Analysis of the limit cycle properties of a fast–slow predator–prey system

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Abstract. We consider fast–slow planar systems of predator-prey models with the prey growing much faster than the predator. By using basic differential and integral calculus, Lyapunov functions and phase plane analysis, other than the geometric singular perturbation theory, we derive that the limit cycle exhibits the temporal pattern of a stable relaxation oscillator as a parameter tends to 0, such result shows the coexistence of the predator and the prey with quite diversified time response, which typically happens when the prey population grows much faster than those of predator.

Keywords: predator–prey models, fast–slow, limit cycle, relaxation oscillator.

2010 Mathematics Subject Classification: 34C26, 34C05.

1   Introduction

The existence and uniqueness of the limit cycle of the biological mathematical model, which is not only of important theoretical meaning but also has certain application background, has attracted the interest of scholars for a long time. For the research of limit cycle, Cheng [2] proved the first uniqueness result, and more general uniqueness results for limit cycle in Rosenzweig–MacArthur predator–prey systems have been reached comprehensively [5,6,9,10].

It is increasingly recognized that changes in ecologically important species' traits can occur fast enough to affect interspecific interactions while they are taking place [1], which suggests that these two means of adaptive change, although both occurring at ecological rates, do not have the same effects on community dynamics. Cortez and Ellner [3] considered a modified Rosenzweig–MacArthur model, which implies that the prey responds to ecological conditions faster than the predator does:

$$\begin{cases} \varepsilon \dfrac{dx}{d\tau} = x \left(r - \dfrac{x}{K} - \dfrac{ay}{1 + hx} \right), \\ \dfrac{dy}{d\tau} = \left(-d + \dfrac{ax}{1 + hx} \right) y, \end{cases} \tag{1.1}$$

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where x is prey population, y is predator population, r is the exponential growth rate of the prey without density limitation, K is the prey carrying capacity, a is the encounter rate, h is the handling time, and d is the per capita death rate of the predator, ε is a small positive number that represents the difference in timescales between the prey and predator species, and the prey population has fast dynamics, that is, the prey grows much faster than the predator, which also makes (1.1) a fast–slow system. The numerical simulation results in [3] show that rapidly induced defences tend to stabilize community dynamics and that some behaviors observed in rapidly evolving systems cannot be produced by phenotypic plasticity. It is well known that many predator–prey systems with oscillatory behavior possess a unique limit cycle which is globally asymptotically stable. Our goal in the paper is to give the rigorous mathematical verification for the changes of the limit cycle as ε tends to 0, we find that the limit cycle exhibits the temporal pattern of a relaxation oscillator as ε tends to 0, which coincides with the numerical results in [3] completely.

It is noticed that Hsu and Shi [4] studied relaxation oscillation profile of the limit cycle when $d \to 0$ or $a \to 0$ for the Rosenzweig–MacArthur predator–prey system [7]:

$$\begin{cases} \dfrac{du}{dt} = u(1-u) - \dfrac{muv}{a+u}, \\ \dfrac{dv}{dt} = -dv + \dfrac{muv}{a+u}, \end{cases} \tag{1.2}$$

where $a, m, d > 0$. In the same way, Wang et al. [8] studied the more general class of predator–prey system, here we follow their methods to get the relaxation oscillation profile of limit cycle when $\varepsilon \to 0$. The phase portrait analysis for certain parameters can show that a prey-only or coexistence equilibrium is globally stable; and there exists a periodic solution for other parameters and the periodic solution is the only globally stable limit cycle. The predator population $y(t)$ is near 0 for a long time when ε^{-1} is small for relaxation oscillation profile. Our result is strictly proved by using basic differential and integral calculus, a Lyapunov function, and phase plane analysis.

We review some known results regarding the dynamics of system (1.1) in Section 2.1, and we prove our main results in Section 2.2 for the case $\varepsilon \to 0$. We will use δ_i and C_i, $(i \in \mathbf{N})$, to denote a variety of positive constants, and in Section 2.2 these constants are independent of ε.

2   Asymptotic behavior of limit cycle for ε small

After transformation $\tau = \varepsilon t$, the topologically equivalent form of the system (1.1) is

$$\begin{cases} \dfrac{dx}{dt} = x\left(r - \dfrac{x}{K} - \dfrac{ay}{1+hx}\right), \\ \dfrac{1}{\varepsilon}\dfrac{dy}{dt} = \left(-d + \dfrac{ax}{1+hx}\right)y. \end{cases} \tag{2.1}$$

So we focus on the system (2.1) in the followings to investigate the profile of limit cycle as $\varepsilon \to 0$.

2.1 Existence of limit cycle

It is obvious to obtain the Jacobian matrix of (2.1) at (λ, y_λ):

$$\mathcal{J}_\uparrow = \begin{pmatrix} \frac{\lambda(rhK - 2h\lambda - 1)}{K(1+h\lambda)} & -\frac{a\lambda}{1+h\lambda} \\ \frac{\varepsilon(rK-\lambda)}{K(1+h\lambda)} & 0 \end{pmatrix}. \tag{2.2}$$

When $0 < \lambda < rK$, $\mathrm{Det}(J) = \frac{\varepsilon a\lambda(rK-\lambda)}{K(1+h\lambda)^2} > 0$, $\mathrm{Tr}(J) = \frac{\lambda(rhK-2h\lambda-1)}{K(1+h\lambda)}$. Then we can get the following basic dynamical behavior of (2.1):

1. $\bar{\lambda} = \frac{rhK-1}{2h}$ is the unique Hopf bifurcation value where a locally stable Hopf bifurcation point emerges, and a periodic solution of small amplitude appears near (λ, y_λ);

2. if $\frac{rhK-1}{2h} < \lambda < rK$, (λ, y_λ) is locally asymptotically stable;

3. if $0 < \lambda < \frac{rhK-1}{2h}$, (λ, y_λ) is unstable, and (2.1) has a unique limit cycle which is globally asymptotically orbital stable [4].

2.2 Phase analysis of limit cycle

In this section, we always assume that $a, d > 0$, $rhK > 1$, and $\lambda = \frac{d}{a-hd}$ satisfies $0 < \lambda < \frac{rhK-1}{2h}$, then (2.1) has a unique limit cycle from Section 2.1. Now we define

$$\begin{aligned} f(x,y) &= xf_1(x,y) = x\left(r - \frac{x}{K} - \frac{ay}{1+hx} \right), \\ g(x,y) &= \varepsilon y g_1(x,y) = \varepsilon y\left(-d + \frac{ax}{1+hx} \right). \end{aligned} \tag{2.3}$$

Firstly, we construct an invariant region where the limit cycle is located in the first quadrant. For this reason, an estimate of the unstable manifold $U = \{(x_1(t), y_1(t)) : t \in \mathbf{R}\}$ is given at the saddle point $(rK, 0)$. From the comparison principle, it satisfies $0 < x_1(t) < rK$ for all $t \in \mathbf{R}$; U is above the isocline $y_0(x) = \frac{(1+hx)(rK-x)}{aK}$ when $\lambda < x < rK$. Because it is monotone for $\lambda < x < rK$, denote this part by $\{(x, y_1(x)) : \lambda \le x \le rK\}$ with $y_1(rK) = 0$. Furthermore, we denote

$$y_2(x) = \varepsilon\left(\varepsilon + \frac{1+rhK}{arK} \right)(rK - x), \qquad y_3(x) = \frac{\varepsilon(a-hd)}{a}(rK - x) + \frac{\varepsilon d}{a}\ln x. \tag{2.4}$$

Then we can estimate the unstable manifold U at the saddle point $(rK, 0)$ by $y_2(x)$ and $y_3(x)$.

Lemma 2.1. *The unstable manifold $U = \{(x_1(t), y_1(t)) : t \in \mathbf{R}\}$ at the saddle point $(rK, 0)$ satisfies*

$$\lambda \le x_1(t) \le rK, \qquad y_2(x) \ge y_1(t) = y_1(x(t)) \ge y_3(x). \tag{2.5}$$

Proof. From the two equations in system (2.1), we have

$$\frac{dy}{dx} = \frac{\varepsilon Ky}{(rK-x)(1+hx) - Kay} \cdot \frac{(a-hd)x - d}{x}.$$

Since the unstable manifold satisfies $0 < x_1(t) < rK$ for all $t \in \mathbf{R}$, then along U, we have

$$\frac{dy}{dx} \le \frac{Ky}{-Kay} \cdot \frac{\varepsilon((a-hd)x - d)}{x} = -\frac{\varepsilon((a-hd)x - d)}{ax}.$$

Integrating along the portion of U from $x = rK$ to some $x < \lambda$, we obtain

$$y \geq \frac{\varepsilon(a - hd)}{a}(rK - x) + \frac{\varepsilon d}{a}\ln x = y_3(x),$$

if $(x, y) \in U$ and $\lambda \leq x \leq rK$.

For the upper bound we notice that the tangent line of the unstable manifold is $y = \left(\varepsilon + \frac{(1+rhK)(r-\varepsilon d)}{arK}\right)(rK - x)$, which is below $y = y_2(x)$. So we just have to prove that the vector field $(f(x,y), g(x,y))$ points towards the region below the line $y = y_2(x)$ when $(x,y) = (x, y_2(x))$ and $\lambda < x < rK$, which is equivalent to

$$\left|\frac{dy}{dx}\right| \leq \varepsilon + \frac{1 + rhK}{arK}, \qquad (x, y) = (x, y_2(x)).$$

Let $M = \varepsilon + \dfrac{1 + rhK}{arK}$, then for $(x, y) = (x, y_2(x))$, $\lambda \leq x < rK$,

$$\begin{aligned}
\left|\frac{dy}{dx}\right| &= \frac{\varepsilon KM(rK - x)[(a - hd)x - d]}{x|(rK - x)(1 + hx) - aKM(rK - x)|} \\
&= \frac{\varepsilon KM[(a - hd)x - d]}{x|(1 + hx) - aKM|} \\
&\leq \frac{\varepsilon KM(a - hd)}{|(1 + hx) - aKM|} \\
&\leq \frac{\varepsilon aKM}{|aKM - (1 + h)|} \\
&\leq \varepsilon M.
\end{aligned}$$

That proves that the upper bound $y_1(x) \leq y_2(x)$.   □

From Lemma 2.1, the unstable manifold achieves its maximum y-value when $x = \lambda$, and the maximum value y_* can be estimated as

$$\frac{\varepsilon(a - hd)}{a}(rK - \lambda) + \frac{\varepsilon d}{a}\ln\lambda \leq y_* \leq \varepsilon\left(\varepsilon + \frac{1 + rhK}{arK}\right)(rK - \lambda). \tag{2.6}$$

From the phase portrait, the limit cycle is below the unstable manifold U, then we have the following upper bound for the location of the limit cycle.

Lemma 2.2. *Define*

$$y_4(x) = \begin{cases} y_2(x), & \lambda \leq x \leq rK, \\ y_2(\lambda), & 0 \leq x \leq \lambda. \end{cases}$$

Then the orbit of the limit cycle $\Sigma = \{(x(t), y(t)) : 0 \leq t \leq T\}$ satisfies

$$\Sigma \subset \{(x, y) : 0 < x < rK, 0 < y < y_4(x)\} \equiv R_1.$$

By constructing a more precise region $R_2 \subset R_1$ containing Σ, we prove that for a sub-region R_3 containing (λ, y_λ), $\Sigma \cap R_3 = \emptyset$. Define

$$R_3 = \{(x(t), y(t)) \in \mathbf{R}_+^2 : W(x, y) \leq W(rK - 1/h - \lambda, y_\lambda)\}, \tag{2.7}$$

where

$$W(x, y) = \varepsilon \int_\lambda^x \frac{g(\xi) - d}{g(\xi)}d\xi + \int_{y_\lambda}^y \frac{\eta - y_\lambda}{\eta}d\eta, \tag{2.8}$$

where $g(x) = \frac{ax}{1 + hx}$. Note that $(rK - 1/h - \lambda, y_\lambda)$ is the reflection of (λ, y_λ) with respect to the line $x = \frac{rhK - 1}{2h}$.

Lemma 2.3. *Let R_3 be defined as in (2.7). Then R_3 is a bounded convex subset of $\mathbf{R}_+^2$ containing (λ, y_λ), and $\Sigma \cap R_3 = \varnothing$. In particular $\Sigma \subset R_2 \equiv R_1 \setminus R_3$.*

Proof. From the definition in (2.8), $W(x,y) = W_1(x) + W_2(y)$, where $W_1(x) = \varepsilon \int_\lambda^x \frac{g(\xi)-d}{g(\xi)} d\xi$ and $W_2(y) = \int_{y_\lambda}^y \frac{\eta - v_\lambda}{\eta} d\eta$. Because $W_1'(x) = \varepsilon(g(x)-d)/g(x)$, then $W_1(x)$ is strictly decreasing in $[0, \lambda)$ and is strictly increasing in (λ, ∞); in the same way, because $W_2'(y) = 1 - (y_\lambda/y)$, $W_2(y)$ is strictly decreasing in $[0, y_\lambda)$ and is strictly increasing in (y_λ, ∞). Thus $W(x,y)$ reaches the global minimum at the unique critical point (λ, y_λ), and every level curve of $W(x,y)$ is a bounded closed curve. The level curves have convex boundary since W_1 and W_2 are both convex one-variable functions. For R_3 be defined as in (2.7), $(rK - 1/h - \lambda, y_\lambda)$ is the right-most point of R_3. Hence for any solution orbit $(x(t), y(t))$ passing through $(x,y) \in R_3 \setminus \{(rK - 1/h - \lambda, y_\lambda)\}$, $\dot{W}(x(t), y(t)) = \varepsilon[g(x) - g(\lambda)] \cdot [y_0(x) - y_0(\lambda)] > 0$. Particularly, for $(x,y) \in \partial R_3 \setminus \{(rK - 1/h - \lambda, y_\lambda)\}$, the vector field $(f(x,y), g(x,y))$ points outwards. Thus from the properties of periodic orbit, $\Sigma \cap R_3 = \varnothing$. $\square$

From Lemma 2.2 and Lemma 2.3, we obtain an invariant region R_2 where the limit cycle is located in. Next we give some estimates for extremal points on the orbit of limit cycle as $\varepsilon \to 0^+$. We assume that the other three parameters $rhK > 1$ and $a, d > 0$ are fixed, and $\varepsilon > 0$ is enough small (thus $\varepsilon < 1$). Define

$$\begin{aligned} x_{\lambda,-} &= \min\{x(t) : (x(t), y(t)) \in \Sigma\}, & x_{\lambda,+} &= \max\{x(t) : (x(t), y(t)) \in \Sigma\}, \\ y_{\lambda,-} &= \min\{y(t) : (x(t), y(t)) \in \Sigma\}, & y_{\lambda,+} &= \max\{y(t) : (x(t), y(t)) \in \Sigma\}. \end{aligned} \tag{2.9}$$

We notice that both the upper and lower portions of the limit cycle are monotone functions, hence we define

$$\Sigma = \{(x, y_+(\lambda, x)) : x_{\lambda,-} \leq x \leq x_{\lambda,+}\} \cup \{(x, y_-(\lambda, x)) : x_{\lambda,-} \leq x \leq x_{\lambda,+}\}, \tag{2.10}$$

such that $y_-(\lambda, x) < y_0(x) < y_+(\lambda, x)$ for $x_{\lambda,-} < x < x_{\lambda,+}$. That is, $\{x, y_+(\lambda, x)\}$ is the upper portion of the limit cycle Σ, and $\{x, y_-(\lambda, x)\}$ is the lower portion. From the equations, it is easy to see that $x_{\lambda,-}$ and $x_{\lambda,+}$ are achieved when Σ intersects with the isocline $y = y_0(x)$, and $y_{\lambda,-}$ and $y_{\lambda,+}$ are achieved when Σ intersects with the line $x = \lambda$. The estimation is mainly based on the inner boundary of the region R_2, i.e. the level curve $\Sigma_1 = \{(x,y) : W(x,y) = W(rK - 1/h - \lambda, y_\lambda)\}$. Thus we also define

$$\begin{aligned} x_{1,\lambda} &= \min\{x : (x,y) \in \Sigma_1\}, & x_{2,\lambda} &= \max\{x : (x,y) \in \Sigma_1\}, \\ y_{1,\lambda} &= \min\{y : (x,y) \in \Sigma_1\}, & y_{2,\lambda} &= \max\{y : (x,y) \in \Sigma_1\}, \end{aligned} \tag{2.11}$$

and

$$\Sigma_1 = \{(x, y_5(x)) : x_{1,\lambda} \leq x \leq x_{2,\lambda}\} \cup \{(x, y_6(x)) : x_{1,\lambda} \leq x \leq x_{2,\lambda}\}, \tag{2.12}$$

such that $y_6(x) < y_0(x) < y_5(x)$ for $x_{1,\lambda} < x < x_{2,\lambda}$. Notice that $\nabla W = (\frac{\varepsilon(g(x)-d)}{g(x)}, \frac{y-y_\lambda}{y})$, thus $y_{1,\lambda}$ and $y_{2,\lambda}$ are the two intersects of $W(x,y) = W(rK - 1/h - \lambda, y_\lambda)$ with the line $x = \lambda$. Also $x_{2,\lambda} = rK - 1/h - \lambda$, and $x_{1,\lambda}$ satisfies $W(x_{1,\lambda}, y_\lambda) = W(rK - 1/h - \lambda, y_\lambda)$ with $x_{1,\lambda} < \lambda$.

To obtain the global asymptotical behavior of the limit cycle Σ, we divide the orbit with four reference points (see Figure 2.1):

$$\begin{aligned} Q_1 &= (\lambda, y_{\lambda,+}), & Q_2 &= (\lambda, y_{\lambda,-}), \\ Q_3 &= \left(\frac{rhK - 1}{2h}, y_-\left(\frac{rhK - 1}{2h}\right)\right), & Q_4 &= \left(\frac{rhK - 1}{2h}, y_+\left(\frac{rhK - 1}{2h}\right)\right). \end{aligned} \tag{2.13}$$
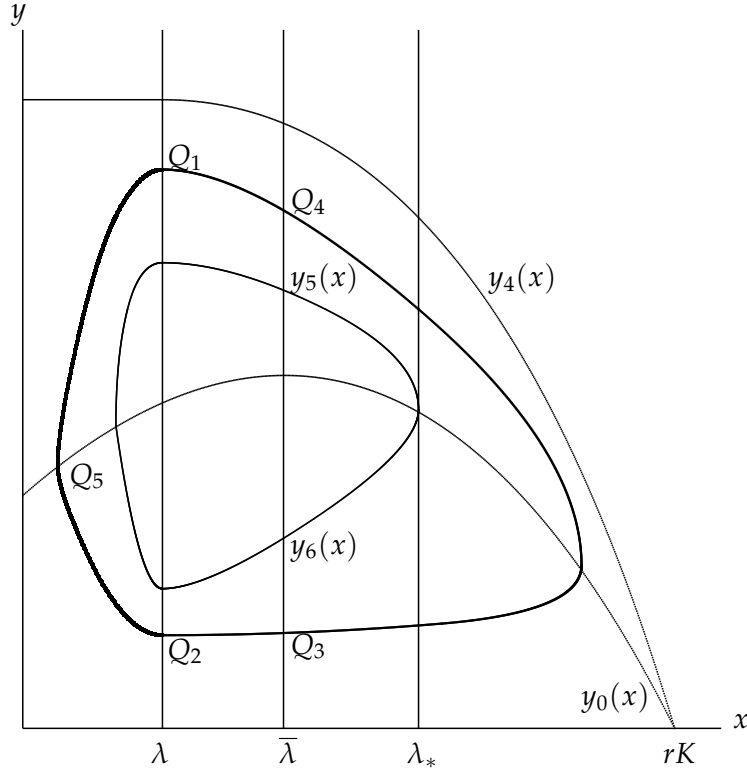
Figure 2.1: Illustration of the phase portrait (not up to scale) and the limit cycle in the proof. The isoclines are the thin solid curves: $x = 0$, $y = 0$, $x = \lambda$ and the parabola $y = y_0(x)$; the limit cycle is the thick solid curve $Q_1Q_2Q_3Q_4$; the boundary of the invariant region R_3: $y = y_4(x)$ is the outer boundary (together with $x = 0$ and $y = 0$); $y = y_5(x)$ and $y = y_6(x)$ are the upper and lower portions of inner boundary respectively; the line $\lambda_* = rK - 1/h - \lambda$ is the reflection of $x = \lambda$ with respect to $\overline{\lambda} = (rhK - 1)/2h$. (This graph is essentially from [4].)

Let $T = T(\lambda)$ be the period of Σ. Then $T = T_1 + T_2 + T_3 + T_4$, where T_i is the time taken from Q_i to Q_{i+1} (with $Q_5 = Q_1$). We also assume that $x(0) = \lambda$ and $y(0) = y_{\lambda,+}$, *i.e.* the orbit starts from the highest point of $y(t)$. Now we obtain our main result.

Theorem 2.4. *Let $\Sigma = \{(x(t), y(t)) : t \in R\}$ be the orbit of the unique periodic solution of (2.1) when $0 < \lambda < \frac{rhK-1}{2h}$. Assume that $rhK > 1$ and $a > d > 0$ are fixed, the extremal points of Σ are defined as in (2.10) and Q_i, T_i ($i = 1,2,3,4$) and the period T are defined as above. When $\varepsilon > 0$ is sufficiently small, then there exist constants $C_1 > 0$, $C_2 > 0$ independent of ε, such that $\varepsilon^{-1}C_1 \geq T \geq \varepsilon^{-1}C_2$. Moreover, for $\varepsilon > 0$ is sufficiently small, there exists some $C_i > 0$, such that*

$$\varepsilon^{-1}C_3 \geq T_1 \geq \varepsilon^{-1}C_4, \quad C_5 \geq T_2 \geq C_6, \quad \varepsilon^{-1}C_7 \geq T_3 \geq \varepsilon^{-1}C_8, \quad C_9 \geq T_4 \geq C_{10}, \qquad (2.14)$$

as $\varepsilon \to 0^+$.

Proof. We prove the theorem in several steps.

Step 1: We show that

$$T_1 \geq \varepsilon^{-1}d^{-1}\left(1 - \frac{x_{\lambda,-}}{\lambda}\right)^{-1}\ln\left(\frac{y_{\lambda,+}}{y_{\lambda,-}}\right). \qquad (2.15)$$

We define $x_{\lambda,-} = \lambda(1 - \delta_1)$ for some $0 < \delta_1 < 1$. Then for $0 < t < T_1$, $\lambda(1 - \delta_1) \leq x(t) \leq \lambda$,

and from the equation of $y(t)$,

$$y' = \varepsilon y \left(-d + \frac{ax}{1 + hx} \right) \geq \varepsilon y \left(-d + \frac{a\lambda(1 - \delta_1)}{1 + h\lambda(1 - \delta_1)} \right) = -\varepsilon y \left(\frac{d\delta_1(a - hd)}{a - hd\delta_1} \right)$$
$$\geq -\varepsilon d\delta_1 y. \tag{2.16}$$

Thus $y(t) \geq y(0) \exp(-\varepsilon d\delta_1 t)$, which leads to

$$T_1 \geq \varepsilon^{-1} d^{-1} \delta_1^{-1} \ln \left(\frac{y_{\lambda,+}}{y_{\lambda,-}} \right) \geq \varepsilon^{-1} d^{-1} \left(1 - \frac{x_{\lambda,-}}{\lambda} \right)^{-1} \ln \left(\frac{y_{\lambda,+}}{y_{\lambda,-}} \right). \tag{2.17}$$

Step 2: We show that for any $0 < \delta_6 < 1$, there exists constant $C_{11} > 0$ such that

$$T_1 \leq \varepsilon^{-1} (\delta_6 d)^{-1} \ln \left(\frac{y_{\lambda,+}}{y_{\lambda,-}} \right) + C_{11}. \tag{2.18}$$

We reconsider the portion of Σ in $(0, T_1)$ again, the orbit does reach $x = \lambda(1 - \delta_6)$.
For $t \in (0, T_1)$, similar to Step 1,

$$y' = \varepsilon y \left(-d + \frac{ax}{1 + hx} \right) \leq \varepsilon y \left(-d + \frac{a\lambda(1 - \delta_6)}{1 + h\lambda(1 - \delta_6)} \right) = -\varepsilon y \left(\frac{d\delta_6(a - hd)}{a - hd\delta_6} \right)$$
$$\leq \frac{-\varepsilon d\delta_6 y}{1 + \sigma}, \tag{2.19}$$

for any small $\sigma > 0$. Since we can choose δ_6 arbitrarily, without lose of generality we can take $\sigma = 0$. Thus we have $y(t) \leq y(0) \exp(-\varepsilon d\delta_6 t)$, and

$$T_1 \leq \varepsilon^{-1} (\delta_6 d)^{-1} \ln \left(\frac{y_{\lambda,+}}{y_{\lambda,-}} \right) + C_{11}. \tag{2.20}$$

Step 3: We show there exist constants $\delta_2, \delta_3 > 0$ such that when $0 < \lambda < \delta_3$,

$$\ln \left(\frac{K(rhK - 1)}{rhK + 1} \right) - \ln \left(\frac{K\lambda}{rK - \lambda} \right) \leq T_2 \leq (\delta_2 a)^{-1} \ln \left(\frac{rhK - 1}{2h\lambda} \right) + \left(\frac{rhK - 1}{2h} - \lambda \right). \tag{2.21}$$

For $T_1 \leq t \leq T_1 + T_2$. We have $\lambda \leq x(t) \leq \frac{rhK-1}{2h}$. From the equation of $x(t)$,

$$x' = g(x)[y_0(x) - y] \geq g(x)[y_0(x) - y_6(x)], \tag{2.22}$$

which follows from Lemma 2.3 that the limit cycle is below the level curve $(x, y_6(x))$ in this portion. Since $y_0(x)$ is concave while $y_6(x)$ is convex, then the minimum of $y_0(x) - y_6(x)$ on the interval $(\lambda, \frac{rhK-1}{2h})$ must achieve at either $x = \lambda$ or $x = \frac{rhK-1}{2h}$. Thus there exist $\delta_2, \delta_3 > 0$ such that when $0 < \lambda < \delta_3$, then

$$y_0(x) - y_6(x) \geq \min \left\{ y_0(\lambda) - y_6(\lambda), y_0 \left(\frac{rhK - 1}{2h} \right) - y_6 \left(\frac{rhK - 1}{2h} \right) \right\} \geq \delta_2 > 0. \tag{2.23}$$

Now from (2.22) and (2.23), we have

$$\frac{1 + hx}{x} \frac{dx}{dt} \geq a\delta_2 \quad \text{and} \quad \ln \left(\frac{rhK - 1}{2h\lambda} \right) + \left(\frac{rhK - 1}{2h} - \lambda \right) \geq a\delta_2 T_2. \tag{2.24}$$

On the other hand, from the equation of $x(t)$,

$$x' = g(x)[y_0(x) - y] \leq x \left(r - \frac{x}{K} \right).$$ (2.25)

then an integration of (2.25) gives

$$\ln \left(\frac{K(rhK-1)}{rhK+1} \right) - \ln \left(\frac{K\lambda}{rK-\lambda} \right) \leq T_2.$$ (2.26)

Step 4: We show that

$$\varepsilon^{-1} \left(\frac{arK}{1+rhK} - d \right)^{-1} \ln \left(\frac{y_+ \left(\frac{rhK-1}{2h} \right)}{y_- \left(\frac{rhK-1}{2h} \right)} \right)$$

$$\geq T_3 \geq \varepsilon^{-1} \left(\frac{a(rhK-1)}{h(rhK+1)} - d \right)^{-1} \ln \left(\frac{y_+ \left(\frac{rhK-1}{2h} \right)}{y_- \left(\frac{rhK-1}{2h} \right)} \right).$$ (2.27)

For this portion, $x(t) \geq \frac{rhK-1}{2h}$, and from the equation of $y(t)$, we obtain

$$y' = \varepsilon y(-d + g(x)) \geq \varepsilon y \left(-d + g \left(\frac{rhK-1}{2h} \right) \right) = \varepsilon y \left(\frac{a(rhK-1)}{h(rhK+1)} - d \right).$$ (2.28)

Hence $y(t) \geq y(T_1 + T_2) \exp \left(\varepsilon \left(\frac{a(rhK-1)}{h(rhK+1)} - d \right) t \right)$, in particular

$$y_+ \left(\frac{rhK-1}{2h} \right) \geq y_- \left(\frac{rhK-1}{2h} \right) \exp \left(\varepsilon \left(\frac{a(rhK-1)}{h(rhK+1)} - d \right) T_3 \right).$$ (2.29)

On the other hand, $x(t) < rK$, and from the equation of $y(t)$, we obtain

$$y' = \varepsilon y(-d + g(x)) \leq \varepsilon y(-d + g(rK)) = \varepsilon y \left(\frac{arK}{1+rhK} - d \right).$$ (2.30)

Hence $y(t) \leq y(T_1 + T_2) \exp \left(\varepsilon \left(\frac{arK}{1+rhK} - d \right) t \right)$, in particular

$$y_+ \left(\frac{rhK-1}{2h} \right) \leq y_- \left(\frac{rhK-1}{2h} \right) \exp \left(\varepsilon \left(\frac{arK}{1+rhK} - d \right) T_3 \right).$$ (2.31)

Step 5: We show there exist constants $\delta_4, \delta_5 > 0$ such that when $0 < \lambda < \delta_5$,

$$\ln \left(\frac{K(rhK-1)}{rhK+1} \right) - \ln \left(\frac{K\lambda}{rK-\lambda} \right) \leq T_4 \leq (\delta_5 a)^{-1} \ln \left(\frac{rhK-1}{2h\lambda} \right) + \left(\frac{rhK-1}{2h} - \lambda \right).$$ (2.32)

This is similar to Step 3. Now we have

$$x' = g(x)[y_0(x) - y] \leq g(x)[y_0(x) - y_5(x)] \leq g(x) \left[y_0 \left(\frac{rhK-1}{2h} \right) - y_5 \left(\frac{rhK-1}{2h} \right) \right],$$ (2.33)

which follows from Lemma 2.3 that the limit cycle is above the level curve $(x, y_5(x))$ in this portion. $y_0(x)$ is increasing while $y_5(x)$ is decreasing in $[\lambda, \frac{rhK-1}{2h})$, and $y_0(x) < y_5(x)$. Similar to Step 3. We obtain that when $0 < \lambda < \delta_5$,

$$|x'| \geq \delta_5 g(x).$$ (2.34)

The remaining part is same as Step 3.

Step 6: We show that there exist constants $y_1, y_2 > 0$ such that $y_{\lambda,+} < y_1$ and $y_2 < y_{\lambda,-}$.

From Lemma 2.1 and (2.6), we obtain the estimate of upper bound of $y_{\lambda,+}$ by letting $y_1 = \varepsilon\left(\varepsilon + \frac{1+rhK}{arK}\right)$. For the estimate of $y_2 > 0$, we notice that any solution orbit satisfies

$$\frac{dx}{dy} = \frac{g(x)}{g(x) - d} \cdot \frac{y_0(x) - y}{\varepsilon y}. \tag{2.35}$$

Recall that $Q_1 = (\lambda, y_{\lambda,+})$ and $Q_2 = (\lambda, y_{\lambda,-})$ are the highest and lowest points on the orbit of the limit cycle Σ. Let the leftmost point on Σ be $Q_5 = (x_{\lambda,-}, y_*)$. Then from (2.35), we obtain that

$$\int_{y_{\lambda,-}}^{y_*} \frac{y_0(x_2(y)) - y}{\varepsilon y} dy = \int_{\lambda}^{x_{\lambda,-}} \frac{g(x) - d}{g(x)} dx = \int_{y_{\lambda,+}}^{y_*} \frac{y_0(x_1(y)) - y}{\varepsilon y} dy, \tag{2.36}$$

where $(x_1(y), y)$ for $y_* \le y \le y_{\lambda,+}$, represents the orbit $Q_1 Q_5$, and $(x_2(y), y)$ for $y_{\lambda,-} \le y_* \le y$, represents the orbit $Q_5 Q_2$. For the last integral in (2.36),

$$\int_{y_{\lambda,+}}^{y_*} \frac{y_0(x) - y}{\varepsilon y} dy = \int_{y_*}^{y_{\lambda,+}} \frac{y - y_0(x)}{\varepsilon y} dy \le \int_{y_*}^{y_{\lambda,+}} \frac{y - y_*}{\varepsilon y} dy$$

$$= \frac{1}{\varepsilon}(y_{\lambda,+} - y_* - y_* \ln y_{\lambda,+} + y_* \ln y_*), \tag{2.37}$$

Since $0 < y_{\lambda,+} < y_1$ for small ε, then the right-hand side of (2.37) is bounded. On the other hand, for the first integral in (2.36),

$$\int_{y_{\lambda,-}}^{y_*} \frac{y_0(x) - y}{\varepsilon y} dy \ge \int_{y_{\lambda,-}}^{y_*} \frac{y_* - y}{\varepsilon y} dy = \frac{1}{\varepsilon}(y_{\lambda,-} - y_* - y_* \ln y_{\lambda,-} + y_* \ln y_*). \tag{2.38}$$

Thus $-\ln y_{\lambda,-}$ is bounded from above from (2.36), (2.37), (2.38), and consequently $y_{\lambda,-}$ is bounded from below by some $y_2 > 0$ for all small $\varepsilon > 0$.

Step 7: The completion of the proof.

From Step 1 and Step 2, for any $\delta_7 > 0$,

$$\varepsilon^{-1}((1-\delta_6)d)^{-1} \ln\left(\frac{y_{\lambda,+}}{y_{\lambda,-}}\right) \le T_1 \le \varepsilon^{-1}\left(\frac{(1+\delta_7)d}{\delta_6}\right)^{-1} \ln\left(\frac{y_1}{y_2}\right) + C_{11}. \tag{2.39}$$

Hence we obtain the estimate for T_1 in the theorem, since all constants except ε are independent of ε. The estimate for T_3 can also be obtained from Step 4 and Step 6 since $y_+\left(\frac{rhK-1}{2h}\right) < y_{\lambda,+} < y_1$ and $y_-\left(\frac{rhK-1}{2h}\right) > y_{\lambda,-} > y_2$. The estimates for T_i for $i = 2, 4$ are clear from Step 3 and Step 5. This completes the proof. $\qquad\square$

2.3 Conclusion

The periodic solution of the system (2.1) is unique thus exists a globally stable limit cycle. It is assumed that the prey population has fast dynamics in this paper and it studies the asymptotic behavior of the limit cycle of (2.1) when ε tends to zero. The predator population $y(t)$ is near 0 for a long time when ε^{-1} is small for relaxation oscillation profile (see Figure 2.2). We show that the period T of Σ tends to ∞ as $\varepsilon \to 0$, see Theorem 2.4 for a more mathematical description.
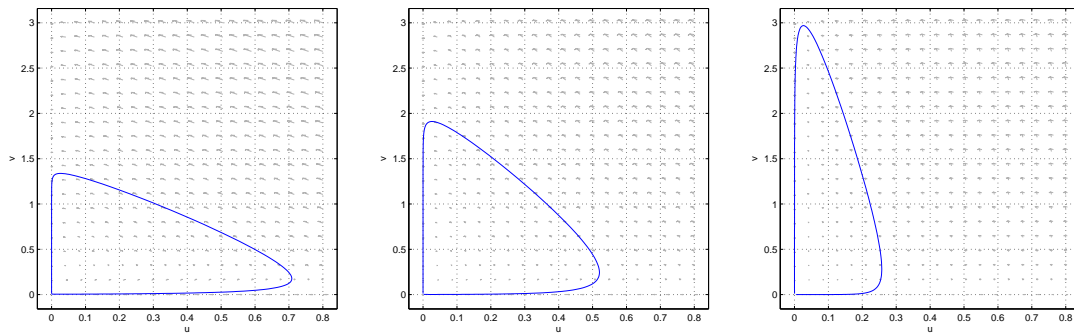
Figure 2.2: Phase portraits of (2.1) with parameters $a = 4, K = 1, r = 1, h = 0.5, d = 0.1$. (Left): $\varepsilon = 0.05$, (Middle): $\varepsilon = 0.35$, (Right): $\varepsilon = 0.8$.

Acknowledgements

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