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Representation of solutions of a solvable nonlinear difference equation of second order

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Abstract. We present a representation of well-defined solutions to the following nonlinear second-order difference equation

$$x_{n+1} = a + \frac{b}{x_n} + \frac{c}{x_n x_{n-1}}, \quad n \in \mathbb{N}_0,$$

where parameters a, b, c, and initial values x_{-1} and x_0 are complex numbers such that $c \neq 0$, in terms of the parameters, initial values, and a special solution to a third-order homogeneous linear difference equation with constant coefficients associated to the nonlinear difference equation, generalizing a recent result in the literature, completing the proof therein by using an essentially constructive method, and giving some theoretical explanations related to the method for solving the difference equation. We also give a more concrete representation of the solutions to the nonlinear difference equation by calculating the special solution to the third-order homogeneous linear difference equation in terms of the zeros of the characteristic polynomial associated to the linear difference equation.

Keywords: nonlinear second-order difference equation, solvable difference equation, linear difference equation, representation of solutions.

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1 Introduction

In this paper, by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$ and $\mathbb{C}$ we denote the set of positive, nonnegative, whole and complex numbers, respectively.

As it is well-known nonhomogeneous linear difference equation with constant coefficients of kth order, that is, the following difference equation

$$c_k x_{n+k} + c_{k-1} x_{n+k-1} + \cdots + c_0 x_n = f_n, \qquad n \geq l, \tag{1.1}$$

where $k \in \mathbb{N}$, $l \in \mathbb{Z}$, c_j, $j = \overline{0,k}$ are given constants, such that

$$c_0 \neq 0 \neq c_k, \tag{1.2}$$

and $(f_n)_{n\geq l}$ is a given sequence of real or complex numbers, is a basic example of a difference equation solvable in closed-form (see, for example, books [5,7,10,13,15,16,19]). If one of the conditions in (1.2) does not hold then the equation is of order less than k, which is why these conditions are posed.

If

$$f_n = 0 \quad \text{for } n \geq l,$$

the equation is called homogeneous linear difference equation with constant coefficients of kth order.

Several methods for finding general solution to difference equation (1.1) have been known yet in the 18th century (for example, the method of generating functions which has been first used for solving the difference equation [8], and the method of guessing solution in the form of a geometric progression).

To find a concrete solution to the equation, initial values $x_l, x_{l+1}, \ldots, x_{l+k-1}$, must be given. Value of index l depends on a specific problem which is studied. Frequently is $l = 0$ or $l = 1$, but it can be any other integer.

Since $c_k \neq 0$, by dividing difference equation (1.1) by c_k we obtain the following equation

$$x_{n+k} + \widehat{c_{k-1}} x_{n+k-1} + \cdots + \widehat{c_0} x_n = \widehat{f_n}, \tag{1.3}$$

for $n \geq l$, where

$$\widehat{c_j} = c_j / c_k, \qquad j = 0, 1, \ldots, k-1,$$

and

$$\widehat{f_n} = f_n / c_k,$$

for $n \geq l$, which is a difference equation of the form in (1.1). Hence, it is usually assumed that

$$c_k = 1. \tag{1.4}$$

For linear difference equations which satisfy condition (1.4), we will say that they are *normalized*.

Many nonlinear difference equations are solved by transforming them by using some suitable changes of variables to linear difference equations with constant coefficients. Somewhat bigger recent interest in solving some concrete nonlinear difference equations in this way, appeared after publication of a note by S. Stević in 2004, in which a special case of the following nonlinear difference equation of second order was solved

$$x_n = \frac{x_{n-2}}{b_n + c_n x_{n-1} x_{n-2}}, \tag{1.5}$$

where $n \in \mathbb{N}_0$ (see, e.g., [29] and the references therein).

The study was, among other papers, continued in [30, 46], where the following nonlinear difference equation with variable coefficients was studied

$$x_n = \frac{a_n x_{n-k}}{b_n + c_n x_{n-1} \cdots x_{n-k}},$$

(see, also [26]). For some related systems see, e.g., [31] and the references therein. Many different ideas and methods for solving nonlinear difference equations, can be found, also in the representative paper [36].

Motivated by some investigations of usually symmetric systems of difference equations (see, e.g., [21–24]), approximately at the same time started a renewed considerable interest in investigation of some related classes of solvable systems of nonlinear difference equations, usually those which are nowadays called close-to-symmetric, or close-to-cyclic systems of nonlinear difference equations (see, for example, [4, 20, 45] and the related references cited therein).

As it is also well-known, the bilinear difference equation

$$z_{n+1} = \frac{\alpha z_n + \beta}{\gamma z_n + \delta}, \qquad n \in \mathbb{N}_0, \tag{1.6}$$

where parameters $\alpha, \beta, \gamma, \delta$, and the initial value z_0 are complex numbers, is one of the first examples of nonlinear difference equations solvable in closed-form (see, for example, books [5, 14, 15, 19, 27]). To get a nonlinear difference equation it must be additionally assumed that

$$\gamma \neq 0 \quad \text{and} \quad \alpha\delta \neq \beta\gamma,$$

otherwise the equation becomes a first-order linear difference equation with constant coefficients or the most simplest constant difference equation, respectively. Based on closed-form formulas for solutions to equation (1.6), which have been known for a long time, in some papers, such as [1, 6, 18] was studied the asymptotic behaviour of their solutions.

Literature on difference equations usually suggests solving equation (1.6) by the change of variables

$$z_n = \widetilde{c_1} \frac{y_{n+1}}{y_n} + \widetilde{c_2}, \qquad n \in \mathbb{N}_0, \tag{1.7}$$

where $\widetilde{c_1}$ and $\widetilde{c_2}$ are two undetermined constants, which are chosen such that the change of variables transforms difference equation (1.6) to a linear difference equation (see, for example, [1, 5, 6, 18, 19, 27]). Equation (1.6) can be also solved by using a system of linear difference equations [14, 15].

Since the first method is closer to the topic of this paper and serves as a good motivation for the method which will be used here, we will give a few words on it.

Using (1.7) in (1.6) gives

$$\left(\widetilde{c_1}\frac{y_{n+2}}{y_{n+1}} + \widetilde{c_2}\right)\left(\gamma\widetilde{c_1}\frac{y_{n+1}}{y_n} + \gamma\widetilde{c_2} + \delta\right) = \alpha\widetilde{c_1}\frac{y_{n+1}}{y_n} + \alpha\widetilde{c_2} + \beta,$$

for $n \in \mathbb{N}_0$, that is,

$$\gamma\widetilde{c_1}^2\frac{y_{n+2}}{y_n} + \widetilde{c_1}(\gamma\widetilde{c_2} - \alpha)\frac{y_{n+1}}{y_n} + \widetilde{c_1}(\gamma\widetilde{c_2} + \delta)\frac{y_{n+2}}{y_{n+1}} + \widetilde{c_2}\,(\gamma\widetilde{c_2} + \delta) - \alpha\widetilde{c_2} - \beta = 0,$$

for $n \in \mathbb{N}_0$.

Multiplying the last equality by y_n, we obtain

$$\gamma\widetilde{c_1}^2 y_{n+2} + \widetilde{c_1}(\gamma\widetilde{c_2} - \alpha)y_{n+1} + (\gamma\widetilde{c_2}^2 + \widetilde{c_2}(\delta - \alpha) - \beta)y_n + \widetilde{c_1}(\gamma\widetilde{c_2} + \delta)\frac{y_{n+2}y_n}{y_{n+1}} = 0,$$

$n \in \mathbb{N}_0$, from which it is easy to see that the equation will be linear with constant coefficients if $\gamma\widetilde{c_2} + \delta = 0$, that is, if

$$\widetilde{c_2} = -\delta/\gamma. \tag{1.8}$$

If so, we have

$$(\gamma\widetilde{c_1})^2 y_{n+2} - \widetilde{c_1}\gamma(\alpha + \delta)y_{n+1} + (\alpha\delta - \beta\gamma)y_n = 0, \qquad n \in \mathbb{N}_0. \tag{1.9}$$

To make difference equation (1.9) normalized, then, clearly, it should be chosen

$$\widetilde{c_1} = 1/\gamma. \tag{1.10}$$

Using (1.8) and (1.10) in (1.7), we see that the following change of variables

$$z_n = \frac{y_{n+1}}{\gamma y_n} - \frac{\delta}{\gamma}, \qquad n \in \mathbb{N}_0, \tag{1.11}$$

transforms equation (1.6) to the following normalized homogeneous linear difference equation of second order

$$y_{n+2} - (\alpha + \delta)y_{n+1} + (\alpha\delta - \beta\gamma)y_n = 0, \tag{1.12}$$

for $n \in \mathbb{N}_0$.

Anyone who see the change of variables (1.7) for the first time should be certainly intrigued by the choice. So, let us say that one of the points is that equation (1.6) can be written as

$$\gamma z_{n+1} + \delta = \alpha + \delta + \frac{\beta\gamma - \alpha\delta}{\gamma z_n + \delta}, \quad n \in \mathbb{N}_0,$$

[1,18,32] from which it follows that the following equation

$$x_{n+1} = \alpha + \delta + \frac{\beta\gamma - \alpha\delta}{x_n}, \quad n \in \mathbb{N}_0,$$

should be solved, or equivalently, the following one

$$x_{n+1} = a + \frac{b}{x_n}, \quad n \in \mathbb{N}_0, \tag{1.13}$$

where $a \in \mathbb{C}$ and $b \in \mathbb{C} \setminus \{0\}$.

What the above procedure says is that difference equation (1.13) is solved by the change of variables

$$x_n = \frac{y_{n+1}}{y_n}, \quad n \in \mathbb{N}_0. \tag{1.14}$$

Motivated by some representations of general solutions to some nonlinear difference equations (for example, those ones in [50]), in terms of the Fibonacci sequence (for some basics of the sequence, see, e.g., [2,51]), recently in [32], S. Stević has given a representation of solutions to equation (1.6) in terms of the solution to a linear second-order homogeneous difference equation with constant coefficients, satisfying the following initial conditions

$$x_{-1} = 0 \quad \text{and} \quad x_0 = 1.$$

Soon after that it turned out that such solutions play some interesting and important roles in solving some other nonlinear difference equations and systems of nonlinear difference equations, for example, the product-type ones, and frequently appear (see, e.g., [34, 37, 41, 43, 44, 48, 49] and the references therein on such systems).

Some classical applications of solvable difference equations can be found, for example, in [15, 17, 28]. For some other recent results on finding closed-form formulas or invariants for solutions to other linear or nonlinear difference equations and systems of nonlinear difference equations, and their applications, see, for example [3, 11, 12, 23–25, 38–40].

Having solved equation (1.13) long time ago, mathematicians started looking for some generalizations of the equation which can be also solved by using the change of variables in (1.14).

The following nonlinear second-order difference equation

$$x_{n+1} = a + \frac{b}{x_n} + \frac{c}{x_n x_{n-1}}, \qquad n \in \mathbb{N}_0, \tag{1.15}$$

is a natural extension of equation (1.13) which is solved by the change of variables (1.14) which transforms it to the following third-order linear difference equation with constant coefficients

$$y_{n+2} = a y_{n+1} + b y_n + c y_{n-1}, \tag{1.16}$$

for $n \in \mathbb{N}_0$.

A special case of equation (1.15), that is, the one with $a = 0$, has been recently studied in [11]. Instead of (1.14), in [11] was used the following (backward shifted) change of variables

$$x_n = \frac{y_n}{y_{n-1}}, \qquad n \geq -1, \tag{1.17}$$

so that equation (1.15) is transformed to

$$y_{n+1} = a y_n + b y_{n-1} + c y_{n-2}, \qquad n \in \mathbb{N}_0, \tag{1.18}$$

and, similar to [32], a representation of general solution to equation (1.15), with $a = 0$, was given in terms of the solution $(s_n)_{n\geq -2}$ of equation (1.18) satisfying the following initial conditions

$$s_{-2} = s_{-1} = 0, \qquad s_0 = 1. \tag{1.19}$$

The shifted change of variables is usually used so that the initial values of the transformed equation (1.18) are y_{-2}, y_{-1} and y_0. The initial values seem more natural than, for example, the following ones: y_{-1}, y_0, y_1, which appear for the case of equation (1.16).

The following theorem was proved in [11].

Theorem 1.1. *Consider equation* (1.15) *with* $a = 0$*. Let* $(x_n)_{n\geq -1}$ *be a well-defined solution to the equation. Then, it has the following representation*

$$x_n = \frac{x_{-1}s_{n+1} + x_0 x_{-1} s_n + c s_{n-1}}{x_{-1}s_n + x_0 x_{-1} s_{n-1} + c s_{n-2}}, \tag{1.20}$$

for $n \in \mathbb{N}_0$.

The theorem is true, but the proof given in [11] is not complete, since it omits an inductive argument, which is a small deficiency of the paper. A bigger deficiency of paper [11] is of theoretical type, since it does not give or use a specific constructive or half-constructive method for getting representation of solutions to equations and systems studied therein. The solutions therein are simply guessed based on some calculations.

Here, we present a more direct half-constructive approach in getting a representation of solutions to difference equation (1.15) and extend Theorem 1.1. One of the motivations for the approach stems from some recent papers by S. Stević and his collaborators, mostly on product-type difference equations and systems of difference equations [34,37,41,43,44,48,49]. The idea is to iterate some associated third-order homogeneous linear difference equations in a suitable chosen way to get representation of solutions to the original difference equation. We also give a more concrete representation of the solutions to the nonlinear difference equation by calculating the special solution to the third-order homogeneous linear difference equation in terms of the zeros of the characteristic polynomial associated to the linear difference equation.

2 Main results

The main results in this paper are proved in this section. The first one generalizes Theorem 1.1.

Theorem 2.1. *Let parameters a, b, c be complex numbers such that $c \neq 0$, and let $(s_n)_{n\geq -2}$ be the solution to equation* (1.18) *satisfying initial conditions* (1.19)*. Then, every well-defined solution to equation* (1.15) *has the following representation*

$$x_n = \frac{x_{-1}(s_{n+1} - as_n) + x_0x_{-1}s_n + cs_{n-1}}{x_{-1}(s_n - as_{n-1}) + x_0x_{-1}s_{n-1} + cs_{n-2}}, \tag{2.1}$$

for every $n \in \mathbb{N}_0$.

Proof. As it has been said in the previous section, by using the change of variables (1.17), equation (1.15) is transformed to equation (1.18).

Now we define initial values of three sequences which will be recursively defined and used in the rest of the proof, in the way as it was done, for example, in [33,37].

Let

$$a_1 := a, \qquad b_1 := b, \qquad c_1 := c. \tag{2.2}$$

In what follows the following three cases will be considered separately: 1) $a \neq 0$, 2) $a = 0$, $b \neq 0$, 3) $a = b = 0$, since the proofs, although follow the same idea, are not the same.

Case $a \neq 0$. We use an iterative procedure which has been recently applied, for example, in papers [34] and [41]. By using the equality (1.18) where n is replaced by $n-2$, in the equation (1.18) where n is replaced by $n-1$, as well as the notation in (2.2), we have

$$\begin{aligned} y_n &= a_1y_{n-1} + b_1y_{n-2} + c_1y_{n-3} \\ &= a_1(ay_{n-2} + by_{n-3} + cy_{n-4}) + b_1y_{n-2} + c_1y_{n-3} \\ &= (aa_1 + b_1)y_{n-2} + (ba_1 + c_1)y_{n-3} + ca_1y_{n-4} \\ &= a_2y_{n-2} + b_2y_{n-3} + c_2y_{n-4}, \end{aligned} \tag{2.3}$$

where

$$a_2 := aa_1 + b_1, \qquad b_2 := ba_1 + c_1, \qquad c_2 := ca_1. \tag{2.4}$$

Using now the equation (1.18) where n is replaced by $n-3$ in (2.3), it follows that

$$\begin{aligned} y_n &= a_2 y_{n-2} + b_2 y_{n-3} + c_2 y_{n-4} \\ &= a_2(a y_{n-3} + b y_{n-4} + c y_{n-5}) + b_2 y_{n-3} + c_2 y_{n-4} \\ &= (a a_2 + b_2) y_{n-3} + (b a_2 + c_2) y_{n-4} + c a_2 y_{n-5} \\ &= a_3 y_{n-3} + b_3 y_{n-4} + c_3 y_{n-5}, \end{aligned} \tag{2.5}$$

where

$$a_3 := a a_2 + b_2, \qquad b_3 := b a_2 + c_2, \qquad c_3 := c a_2. \tag{2.6}$$

Based on relations (2.3)–(2.6), we assume that for some $k \in \mathbb{N}$ such that $2 \le k \le n-1$, we have that

$$y_n = a_k y_{n-k} + b_k y_{n-k-1} + c_k y_{n-k-2}, \tag{2.7}$$

and

$$a_k := a a_{k-1} + b_{k-1}, \qquad b_k := b a_{k-1} + c_{k-1}, \qquad c_k := c a_{k-1}. \tag{2.8}$$

Using further the equality (1.18) where n is replaced by $n-k-1$ in (2.7), it follows that

$$\begin{aligned} y_n &= a_k y_{n-k} + b_k y_{n-k-1} + c_k y_{n-k-2} \\ &= a_k(a y_{n-k-1} + b y_{n-k-2} + c y_{n-k-3}) + b_k y_{n-k-1} + c_k y_{n-k-2} \\ &= (a a_k + b_k) y_{n-k-1} + (b a_k + c_k) y_{n-k-2} + c a_k y_{n-k-3} \\ &= a_{k+1} y_{n-k-1} + b_{k+1} y_{n-k-2} + c_{k+1} y_{n-k-3}, \end{aligned} \tag{2.9}$$

where

$$a_{k+1} := a a_k + b_k, \qquad b_{k+1} := b a_k + c_k, \qquad c_{k+1} := c a_k. \tag{2.10}$$

From (2.3), (2.4), (2.9), (2.10), and by using the induction we see that (2.7) and (2.8) must hold for every $2 \le k \le n$.

Now we prolong sequences a_k, b_k and c_k for some non-positive values of index k, as it was done, for example, in [43, 44]. Note that since $c \ne 0$, the recurrent relations in (2.8) can be really used for calculating values of sequences a_k, b_k and c_k for every $k \le 0$.

Using the recurrent relations with the indices $k = 1$, $k = 0$ and $k = -1$, respectively, after some calculations, it follows that

$$a_0 = \frac{c_1}{c} = 1, \qquad \text{(from (2.2), (2.8))}, \tag{2.11}$$

$$b_0 = a_1 - a a_0 = a - a \cdot 1 = 0, \qquad \text{(from (2.2), (2.8), (2.11))}, \tag{2.12}$$

$$c_0 = b_1 - b a_0 = b - b \cdot 1 = 0, \qquad \text{(from (2.2), (2.8), (2.11))}, \tag{2.13}$$

$$a_{-1} = \frac{c_0}{c} = 0, \qquad \text{(from (2.8), (2.13))}, \tag{2.14}$$

$$b_{-1} = a_0 - a a_{-1} = 1 - a \cdot 0 = 1, \qquad \text{(from (2.8), (2.11), (2.14))}, \tag{2.15}$$

$$c_{-1} = b_0 - b a_{-1} = 0 - b \cdot 0 = 0, \qquad \text{(from (2.8), (2.12), (2.14))}, \tag{2.16}$$

$$a_{-2} = \frac{c_{-1}}{c} = 0, \qquad \text{(from (2.8), (2.16))}, \tag{2.17}$$

$$b_{-2} = a_{-1} - a a_{-2} = 0 - a \cdot 0 = 0, \qquad \text{(from (2.8), (2.14), (2.17))}, \tag{2.18}$$

$$c_{-2} = b_{-1} - b a_{-2} = 1 - b \cdot 0 = 1, \qquad \text{(from (2.8), (2.15), (2.17))}. \tag{2.19}$$

Hence, we have

$$\begin{aligned} &a_0 = 1, && a_{-1} = 0, && a_{-2} = 0,\\ &b_0 = 0, && b_{-1} = 1, && b_{-2} = 0,\\ &c_0 = 0, && c_{-1} = 0, && c_{-2} = 1. \end{aligned}$$

From (2.8), we also have

$$a_n = aa_{n-1} + ba_{n-2} + ca_{n-3}, \tag{2.20}$$

$$b_n = a_{n+1} - aa_n, \tag{2.21}$$

$$c_n = ca_{n-1}, \tag{2.22}$$

for $n \in \mathbb{N}$. In fact, since $c \neq 0$, it is not difficult to see that equalities (2.20)–(2.22) hold for every $n \in \mathbb{Z}$.

If we take $k = n$ in (2.7), we obtain

$$y_n = a_n y_0 + b_n y_{-1} + c_n y_{-2}, \tag{2.23}$$

for $n \in \mathbb{N}_0$.

From (2.21)–(2.23), we get

$$y_n = a_n y_0 + (a_{n+1} - aa_n) y_{-1} + ca_{n-1} y_{-2}, \tag{2.24}$$

for $n \in \mathbb{N}_0$.

Using (2.24) in (1.17), we obtain

$$x_n = \frac{a_n y_0 + (a_{n+1} - aa_n) y_{-1} + ca_{n-1} y_{-2}}{a_{n-1} y_0 + (a_n - aa_{n-1}) y_{-1} + ca_{n-2} y_{-2}}, \tag{2.25}$$

for $n \in \mathbb{N}_0$.

Case $a = 0$, $b \neq 0$. In this case, equation (1.18) becomes

$$y_{n+1} = by_{n-1} + cy_{n-2}, \tag{2.26}$$

for $n \in \mathbb{N}_0$.

Let

$$\widetilde{b}_1 := b, \qquad \widetilde{c}_1 := c, \qquad \widetilde{d}_1 = 0. \tag{2.27}$$

Now we use a modified procedure, which has been used in the case $a \neq 0$ (the modification is necessary since the coefficient at y_n is zero and the same procedure does not have an effect in the case).

Using the equality (2.26) where n is replaced by $n - 3$, in the equality (2.26) where n is replaced by $n - 1$, we obtain

$$\begin{aligned} y_n &= \widetilde{b}_1 y_{n-2} + \widetilde{c}_1 y_{n-3} + \widetilde{d}_1 y_{n-4}\\ &= \widetilde{b}_1 (by_{n-4} + cy_{n-5}) + \widetilde{c}_1 y_{n-3} + \widetilde{d}_1 y_{n-4}\\ &= \widetilde{c}_1 y_{n-3} + (b\widetilde{b}_1 + \widetilde{d}_1) y_{n-4} + c\widetilde{b}_1 y_{n-5}\\ &= \widetilde{b}_2 y_{n-3} + \widetilde{c}_2 y_{n-4} + \widetilde{d}_2 y_{n-5}, \end{aligned} \tag{2.28}$$

where

$$\widetilde{b}_2 := \widetilde{c}_1, \qquad \widetilde{c}_2 := b\widetilde{b}_1 + \widetilde{d}_1, \qquad \widetilde{d}_2 := c\widetilde{b}_1. \tag{2.29}$$

Using now (2.26) where n is replaced by $n-4$ in (2.28), we obtain

$$\begin{aligned} y_n &= \widetilde{b}_2 y_{n-3} + \widetilde{c}_2 y_{n-4} + \widetilde{d}_2 y_{n-5} \\ &= \widetilde{b}_2 (b y_{n-5} + c y_{n-6}) + \widetilde{c}_2 y_{n-4} + \widetilde{d}_2 y_{n-5} \\ &= \widetilde{c}_2 y_{n-4} + (b\widetilde{b}_2 + \widetilde{d}_2) y_{n-5} + c\widetilde{b}_2 y_{n-6} \\ &= \widetilde{b}_3 y_{n-4} + \widetilde{c}_3 y_{n-5} + \widetilde{d}_3 y_{n-6}, \end{aligned} \tag{2.30}$$

where

$$\widetilde{b}_3 := \widetilde{c}_2, \qquad \widetilde{c}_3 := b\widetilde{b}_2 + \widetilde{d}_2, \qquad \widetilde{d}_3 := c\widetilde{b}_2. \tag{2.31}$$

Based on equalities (2.28)–(2.31), assume that for some $k \in \mathbb{N}$ such that $2 \le k \le n-2$, we have proved that

$$y_n = \widetilde{b}_k y_{n-k-1} + \widetilde{c}_k y_{n-k-2} + \widetilde{d}_k y_{n-k-3}, \tag{2.32}$$

and

$$\widetilde{b}_k := \widetilde{c}_{k-1}, \qquad \widetilde{c}_k := b\widetilde{b}_{k-1} + \widetilde{d}_{k-1}, \qquad \widetilde{d}_k := c\widetilde{b}_{k-1}. \tag{2.33}$$

Then, by using the equality (2.26) where n is replaced by $n-k-2$ in (2.32), we have

$$\begin{aligned} y_n &= \widetilde{b}_k y_{n-k-1} + \widetilde{c}_k y_{n-k-2} + \widetilde{d}_k y_{n-k-3} \\ &= \widetilde{b}_k (b y_{n-k-3} + c y_{n-k-4}) + \widetilde{c}_k y_{n-k-2} + \widetilde{d}_k y_{n-k-3} \\ &= \widetilde{c}_k y_{n-k-2} + (b\widetilde{b}_k + \widetilde{d}_k) y_{n-k-3} + c\widetilde{b}_k y_{n-k-4} \\ &= \widetilde{b}_{k+1} y_{n-k-2} + \widetilde{c}_{k+1} y_{n-k-3} + \widetilde{d}_{k+1} y_{n-k-4}, \end{aligned} \tag{2.34}$$

where

$$\widetilde{b}_{k+1} := \widetilde{c}_k, \qquad \widetilde{c}_{k+1} := b\widetilde{b}_k + \widetilde{d}_k, \qquad d_{k+1} := c\widetilde{b}_k. \tag{2.35}$$

From relations (2.28), (2.29), (2.34), (2.35), and by using the method of mathematical induction we see that relations (2.32) and (2.33) hold for every $k, n \in \mathbb{N}$ such that $2 \le k \le n-1$.

As above, since $c \ne 0$, by using all the relations in (2.33), it is not difficult to see that the sequences $\widetilde{b}_k$, $\widetilde{c}_k$ and $\widetilde{d}_k$, can be also calculated for all nonpositive values of index k.

By using the three recurrent relations in (2.33) with indices $k = 1$, $k = 0$, $k = -1$ and $k = -2$, respectively, after some calculations and repeating use of already calculated terms, we have

$$\widetilde{b}_0 = \frac{\widetilde{d}_1}{c} = 0, \qquad \text{(from (2.27), (2.33))}, \tag{2.36}$$

$$\widetilde{c}_0 = \widetilde{b}_1 = b, \qquad \text{(from (2.27), (2.33))}, \tag{2.37}$$

$$\widetilde{d}_0 = \widetilde{c}_1 - b\widetilde{b}_0 = c - b \cdot 0 = c, \qquad \text{(from (2.27), (2.33), (2.36))}, \tag{2.38}$$

$$\widetilde{b}_{-1} = \frac{\widetilde{d}_0}{c} = 1, \qquad \text{(from (2.33), (2.38))}, \tag{2.39}$$

$$\widetilde{c}_{-1} = \widetilde{b}_0 = 0, \qquad \text{(from (2.33), (2.36))}, \tag{2.40}$$

$$\widetilde{d}_{-1} = \widetilde{c}_0 - b\widetilde{b}_{-1} = b - b \cdot 1 = 0, \qquad \text{(from (2.33), (2.37), (2.39))}, \tag{2.41}$$

$$\widetilde{b}_{-2} = \frac{\widetilde{d}_{-1}}{c} = 0, \qquad \text{(from (2.33), (2.41))}, \tag{2.42}$$

$$\widetilde{c}_{-2} = \widetilde{b}_{-1} = 1, \qquad \text{(from (2.33), (2.39))}, \tag{2.43}$$

$$\widetilde{d}_{-2} = \widetilde{c}_{-1} - b\widetilde{b}_{-2} = 0 - b \cdot 0 = 0, \qquad \text{(from (2.33), (2.40), (2.42))}, \tag{2.44}$$

$$\widetilde{b}_{-3} = \frac{\widetilde{d}_{-2}}{c} = 0, \qquad (\text{from } (2.33), (2.44)), \tag{2.45}$$

$$\widetilde{c}_{-3} = \widetilde{b}_{-2} = 0, \qquad (\text{from } (2.33), (2.42)), \tag{2.46}$$

$$\widetilde{d}_{-3} = \widetilde{c}_{-2} - b\widetilde{b}_{-3} = 1 - b \cdot 0 = 1, \qquad (\text{from } (2.33), (2.43), (2.45)). \tag{2.47}$$

So, from (2.39)–(2.47), we have

$$\begin{aligned} &\widetilde{b}_{-1} = 1, \quad \widetilde{b}_{-2} = 0, \quad \widetilde{b}_{-3} = 0, \\ &\widetilde{c}_{-1} = 0, \quad \widetilde{c}_{-2} = 1, \quad \widetilde{c}_{-3} = 0, \\ &\widetilde{d}_{-1} = 0, \quad \widetilde{d}_{-2} = 0, \quad \widetilde{d}_{-3} = 1. \end{aligned} \tag{2.48}$$

If we choose $k = n - 1$ in (2.32), we obtain

$$y_n = \widetilde{b}_{n-1} y_0 + \widetilde{c}_{n-1} y_{-1} + \widetilde{d}_{n-1} y_{-2}, \tag{2.49}$$

for $n \in \mathbb{N}_0$.

By using (2.33) in (2.49), it follows that

$$y_n = \widetilde{b}_{n-1} y_0 + \widetilde{b}_n y_{-1} + c\widetilde{b}_{n-2} y_{-2}, \tag{2.50}$$

for $n \in \mathbb{N}_0$.

From initial conditions (2.48) we see that sequence $\widetilde{b}_n$ is the solution to the equation (1.18) with $a = 0$, $b \neq 0$, with the backward shifted initial conditions of the sequence a_n defined in (2.2) and (2.8).

Hence,

$$\widetilde{b}_{n-1} = a_n,$$

so from (2.50) we see that (2.24) also holds in the case $a = 0$, $b \neq 0$, and consequently the formula in (2.25).

Case $a = b = 0$. In this case, equation (1.18) becomes

$$y_{n+1} = c y_{n-2}, \tag{2.51}$$

for $n \in \mathbb{N}_0$.

From (2.51) it easily follows that

$$s_{3m-i} = c^m s_{-i},$$

for $m \in \mathbb{N}_0$ and $i = \overline{0,2}$, from which it is easily verified that (2.24) also holds in this case, and consequently formula (2.25).

From (2.25) we have

$$\begin{aligned} x_n &= \frac{a_n \frac{y_0}{y_{-1}} + a_{n+1} - a a_n + c a_{n-1} \frac{y_{-2}}{y_{-1}}}{a_{n-1} \frac{y_0}{y_{-1}} + a_n - a a_{n-1} + c a_{n-2} \frac{y_{-2}}{y_{-1}}} \\ &= \frac{a_n x_0 + a_{n+1} - a a_n + c a_{n-1} x_{-1}^{-1}}{a_{n-1} x_0 + a_n - a a_{n-1} + c a_{n-2} x_{-1}^{-1}} \\ &= \frac{a_n x_{-1} x_0 + (a_{n+1} - a a_n) x_{-1} + c a_{n-1}}{a_{n-1} x_{-1} x_0 + (a_n - a a_{n-1}) x_{-1} + c a_{n-2}}, \end{aligned} \tag{2.52}$$

for every $n \in \mathbb{N}_0$.

Now note that by definition of sequences a_n and s_n we have that

$$a_n = s_n,$$

for $n \geq -2$.

Using this fact in (2.52), we immediately obtain formula (2.1), completing the proof of the theorem. □

Remark 2.2. The method used here is inductive, since in proving hypotheses (2.7), (2.8), (2.32) and (2.33) the method was used. However, it is also a constructive one, since we define sequences a_k, b_k, c_k, $\widetilde{b}_k$, $\widetilde{c}_k$, $\widetilde{d}_k$, appearing in the proof of Theorem 2.1 in a clear constructive way, by using some initial conditions and recurrent relations. Hence, the above method is half-constructive.

To conduct further investigations we need a lemma, which follows, for example, from the Lagrange interpolation formula (see, e.g., [9]), or by using the residue theorem (see, e.g., [47]).

Lemma 2.3. *Let*

$$p(t) = \widetilde{a}_k t^k + \widetilde{a}_{k-1} t^{k-1} + \cdots + \widetilde{a}_1 t + \widetilde{a}_0,$$

and t_j, $j = 1, 2, \ldots, k$, *be the zeros of* $p(t)$, *which are distinct, that is,*

$$t_i \neq t_j, \qquad i \neq j.$$

Then

$$\sum_{j=1}^{k} \frac{t_j^s}{p'(t_j)} = 0$$

for $0 \leq s \leq k-2$, *and*

$$\sum_{j=1}^{k} \frac{t_j^{k-1}}{p'(t_j)} = \frac{1}{\widetilde{a}_k}.$$

Since linear difference equation (1.18) is of the third-order it is practically solvable, since the characteristic equation

$$\lambda^3 - a\lambda^2 - b\lambda - c = 0 \tag{2.53}$$

associated to difference equation (1.18) is a polynomial equation of the third order, so solvable by radicals. From this it follows that difference equation (1.15) is also practically solvable.

The zeros of polynomial equation (2.53) can be obtained by using a standard procedure (see, e.g., [9]). By using the change of variables

$$\lambda = s + \frac{a}{3}, \tag{2.54}$$

in equation (2.53), after some calculation, we obtain

$$s^3 - \left(\frac{a^2}{3} + b\right) s - \frac{2a^3}{27} - \frac{ab}{3} - c = 0. \tag{2.55}$$

Let

$$p := -\frac{a^2}{3} - b \tag{2.56}$$

and

$$q := -\frac{2a^3}{27} - \frac{ab}{3} - c, \tag{2.57}$$

then equation (2.55) is written as

$$s^3 + ps + q = 0. \tag{2.58}$$

We find solution to equation (2.58) in the form

$$s = u + v. \tag{2.59}$$

Substituting (2.59) in (2.58) and requesting that

$$3uv + p = 0, \tag{2.60}$$

we obtain

$$u^3 + v^3 = -q. \tag{2.61}$$

From (2.60) and (2.61), we see that u^3 and v^3 are the zeros of the following polynomial

$$P_2(t) = t^2 + qt - \frac{p^3}{27},$$

that is, they are equal to

$$t_{1,2} = -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}.$$

Hence, the zeros of polynomial (2.53) are

$$\lambda_1 = \frac{a}{3} + \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}, \tag{2.62}$$

$$\lambda_2 = \frac{a}{3} + \varepsilon\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \bar{\varepsilon}\sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}, \tag{2.63}$$

$$\lambda_3 = \frac{a}{3} + \bar{\varepsilon}\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \varepsilon\sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}, \tag{2.64}$$

where ε is a complex zero different from 1, of the polynomial equation

$$z^3 = 1.$$

The character of zeros λ_j, $j = 1, 2, 3$, depends on the sign of the following quantity

$$\Delta := \frac{q^2}{4} + \frac{p^3}{27}, \tag{2.65}$$

the, so called, discriminant.

The following three cases are possible:

1. if $\Delta \neq 0$, then all zeros of equation (2.53) are different. More precisely, if $\Delta < 0$, then they are real and different, while if $\Delta > 0$ then two zeros are complex-conjugate and one is real;
2. if $\Delta = 0$ and $a^2 \neq -3b$, then all zeros of equation (2.53) are real, but two of them are equal;
3. if $\Delta = 0$ and $a^2 = -3b$, then all zeros of equation (2.53) are real and equal.

Case $\Delta \neq 0$. In this case zeros (2.62)–(2.64) of polynomial (2.53) are distinct. Hence, equation (1.18) has general solution in the following form

$$y_n = g_1\lambda_1^n + g_2\lambda_2^n + g_3\lambda_3^n, \qquad n \geq -2, \tag{2.66}$$

where $g_j \in \mathbb{C}$, $j = 1, 2, 3$.

If we apply Lemma 2.3 to the following polynomial

$$p_3(t) = \prod_{j=1}^{3}(t - \lambda_j),$$

we have

$$\sum_{j=1}^{3} \frac{\lambda_j^l}{p_3'(\lambda_j)} = 0, \quad \text{for } l = 0, 1, \tag{2.67}$$

and

$$\sum_{j=1}^{3} \frac{\lambda_j^2}{p_3'(\lambda_j)} = 1. \tag{2.68}$$

Since we need the solution to equation (1.18) satisfying the initial conditions in (1.19), from (2.66)–(2.68), we obtain

$$s_n = \frac{\lambda_1^{n+2}}{p_3'(\lambda_1)} + \frac{\lambda_2^{n+2}}{p_3'(\lambda_2)} + \frac{\lambda_3^{n+2}}{p_3'(\lambda_3)},$$

which can be also written in the following form

$$s_n = \frac{\lambda_1^{n+2}}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} + \frac{\lambda_2^{n+2}}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} + \frac{\lambda_3^{n+2}}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}, \tag{2.69}$$

for $n \geq -2$ (see, e.g., [42]).

From the above consideration and Theorem 2.1 we obtain the following corollary.

Corollary 2.4. *Let parameters a, b, c be complex numbers such that $c \neq 0$ and*

$$27\left(\frac{2a^3}{27} + \frac{ab}{3} + c\right)^2 - 4\left(\frac{a^2}{3} + b\right)^3 \neq 0. \tag{2.70}$$

Then, every well-defined solution to equation (1.15) has the representation given by formula (2.1), where the sequence $(s_n)_{n \geq -2}$ therein is given by (2.69), where λ_j, $j = 1, 2, 3$, are given by (2.62)–(2.64), while p and q are given by (2.56) and (2.57), respectively.

Remark 2.5. By a simple calculation it is proved that condition (2.70) is equivalent to the following one

$$(ab)^2 + 4b^3 - 4a^3c - 18abc - 27c^2 \neq 0.$$

Case $\Delta = 0$, $a^2 \neq -3b$. In this case all zeros of equation (2.53) are real, but two of them are equal. We may assume that $\lambda_1 \neq \lambda_2 = \lambda_3$. General solution to equation (1.18) has the form

$$y_n = \widehat{g}_1 \lambda_1^n + (\widehat{g}_2 + \widehat{g}_3 n)\lambda_2^n, \qquad n \in \mathbb{N}, \tag{2.71}$$

where $\widehat{g}_j \in \mathbb{C}$, $j = 1, 2, 3$.

The solution to (1.18) satisfying the initial conditions in (1.19) in this case can be obtained by a limiting argument (see [35]), and it is given by

$$\begin{aligned} s_n &= \lim_{\lambda_3 \to \lambda_2}\left(\frac{\lambda_1^{n+2}}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} + \frac{\lambda_2^{n+2}}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} + \frac{\lambda_3^{n+2}}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}\right) \\ &= \frac{\lambda_1^{n+2} - (n+2)\lambda_1\lambda_2^{n+1} + (n+1)\lambda_2^{n+2}}{(\lambda_2 - \lambda_1)^2}, \end{aligned}$$

for $n \geq -2$, that is,

$$s_n = \frac{\lambda_1^{n+2} + \big(\lambda_2 - 2\lambda_1 + n(\lambda_2 - \lambda_1)\big)\lambda_2^{n+1}}{(\lambda_2 - \lambda_1)^2}, \tag{2.72}$$

for $n \geq -2$.

From the consideration and Theorem 2.1 we get the following corollary in the case.

Corollary 2.6. *Let parameters a, b, c be complex numbers such that $c \neq 0$, $a^2 \neq -3b$ and*

$$27\left(\frac{2a^3}{27} + \frac{ab}{3} + c\right)^2 - 4\left(\frac{a^2}{3} + b\right)^3 = 0. \tag{2.73}$$

Then, every well-defined solution to equation (1.15) has the representation given by formula (2.1), where the sequence $(s_n)_{n\geq -2}$ therein is given by (2.72), where λ_j, $j = 1, 2, 3$, are given by

$$\lambda_1 = \frac{a}{3} - 2\sqrt[3]{\frac{q}{2}}, \tag{2.74}$$

$$\lambda_{2,3} = \frac{a}{3} - \varepsilon\sqrt[3]{\frac{q}{2}} - \bar{\varepsilon}\sqrt[3]{\frac{q}{2}} = \frac{a}{3} + \sqrt[3]{\frac{q}{2}}, \tag{2.75}$$

while q is given in (2.57).

Case $\Delta = 0$, $a^2 = -3b$. In this case all zeros of equation (2.53) are real and equal to $a/3$. General solution to equation (1.18) has the form

$$y_n = (\tilde{g}_1 + \tilde{g}_2 n + \tilde{g}_3 n^2)\left(\frac{a}{3}\right)^n, \qquad n \in \mathbb{N}, \tag{2.76}$$

where $\tilde{g}_j \in \mathbb{C}$, $j = 1, 2, 3$.

The solution to (1.18) satisfying the initial conditions in (1.19) in this case can be also obtained by a limiting argument (see [35]), and it is given by

$$s_n = \frac{(n+1)(n+2)}{2}\left(\frac{a}{3}\right)^n, \tag{2.77}$$

for $n \geq -2$.

From the consideration and Theorem 2.1 we get the following corollary in this case.

Corollary 2.7. *Let parameters a, b, c be complex numbers such that $c \neq 0$, $a^2 = -3b$ and that condition (2.73) holds. Then, every well-defined solution to equation (1.15) has the representation given by formula (2.1), where the sequence $(s_n)_{n\geq -2}$ therein is given by (2.77).*

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