments of the vector field rotation theory lead to the existence of at least one nontrivial solution to the boundary value problem whenever the indices at zero and infinity are different. At the end of the paper, we draw attention to the scalar case and derive from the main theorem sufficient conditions for the existence of a nontrivial solution to the boundary value problem.

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