

# Existence of positive solutions for singular impulsive differential equations with integral boundary conditions on an infinite interval in Banach spaces\*

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**Abstract** In this paper, the Mönch fixed point theorem is used to investigate the existence of positive solutions for the second-order boundary value problem with integral boundary conditions of nonlinear impulsive differential equations on an infinite interval in a Banach space.

**Keywords:** Impulsive singular differential equations; Positive solutions; Mönch fixed point theorem; Measure of non-compactness

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## 1 Introduction

The theory of boundary-value problems with integral boundary conditions for ordinary differential equations arises in different areas of applied mathematics and physics. For example, heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics can be reduced to the nonlocal problems with integral boundary conditions. In recent years, the theory of ordinary differential equations in Banach space has become a new important branch of investigation (see, for example, [1-4] and references therein). In a recent paper [7], using the cone theory and monotone iterative technique, Zhang et al investigated the existence of minimal nonnegative solution of the following nonlocal boundary value problems for second-order nonlinear impulsive differential equations on an infinite interval with an infinite number of impulsive times

$$\begin{cases} -x''(t) = f(t, x(t), x'(t)), & t \in J, t \neq t_k, \\ \Delta x|_{t=t_k} = I_k(x(t_k)), & k = 1, 2, \dots, \\ \Delta x'|_{t=t_k} = \bar{I}_k(x(t_k)), & k = 1, 2, \dots, \\ x(0) = \int_0^\infty g(t)x(t)dt, & x'(\infty) = 0, \end{cases}$$

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where  $J = [0, +\infty)$ ,  $f \in C(J \times R^+ \times R^+, R^+)$ ,  $R^+ = [0, +\infty]$ ,  $0 < t_1 < t_2 < \dots < t_k < \dots$ ,  $t_k \rightarrow \infty$ ,  $I_k \in C[R^+, R^+]$ ,  $\bar{I}_k \in C[R^+, R^+]$ ,  $g(t) \in C[R^+, R^+]$ , with  $\int_0^\infty g(t)dt < 1$ .

Very recently, by using Schauder fixed point theorem, Guo [6] obtained the existence of positive solutions for a class of  $n$ th-order nonlinear impulsive singular integro-differential equations in a Banach space. Motivated by Guo's work, in this paper, we shall use the cone theory and the Mönch fixed point theorem to investigate the positive solutions for a class of second-order nonlinear impulsive integro-differential equations in a Banach space.

Consider the following boundary value problem with integral boundary conditions for second-order nonlinear impulsive integro-differential equation of mixed type in a real Banach space  $E$ :

$$\begin{cases} -x''(t) = f(t, x(t), x'(t)), & t \in J, t \neq t_k, \\ \Delta x|_{t=t_k} = I_{0k}(x(t_k), x'(t_k)), \\ \Delta x'|_{t=t_k} = I_{1k}(x(t_k), x'(t_k)), & k = 1, 2, \dots, \\ x(0) = \int_0^\infty g(t)x(t)dt, & x'(\infty) = x_\infty, \end{cases} \quad (1)$$

where  $J = [0, \infty)$ ,  $J_+ = (0, \infty)$ ,  $0 < t_1 < t_2 < \dots < t_k < \dots, t_k \rightarrow \infty$ ,  $J_k = (t_k, t_{k+1}]$  ( $k = 1, 2, \dots$ ),  $J'_+ = J_+ \setminus \{t_1, \dots, t_k, \dots\}$ ,  $f$  may be singular at  $t = 0$  and  $x = \theta$  or  $x' = \theta$ .  $I_{0k}$  and  $I_{1k}$  may be singular at  $x = \theta$  or  $x' = \theta$ ,  $\theta$  is the zero element of  $E$ ,  $g(t) \in L[0, \infty)$  with  $\int_0^\infty g(t)dt < 1$ ,  $\int_0^\infty tg(t)dt < \infty$ ,  $x(\infty) = \lim_{t \rightarrow \infty} x'(t)$ ,  $x_\infty \geq x_0^*$ ,  $x_0^* \in P_+$ ,  $P_+$  is the same as that defined in Section 2.  $\Delta x|_{t=t_k}$  denotes the jump of  $x(t)$  at  $t = t_k$ , i.e.,  $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$ , where  $x(t_k^+)$ ,  $x(t_k^-)$  represent the right and left limits of  $x(t)$  at  $t = t_k$  respectively.  $\Delta x'|_{t=t_k}$  has a similar meaning for  $x'(t)$ .

The main features of the present paper are as follows: Firstly, compared with [7], the second-order boundary value problem we discussed here is in Banach spaces and nonlinear term permits singularity not only at  $t = 0$  but also at  $x, x' = \theta$ . Secondly, compared with [6], the relative compact conditions we used are weaker.

## 2 Preliminaries and several lemmas

Let  $PC[J, E] = \{x|x(t) : J \rightarrow E, x \text{ is continuous at } t \neq t_k \text{ and left continuous at } t = t_k, x(t_k^+) \text{ exists, } k = 1, 2, \dots\}$ .  $PC^1[J, E] = \{x|x \in PC[J, E], x'(t) \text{ exists at } t \neq t_k \text{ and } x'(t_k^+), x'(t_k^-) \text{ exist } k = 1, 2, \dots\}$ .

$$FPC[J, E] = \left\{ x \in PC[J, E] : \sup_{t \in J} \frac{\|x(t)\|}{t+1} < +\infty \right\},$$

$$DPC^1[J, E] = \left\{ x \in PC^1[J, E] : \sup_{t \in J} \frac{\|x(t)\|}{t+1} < +\infty, \text{ and } \sup_{t \in J} \|x'(t)\| < +\infty \right\}.$$

Obviously,  $FPC[J, E]$  is a Banach space with norm

$$\|x\|_F = \sup_{t \in J} \frac{\|x(t)\|}{t+1}.$$

and  $DPC^1[J, E]$  is also a Banach space with norm

$$\|x\|_D = \max\{\|x\|_F, \|x'\|_1\},$$

where

$$\|x'\|_1 = \sup_{t \in J} \|x'(t)\|.$$

The basic space using in this paper is  $DPC^1[J, E]$ .

Let  $P$  be a normal cone in  $E$  with normal constant  $N$  which defines a partial ordering in  $E$  by  $x \leq y$ . If  $x \leq y$  and  $x \neq y$ , we write  $x < y$ . Let  $P_+ = P \setminus \{\theta\}$ . So,  $x \in P_+$  if and only if  $x > \theta$ . For details on cone theory, see [4].

Let  $P_{0\lambda} = \{x \in P : x \geq \lambda x_0^*\}$ , ( $\lambda > 0$ ). Obviously,  $P_{0\lambda} \subset P_+$  for any  $\lambda > 0$ . When  $\lambda = 1$ , we write  $P_0 = P_{01}$ , i.e.  $P_0 = \{x \in P : x \geq x_0^*\}$ . Let  $P(F) = \{x \in FPC[J, E] : x(t) \geq \theta, \forall t \in J\}$ , and  $P(D) = \{x \in DPC^1[J, E] : x(t) \geq \theta, x'(t) \geq \theta, \forall t \in J\}$ . It is clear,  $P(F)$ ,  $P(D)$  are cones in  $FPC[J, E]$  and  $DPC^1[J, E]$ , respectively. A map  $x \in DPC^1[J, E] \cap C^2[J_+, E]$  is called a positive solution of BVP (1) if  $x \in P(D)$  and  $x(t)$  satisfies BVP (1).

Let  $\alpha, \alpha_F, \alpha_D$  denote the Kuratowski measure of non-compactness in  $E, FPC[J, E], DPC^1[J, E]$ . For details on the definition and properties of the measure of non-compactness, the reader is referred to references [1-4].

Denote

$$\lambda^* = \min \left\{ \frac{\int_0^\infty tg(t)dt}{1 - \int_0^\infty g(t)dt}, 1 \right\}.$$

Let us list the following assumptions, which will stand throughout this paper.

(H<sub>1</sub>)  $f \in C[J_+ \times P_{0\lambda} \times P_{0\lambda}, P]$  for any  $\lambda > 0$  and there exist  $a, b, c \in L[J_+, J]$  and  $z \in C[J_+ \times J_+, J]$  such that

$$\|f(t, x, y)\| \leq a(t) + b(t)z(\|x\|, \|y\|), \quad \forall t \in J_+, x \in P_{0\lambda^*}, y \in P_{0\lambda^*}$$

and

$$\frac{\|f(t, x, y)\|}{c(t)(\|x\| + \|y\|)} \rightarrow 0, \quad \text{as } x \in P_{0\lambda^*}, y \in P_{0\lambda^*}, \|x\| + \|y\| \rightarrow \infty,$$

uniformly for  $t \in J_+$ , and

$$\int_0^\infty a(t)dt = a^* < \infty, \quad \int_0^\infty b(t)dt = b^* < \infty, \quad \int_0^\infty c(t)(1+t)dt = c^* < \infty.$$

(H<sub>2</sub>)  $I_{ik} \in C[P_{0\lambda} \times P_{0\lambda}, P]$  for any  $\lambda > 0$  and there exist  $F_i \in L[J_+ \times J_+, J_+]$  and constants  $\eta_{ik}, \gamma_{ik}$ , ( $i = 0, 1, k = 1, 2, \dots$ ) such that

$$\|I_{ik}(x, y)\| \leq \eta_{ik}F_i(\|x\|, \|y\|), \quad x \in P_{0\lambda^*}, y \in P_{0\lambda^*} \quad (i = 0, 1),$$

and

$$\frac{\|I_{ik}(t, x, y)\|}{\gamma_{ik}(\|x\| + \|y\|)} \rightarrow 0, \quad \text{as } x \in P_{0\lambda^*}, y \in P_{0\lambda^*}, \|x\| + \|y\| \rightarrow \infty,$$

uniformly for ( $i = 0, 1, k = 1, 2, \dots$ ), here

$$0 < \eta_i^* = \sum_{k=1}^{\infty} \eta_{ik} < \infty, \quad 0 < \gamma_i^* = \sum_{k=1}^{\infty} \gamma_{ik}(1+t_k) < \infty.$$

(H<sub>3</sub>) For any  $t \in J_+$ ,  $R > 0$  and countable bounded set  $V_i \subset DPC^1[J, P_{0\lambda^*R}]$  ( $i = 0, 1$ ), there exist  $h_i(t) \in L[J, J]$  ( $i = 0, 1$ ) and positive constants  $m_{ikj}$  ( $i, j = 0, 1, k = 1, 2 \dots$ ) such that

$$\alpha(f(t, V_0(t), V_1(t))) \leq \sum_{i=0}^1 h_i(t)\alpha(V_i(t)), \quad \alpha(I_{ik}(V_0(t), V_1(t))) \leq \sum_{j=0}^1 m_{ikj}\alpha(V_j(t)),$$

$$h^* = \int_0^\infty h_0(t)(1+t) + h_1(t)dt < \infty, \quad m^* = \sum_{k=1}^\infty \sum_{i=0}^1 (m_{ik0}(1+t_k) + m_{ik1}) < \infty,$$

where

$$P_{0\lambda^*R} = \{x \in P : x \geq \lambda^*x_0^*, \|x\| < R\}.$$

(H<sub>4</sub>)  $t \in J_+$ ,  $\lambda^*x_0^* \leq x_i \leq \bar{x}_i$  ( $i = 0, 1$ ), imply  $f(t, x_0, x_1) \leq f(t, \bar{x}_0, \bar{x}_1)$ .

In what follows, we write  $Q = \{x \in DPC^1[J, P] : x^{(i)}(t) \geq \lambda^*x_0^*, \forall t \in J, i = 0, 1\}$ . Evidently,  $Q$  is a closed convex set in  $DPC^1[J, E]$ . We shall reduce BVP (1) to an impulsive integral equations in  $E$ . To this end, we first consider operator  $A$  defined by

$$\begin{aligned} (Ax)(t) &= \frac{1}{1 - \int_0^\infty g(t)dt} \left\{ x_\infty \int_0^\infty tg(t)dt + \int_0^\infty g(t) \left[ \int_0^\infty G(t, s)f(s, x(s), x'(s))ds \right. \right. \\ &\quad \left. \left. + \sum_{k=1}^\infty G(t, t_k)I_{1k}(x(t_k), x'(t_k)) + \sum_{k=1}^\infty G'_s(t, t_k)I_{0k}(x(t_k), x'(t_k)) \right] dt \right\} + tx_\infty \\ &\quad + \int_0^\infty G(t, s)f(s, x(s), x'(s))ds + \sum_{k=1}^\infty G(t, t_k)I_{1k}(x(t_k), x'(t_k)) \\ &\quad + \sum_{k=1}^\infty G'_s(t, t_k)I_{0k}(x(t_k), x'(t_k)), \end{aligned} \quad (2)$$

where

$$G(t, s) = \begin{cases} t, & 0 \leq t \leq s < +\infty, \\ s, & 0 \leq s \leq t < +\infty, \end{cases} \quad G'_s(t, s) = \begin{cases} 0, & 0 \leq t \leq s < +\infty, \\ 1, & 0 \leq s \leq t < +\infty. \end{cases}$$

**Lemma 1** *If conditions (H<sub>1</sub>) – (H<sub>2</sub>) are satisfied, then operator  $A$  defined by (2) is a continuous operator from  $Q$  into  $Q$ .*

**Proof.** Let

$$\varepsilon_0 = \frac{1}{8c^* \left( 1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt} \right)}, \quad (3)$$

and

$$r = \frac{\lambda^*\|x_0^*\|}{N} > 0. \quad (4)$$

By (H<sub>1</sub>), there exists a  $R > r$  such that

$$\|f(t, x, y)\| \leq \varepsilon_0 c(t)(\|x\| + \|y\|), \quad \forall t \in J_+, x \in P_{0\lambda^*}, y \in P_{0\lambda^*}, \|x\| + \|y\| > R,$$

and

$$\|f(t, x, y)\| \leq a(t) + Mb(t), \quad \forall t \in J_+, \quad x \in P_{0\lambda^*}, \quad y \in P_{0\lambda^*}, \quad \|x\| + \|y\| \leq R,$$

where

$$M = \max\{z(u_0, u_1) : r \leq u_i \leq R \ (i = 0, 1)\}.$$

Hence

$$\|f(t, x, y)\| \leq \varepsilon_0 c(t)(\|x\| + \|y\|) + a(t) + Mb(t), \quad \forall t \in J_+, \quad x \in P_{0\lambda^*}, \quad y \in P_{0\lambda^*}. \quad (5)$$

On the other hand, let

$$\bar{\varepsilon}_i = \frac{1}{8\gamma_i^* \left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt}\right)} \quad (i = 0, 1). \quad (6)$$

We see that, by condition (H<sub>2</sub>), there exists a  $R_1 > r$  such that

$$\|I_{ik}(x, y)\| \leq \bar{\varepsilon}_i \gamma_{ik}(\|x\| + \|y\|), \quad \forall x \in P_{0\lambda^*}, \quad y \in P_{0\lambda^*}, \quad \|x\| + \|y\| > R_1 \quad (i = 0, 1, k = 1, 2, \dots),$$

and

$$\|I_{ik}(x, y)\| \leq \eta_{ik} M_1, \quad \forall x \in P_{0\lambda^*}, \quad y \in P_{0\lambda^*}, \quad \|x\| + \|y\| \leq R_1 \quad (i = 0, 1, k = 1, 2, \dots),$$

where

$$M_1 = \max\{F_i(u_0, u_1) : r \leq u_i \leq R \ (i = 0, 1)\}.$$

Hence

$$\|I_{ik}(x, y)\| \leq \bar{\varepsilon}_i \gamma_{ik}(\|x\| + \|y\|) + \eta_{ik} M_1, \quad \forall x \in P_{0\lambda^*}, \quad y \in P_{0\lambda^*}, \quad i = 0, 1, \quad k = 1, 2, \dots \quad (7)$$

Let  $x \in Q$ , by (5), we can get

$$\begin{aligned} \|f(t, x(t), x'(t))\| &\leq \varepsilon_0 c(t)(1+t) \left( \frac{\|x(t)\|}{t+1} + \frac{\|x'(t)\|}{t+1} \right) + a(t) + Mb(t) \\ &\leq \varepsilon_0 c(t)(1+t)(\|x\|_F + \|x'\|_1) + a(t) + Mb(t) \\ &\leq 2\varepsilon_0 c(t)(1+t)\|x\|_D + a(t) + Mb(t), \quad \forall t \in J_+, \end{aligned} \quad (8)$$

which together with condition (H<sub>1</sub>) implies the convergence of the infinite integral

$$\int_0^\infty \|f(s, x(s), x'(s))\| ds. \quad (9)$$

On the other hand, by (7), we have

$$\begin{aligned} \|I_{ik}(x(t_k), x'(t_k))\| &\leq \bar{\varepsilon}_i \gamma_{ik}(1+t_k) \left( \frac{\|x(t_k)\|}{t_k+1} + \frac{\|x'(t_k)\|}{t_k+1} \right) + \eta_{ik} M_1 \\ &\leq \bar{\varepsilon}_i \gamma_{ik}(1+t_k)(\|x\|_F + \|x'\|_1) + \eta_{ik} M_1 \\ &\leq 2\bar{\varepsilon}_i \gamma_{ik}(1+t_k)\|x\|_D + \eta_{ik} M_1 \quad (i = 0, 1), \end{aligned} \quad (10)$$

which together with (2), (H<sub>1</sub>) and (H<sub>2</sub>) implies that

$$\begin{aligned} \|(Ax)(t)\| &\leq \frac{1}{1 - \int_0^\infty g(t)dt} \left\{ \|x_\infty\| \int_0^\infty tg(t)dt + \int_0^\infty g(t) \left[ \int_0^\infty G(t,s) \|f(s, x(s), x'(s))\| ds \right. \right. \\ &\quad \left. \left. + \sum_{k=1}^\infty G(t, t_k) \|I_{1k}(x(t_k), x'(t_k))\| + \sum_{k=1}^\infty G'_s(t, t_k) \|I_{0k}(x(t_k), x'(t_k))\| \right] dt \right\} \\ &\quad + t \|x_\infty\| + \int_0^\infty G(t,s) \|f(s, x(s), x'(s))\| ds + \sum_{k=1}^\infty G(t, t_k) \|I_{1k}(x(t_k), x'(t_k))\| \quad (11) \\ &\quad + \sum_{k=1}^\infty G'_s(t, t_k) \|I_{0k}(x(t_k), x'(t_k))\|. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\|(Ax)(t)\|}{1+t} &\leq \frac{1}{1 - \int_0^\infty g(t)dt} \left\{ \|x_\infty\| \int_0^\infty tg(t)dt + \int_0^\infty g(t) \left[ \int_0^\infty \|f(s, x(s), x'(s))\| ds \right. \right. \\ &\quad \left. \left. + \sum_{k=1}^\infty \|I_{1k}(x(t_k), x'(t_k))\| + \sum_{k=1}^\infty \|I_{0k}(x(t_k), x'(t_k))\| \right] dt \right\} + \|x_\infty\| \\ &\quad + \int_0^\infty \|f(s, x(s), x'(s))\| ds + \sum_{k=1}^\infty \|I_{1k}(x(t_k), x'(t_k))\| \\ &\quad + \sum_{k=1}^\infty \|I_{0k}(x(t_k), x'(t_k))\| \\ &\leq \left( 1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt} \right) [2\varepsilon_0 c^* \|x\|_D + a^* + Mb^*] + \left( 1 + \frac{\int_0^\infty tg(t)dt}{1 - \int_0^\infty g(t)dt} \right) \|x_\infty\| \\ &\quad + \left( 1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt} \right) \sum_{i=0}^1 \sum_{k=1}^\infty \|I_{ik}(x(t_k), x'(t_k))\| \\ &\leq \frac{1}{2} \|x\|_D + \left( 1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt} \right) (a^* + Mb^* + \eta_0^* M_1 + \eta_1^* M_1) \\ &\quad + \left( 1 + \frac{\int_0^\infty tg(t)dt}{1 - \int_0^\infty g(t)dt} \right) \|x_\infty\|. \quad (12) \end{aligned}$$

Differentiating (2), we get

$$(A'x)(t) = \int_t^\infty f(s, x(s), x'(s)) ds + \sum_{t_k \geq t} I_{1k}(x(t_k), x'(t_k)) + x_\infty. \quad (13)$$

Hence,

$$\begin{aligned} \|(A'x)(t)\| &\leq \int_0^\infty \|f(s, x(s), x'(s))\| ds + \|x_\infty\| + \sum_{k=1}^\infty \|I_{1k}(x(t_k), x'(t_k))\| \\ &\leq 2\varepsilon_0 c^* \|x\|_D + a^* + Mb^* + \|x_\infty\| + 2\bar{\varepsilon}_1 \gamma_1^* \|x\|_D + \eta_1^* M_1 \\ &\leq \frac{1}{2} \|x\|_D + a^* + Mb^* + \|x_\infty\| + \eta_1^* M_1, \quad \forall t \in J. \quad (14) \end{aligned}$$

It follows from (12) and (14) that

$$\begin{aligned} \|Ax\|_D &\leq \frac{1}{2} \|x\|_D + \left( 1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt} \right) (a^* + Mb^* + \eta_0^* M_1 + \eta_1^* M_1) \\ &\quad + \left( 1 + \frac{\int_0^\infty tg(t)dt}{1 - \int_0^\infty g(t)dt} \right) \|x_\infty\|. \quad (15) \end{aligned}$$

So,  $Ax \in DPC^1[J, E]$ . On the other hand, it can be easily seen that

$$(Ax)(t) \geq \left( \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt} \right) x_\infty \geq \lambda^* x_\infty \geq \lambda^* x_0^*, \quad \forall t \in J,$$

$$(A'x)(t) \geq x_\infty \geq \lambda^* x_\infty \geq \lambda^* x_0^*, \quad \forall t \in J.$$

Hence,  $Ax \in Q$ . Thus, we have proved that  $A$  maps  $Q$  into  $Q$  and (15) holds.

Finally, we show that  $A$  is continuous. Let  $(x_m, \bar{x}) \in Q$ ,  $\|x_m - \bar{x}\|_D \rightarrow 0$  ( $m \rightarrow \infty$ ). Then  $\{x_m\}$  is a bounded subset of  $Q$ . Thus, there exists  $r > 0$  such that  $\|x_m\|_D < r$  for  $m \geq 1$  and  $\|\bar{x}\|_D \leq r$ . Similar to (12) and (14), it is easy to get

$$\begin{aligned} \|Ax_m - A\bar{x}\|_D &\leq \left( 1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt} \right) \int_0^\infty \|f(s, x_m(s), x'_m(s)) - f(s, \bar{x}(s), \bar{x}'(s))\| ds \\ &+ \left( 1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt} \right) \left( \sum_{i=0}^1 \sum_{k=1}^\infty \|I_{ik}(x_m(t_k), x'_m(t_k)) - I_{ik}(\bar{x}(t_k), \bar{x}'(t_k))\| \right). \end{aligned} \quad (16)$$

It is clear that,

$$f(t, x_m(t), x'_m(t)) \rightarrow f(t, \bar{x}(t), \bar{x}'(t)) \text{ as } m \rightarrow \infty, \quad \forall t \in J_+. \quad (17)$$

By (8), we get

$$\begin{aligned} \|f(t, x_m(t), x'_m(t)) - f(t, \bar{x}(t), \bar{x}'(t))\| &\leq 4\varepsilon_0 c(t)(1+t)r + 2a(t) + 2Mb(t) \\ &= \sigma(t) \in L[J, J], \quad m = 1, 2, 3, \dots, \quad \forall t \in J_+. \end{aligned} \quad (18)$$

It follows from (17), (18) and the dominated convergence theorem that

$$\lim_{m \rightarrow \infty} \int_0^\infty \|f(s, x_m(s), x'_m(s)) - f(s, \bar{x}(s), \bar{x}'(s))\| ds = 0. \quad (19)$$

It is clear that,

$$I_{ik}(x_m(t_k), x'_m(t_k)) \rightarrow I_{ik}(\bar{x}(t_k), \bar{x}'(t_k)), \text{ as } m \rightarrow \infty, \quad i = 0, 1, \quad k = 1, 2, \dots \quad (20)$$

So,

$$\lim_{m \rightarrow \infty} \left( \sum_{i=0}^1 \sum_{k=1}^\infty \|I_{ik}(x_m(t_k), x'_m(t_k)) - I_{ik}(\bar{x}(t_k), \bar{x}'(t_k))\| \right) = 0. \quad (21)$$

It follows from (16), (19) and (21) that  $\|Ax_m - A\bar{x}\|_D \rightarrow 0$  as  $m \rightarrow \infty$ . Therefore, the continuity of  $A$  is proved.

**Lemma 2** *If condition (H<sub>1</sub>) and (H<sub>2</sub>) are satisfied, then  $x \in Q \cap C^2[J'_+, E]$  is a solution of BVP (1) if and only if  $x \in Q$  is a solution of the following impulsive integral equation:*

$$\begin{aligned}
x(t) &= \frac{1}{1 - \int_0^\infty g(t)dt} \left\{ x_\infty \int_0^\infty tg(t)dt + \int_0^\infty g(t) \left[ \int_0^\infty G(t,s)f(s,x(s),x'(s))ds \right. \right. \\
&+ \sum_{k=1}^\infty G(t,t_k)I_{1k}(x(t_k),x'(t_k)) + \sum_{k=1}^\infty G'_s(t,t_k)I_{0k}(x(t_k),x'(t_k)) \left. \left. \right] dt \right\} + tx_\infty \quad (22) \\
&+ \int_0^\infty G(t,s)f(s,x(s),x'(s))ds + \sum_{k=1}^\infty G(t,t_k)I_{1k}(x(t_k),x'(t_k)) \\
&+ \sum_{k=1}^\infty G'_s(t,t_k)I_{0k}(x(t_k),x'(t_k)).
\end{aligned}$$

**Proof.** Suppose that  $x \in Q \cap C^2[J'_+, E]$  is a solution of BVP (1). For  $t \in J$ , integrating (1) from 0 to  $t$ , we have

$$-x'(t) + x'(0) = \int_0^t f(s, x(s), x'(s))ds + \sum_{t_k < t} I_{1k}(x(t_k), x'(t_k)). \quad (23)$$

Taking limit for  $t \rightarrow \infty$ , we get

$$-x_\infty + x'(0) = \int_0^\infty f(s, x(s), x'(s))ds + \sum_{k=1}^\infty I_{1k}(x(t_k), x'(t_k)). \quad (24)$$

Thus,

$$x'(0) = x_\infty + \int_0^\infty f(s, x(s), x'(s))ds + \sum_{k=1}^\infty I_{1k}(x(t_k), x'(t_k)). \quad (25)$$

$$\begin{aligned}
x'(t) &= x_\infty + \int_0^\infty f(s, x(s), x'(s))ds + \sum_{k=1}^\infty I_{1k}(x(t_k), x'(t_k)) - \int_0^t f(s, x(s), x'(s))ds \\
&- \sum_{t_k < t} I_{1k}(x(t_k), x'(t_k)). \quad (26)
\end{aligned}$$

$$x'(t) = x_\infty + \int_t^\infty f(s, x(s), x'(s))ds + \sum_{k=1}^\infty I_{1k}(x(t_k), x'(t_k)) - \sum_{t_k < t} I_{1k}(x(t_k), x'(t_k)). \quad (27)$$

Integrating (27) from 0 to  $t$ , we obtain

$$\begin{aligned}
x(t) &= x(0) + tx_\infty + \int_0^\infty G(t,s)f(s,x(s),x'(s))ds + \sum_{k=1}^\infty G(t,t_k)I_{1k}(x(t_k),x'(t_k)) \\
&+ \sum_{k=1}^\infty G'_s(t,t_k)I_{0k}(x(t_k),x'(t_k)), \quad (28)
\end{aligned}$$

which together with the boundary value condition implies that

$$\begin{aligned}
x(0) &= \int_0^\infty g(t)x(t)dt = x(0) \int_0^\infty g(t)dt + x_\infty \int_0^\infty tg(t)dt + \sum_{k=1}^\infty G(t,t_k)I_{1k}(x(t_k),x'(t_k)) \\
&+ \int_0^\infty g(t) \left[ \int_0^\infty G(t,s)f(s,x(s),x'(s))ds + \sum_{k=1}^\infty G'_s(t,t_k)I_{0k}(x(t_k),x'(t_k)) \right] dt. \quad (29)
\end{aligned}$$



Thus,

$$\begin{aligned}
 x(0) &= \frac{1}{1 - \int_0^\infty g(t)dt} \left\{ x_\infty \int_0^\infty tg(t)dt + \int_0^\infty g(t) \left[ \int_0^\infty G(t,s)f(s,x(s),x'(s))ds \right. \right. \\
 &\quad \left. \left. + \sum_{k=1}^\infty G(t,t_k)I_{1k}(x(t_k),x'(t_k)) + \sum_{k=1}^\infty G'_s(t,t_k)I_{0k}(x(t_k),x'(t_k)) \right] dt \right\}. \quad (30)
 \end{aligned}$$

Substituting (30) into (28), we have

$$\begin{aligned}
 x(t) &= \frac{1}{1 - \int_0^\infty g(t)dt} \left\{ x_\infty \int_0^\infty tg(t)dt + \int_0^\infty g(t) \left[ \int_0^\infty G(t,s)f(s,x(s),x'(s))ds \right. \right. \\
 &\quad \left. \left. + \sum_{k=1}^\infty G(t,t_k)I_{1k}(x(t_k),x'(t_k)) + \sum_{k=1}^\infty G'_s(t,t_k)I_{0k}(x(t_k),x'(t_k)) \right] dt \right\} + tx_\infty \quad (31) \\
 &\quad + \int_0^\infty G(t,s)f(s,x(s),x'(s))ds + \sum_{k=1}^\infty G(t,t_k)I_{1k}(x(t_k),x'(t_k)) \\
 &\quad + \sum_{k=1}^\infty G'_s(t,t_k)I_{0k}(x(t_k),x'(t_k)).
 \end{aligned}$$

Obviously, integral  $\int_0^t \int_s^\infty f(\tau, x(\tau), x'(\tau))d\tau ds$  is convergent.

Conversely, if  $x$  a solution of integral equation, then direct differentiation gives the proof.

**Lemma 3** Let  $(H_1)$  be satisfied,  $V \subset Q$  be a bounded set. Then  $\frac{(AV)(t)}{1+t}$  and  $(A'V)(t)$  are equicontinuous on any finite subinterval  $J_k$  of  $J$  and for any  $\varepsilon > 0$ , there exists  $N > 0$  such that

$$\left\| \frac{(Ax)(t')}{1+t'} - \frac{(Ax)(t'')}{1+t''} \right\| < \varepsilon, \quad \|(A'x)(t') - (A'x)(t'')\| < \varepsilon \quad (32)$$

uniformly with respect to  $x \in V$  as  $t', t'' \geq N$ .

**Proof.** For  $x \in V$ ,  $t'' > t'$ ,  $t'', t' \in J_k$ , we have

$$\begin{aligned}
 &\left\| \frac{(Ax)(t')}{1+t'} - \frac{(Ax)(t'')}{1+t''} \right\| \\
 &\leq |t' - t''| \cdot \left( 1 + \frac{\int_0^\infty tg(t)dt}{1 - \int_0^\infty g(t)dt} \right) \|x_\infty\| + \left( 1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt} \right) \cdot \\
 &\quad \left\{ \left\| \frac{t'}{1+t'} \int_{t'}^\infty f(s,x(s),x'(s))ds - \frac{t''}{1+t''} \int_{t''}^\infty f(s,x(s),x'(s))ds \right\| \right. \\
 &\quad \left. + \left\| \int_0^{t'} \frac{s}{1+t'} f(s,x(s),x'(s))ds - \int_0^{t''} \frac{s}{1+t''} f(s,x(s),x'(s))ds \right\| \right\} \quad (33) \\
 &\quad + \left| \frac{1}{1+t'} - \frac{1}{1+t''} \right| \int_0^\infty g(t) \left[ \sum_{k=1}^\infty G(t,t_k) \|I_{1k}(x(t_k),x'(t_k))\| \right. \\
 &\quad \left. + \sum_{k=1}^\infty G'_s(t,t_k) \|I_{0k}(x(t_k),x'(t_k))\| \right] dt + \left| \frac{1}{1+t'} - \frac{1}{1+t''} \right| \sum_{k=1}^\infty G(t',t_k) \|I_{1k}(x(t_k),x'(t_k))\| \\
 &\quad + \left| \frac{1}{1+t'} - \frac{1}{1+t''} \right| \sum_{k=1}^\infty G'_s(t',t_k) \|I_{0k}(x(t_k),x'(t_k))\|
 \end{aligned}$$

$$\begin{aligned}
&\leq |t' - t''| \cdot \left(1 + \frac{\int_0^\infty tg(t)dt}{1 - \int_0^\infty g(t)dt}\right) \|x_\infty\| + \left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt}\right) \cdot \\
&\quad \left\{ \left| \frac{t'}{1+t'} - \frac{t''}{1+t''} \right| \cdot \left\| \int_0^\infty f(s, x(s), x'(s))ds \right\| + \left\| \int_{t'}^{t''} sf(s, x(s), x'(s))ds \right\| \right. \\
&\quad + \frac{t''}{1+t''} \left\| \int_{t'}^{t''} f(s, x(s), x'(s))ds \right\| + \left| \frac{1}{1+t'} - \frac{1}{1+t''} \right| \cdot \left\| \int_0^{t'} sf(s, x(s), x'(s))ds \right\| \\
&\quad \left. + \left| \frac{t'}{1+t'} - \frac{t''}{1+t''} \right| \cdot \left\| \int_0^{t'} f(s, x(s), x'(s))ds \right\| \right\} \\
&\quad + \left| \frac{1}{1+t'} - \frac{1}{1+t''} \right| \int_0^\infty g(t) \left[ \sum_{k=1}^\infty G(t, t_k) \|I_{1k}(x(t_k), x'(t_k))\| + \sum_{k=1}^\infty G'_s(t, t_k) \|I_{0k}(x(t_k), x'(t_k))\| \right] dt \\
&\quad + |t' - t''| \left[ \sum_{k=1}^\infty G(t', t_k) \|I_{1k}(x(t_k), x'(t_k))\| + \sum_{k=1}^\infty G'_s(t', t_k) \|I_{0k}(x(t_k), x'(t_k))\| \right],
\end{aligned}$$

which implies that  $\{\frac{AV(t)}{1+t} : x \in V\}$  is equicontinuous on any finite subinterval  $J_k$  of  $J$ .

Since  $V \subset Q$  is bounded, there exists  $r > 0$  such that for any  $x \in V$ ,  $\|x\|_D \leq r$ . By (13),  $t'', t' \in J_k$ , we get

$$\begin{aligned}
\|(A'x)(t') - (A'x)(t'')\| &= \left\| \int_{t'}^{t''} f(s, x(s), x'(s))ds + \sum_{\substack{t_k \geq t' \\ t_k \geq t''}} I_{1k}(x(t_k), x'(t_k)) + x_\infty \right. \\
&\quad \left. - \sum_{t_k \geq t''} I_{1k}(x(t_k), x'(t_k)) - x_\infty \right\| \\
&\leq \int_{t'}^{t''} [2\varepsilon_0 rc(s)(1+s) + a(s) + Mb(s)]ds.
\end{aligned} \tag{34}$$

In the following, we are in position to show that for any  $\varepsilon > 0$ , there exists  $N > 0$  such that

$$\left\| \frac{(Ax)(t')}{1+t'} - \frac{(Ax)(t'')}{1+t''} \right\| < \varepsilon, \quad \|(A'x)(t') - (A'x)(t'')\| < \varepsilon$$

uniformly with respect to  $x \in V$  as  $t', t'' \geq N$ .

Combining with (33), we need only to show that for any  $\varepsilon > 0$ , there exists sufficiently large  $N > 0$  such that

$$\left\| \int_0^{t'} \frac{s}{1+t'} f(s, x(s), x'(s))ds - \int_0^{t''} \frac{s}{1+t''} f(s, x(s), x'(s))ds \right\| < \varepsilon$$

for all  $x \in V$  as  $t', t'' \geq N$ . The rest part of the proof is very similar to Lemma 2.3 in [5], we omit the details.

**Lemma 4** *Let (H<sub>1</sub>) and (H<sub>2</sub>) are satisfied, V be a bounded set in  $DPC^1[J, E]$ . Then*

$$\alpha_D(AV) = \max \left\{ \sup_{t \in J} \alpha \left( \frac{(AV)(t)}{1+t} \right), \sup_{t \in J} \alpha((AV)'(t)) \right\}.$$

**Proof.** The proof is similar to that of Lemma 2.4 in [5], we omit it.

**Lemma 5** ([1,2]) *Mönch Fixed-Point Theorem. Let Q be a closed convex set of E and  $u \in Q$ . Assume that the continuous operator  $F : Q \rightarrow Q$  has the following property:  $V \subset Q$  countable,  $V \subset \overline{co}(\{u\} \cup F(V)) \implies V$  is relatively compact. Then F has a fixed point in Q.*

**Lemma 6** *If (H<sub>4</sub>) is satisfied, then for  $x, y \in Q$ ,  $x^{(i)} \leq y^{(i)}$ ,  $t \in J$  ( $i = 0, 1$ ) imply that  $(Ax)^{(i)} \leq (Ay)^{(i)}$ ,  $t \in J$  ( $i = 0, 1$ ).*

**Proof.** It is easy to see that this lemma follows from (2), (13) and condition (H<sub>4</sub>). The proof is obvious.

### 3 Main results

**Theorem 1** *Assume conditions (H<sub>1</sub>), (H<sub>2</sub>) and (H<sub>3</sub>) are satisfied. If  $\left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt}\right) \cdot (2h^* + m^*) < 1$ , then BVP (1) has a positive solution  $\bar{x} \in DPC^1[J, E] \cap C^2[J'_+, E]$  satisfying  $(\bar{x})^{(i)}(t) \geq \lambda^* x_0^*$  for  $t \in J$  ( $i = 0, 1$ ).*

**Proof.** By Lemma 1, operator  $A$  defined by (2) is a continuous operator from  $Q$  into  $Q$ , and by Lemma 2, we need only to show that  $A$  has a fixed point  $\bar{x}$  in  $Q$ . Choose

$$R > 2 \left\{ \left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt}\right) (a^* + Mb^* + \eta_0^* M_1 + \eta_1^* M_1) + \left(1 + \frac{\int_0^\infty tg(t)dt}{1 - \int_0^\infty g(t)dt}\right) \|x_\infty\| \right\}, \quad (35)$$

and let  $Q_1 = \{x \in Q : \|x\|_D \leq R\}$ . Obviously,  $Q_1$  is a bounded closed convex set in space  $DPC^1[J, E]$ . It is easy to see that  $Q_1$  is not empty since  $\lambda^*(1+t)x_\infty \in Q_1$ . It follows from (15) and (35) that  $x \in Q_1$  implies that  $Ax \in Q_1$ , i.e.,  $A$  maps  $Q_1$  into  $Q_1$ . Now, we are in position to show that  $A(Q_1)$  is relatively compact. Let  $V = \{x_m : m = 1, 2, \dots\} \subset Q_1$  satisfying  $V \subset \overline{\text{co}}\{u\} \cup AV$  for some  $u \in Q_1$ . Then  $\|x_m\|_D \leq R$ . We have, by (2) and (13)

$$\begin{aligned} (Ax_m)(t) &= \frac{1}{1 - \int_0^\infty g(t)dt} \left\{ x_\infty \int_0^\infty tg(t)dt + \int_0^\infty g(t) \left[ \int_0^\infty G(t,s)f(s, x_m(s), x'_m(s))ds \right. \right. \\ &\quad \left. \left. + \sum_{k=1}^\infty G(t, t_k)I_{1k}(x_m(t_k), x'_m(t_k)) + \sum_{k=1}^\infty G'_s(t, t_k)I_{0k}(x_m(t_k), x'_m(t_k)) \right] dt \right\} + tx_\infty \\ &\quad + \int_0^\infty G(t,s)f(s, x_m(s), x'_m(s))ds + \sum_{k=1}^\infty G(t, t_k)I_{1k}(x_m(t_k), x'_m(t_k)) \\ &\quad + \sum_{k=1}^\infty G'_s(t, t_k)I_{0k}(x_m(t_k), x'_m(t_k)), \end{aligned} \quad (36)$$

and

$$(A'x_m)(t) = \int_t^\infty f(s, x_m(s), x'_m(s))ds + \sum_{t_k \geq t} I_{1k}(x_m(t_k)) + x_\infty. \quad (37)$$

By Lemma 4, we have

$$\alpha_D(AV) = \max \left\{ \sup_{t \in J} \alpha((AV)'(t)), \sup_{t \in J} \alpha\left(\frac{(AV)(t)}{1+t}\right) \right\}, \quad (38)$$

where  $(AV)(t) = \{(Ax_m)(t) : m = 1, 2, 3, \dots\}$ , and  $(AV)'(t) = \{(A'x_m)(t) : m = 1, 2, 3, \dots\}$ .

By (9), we know that the infinite integral  $\int_0^\infty \|f(t, x(t), x'(t))\|dt$  is convergent uniformly for  $m = 1, 2, 3, \dots$ . So, for any  $\varepsilon > 0$ , we can choose a sufficiently large  $T > 0$  such that

$$\int_T^\infty \|f(t, x(t), x'(t))\|dt < \varepsilon. \quad (39)$$

Then, by Guo et al. [1, Theorem 1.2.3], (2), (36), (37), (39) and (H<sub>3</sub>), we obtain

$$\begin{aligned}
 \alpha\left(\frac{(AV)(t)}{1+t}\right) &\leq \frac{1}{1+t}\left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt}\right)\left\{2 \int_0^T \alpha(f(t, V(t), V'(t)))dt + 2\varepsilon\right\} \\
 &\quad + \frac{1}{1+t}\left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt}\right)\sum_{k=1}^\infty \sum_{i=0}^1 \alpha(I_{ik}(V(t_k), V'(t_k))) \\
 &\leq 2\left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt}\right)\int_0^\infty \alpha(f(t, V(t), V'(t)))dt + 2\varepsilon \\
 &\quad + \left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt}\right)\sum_{i=0}^1 \sum_{k=1}^\infty \alpha(I_{ik}(V(t_k), V'(t_k))) \tag{40} \\
 &\leq 2\left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt}\right)\alpha_D(V)\int_0^\infty h_0(t)(1+t) + h_1(t)dt \\
 &\quad + \left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt}\right)\alpha_D(V)\sum_{k=1}^\infty \sum_{i=0}^1 (m_{ik0}(1+t_k) + m_{ik1}) + 2\varepsilon. \\
 &\leq 2\left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt}\right)h^*\alpha_D(V) + m^*\left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt}\right)\alpha_D(V) + 2\varepsilon,
 \end{aligned}$$

and

$$\alpha((AV)'(t)) \leq 2 \int_0^\infty \alpha(f(s, V(s), V'(s)))ds + 2\varepsilon \leq 2\left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt}\right)h^*\alpha_D(V) + 2\varepsilon. \tag{41}$$

By (38), (40) and (41) that

$$\alpha_D(AV) \leq \left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt}\right)(2h^* + m^*)\alpha_D(V). \tag{42}$$

On the other hand,  $\alpha_D(V) \leq \alpha_D\{\overline{\text{co}}(\{u\} \cup (AV))\} = \alpha_D(AV)$ . Then, (42) implies  $\alpha_D(V) = 0$ , i.e.,  $V$  is relatively compact in  $DPC^1[J, E]$ . Hence, the Mönch fixed point theorem guarantees that  $A$  has a fixed point  $\bar{x}$  in  $Q_1$ . Thus, Theorem 1 is proved.

**Theorem 2** *Let cone  $P$  be normal and conditions (H<sub>1</sub>) – (H<sub>4</sub>) be satisfied. Then BVP (1) has a positive solution  $y \in Q \cap [J'_+, E]$  which is minimal in the sense that  $x^{(i)}(t) \geq y^{(i)}(t)$ ,  $t \in J$  ( $i = 0, 1$ ) for any positive solution  $x \in Q \cap [J'_+, E]$  of BVP (1). Moreover,  $\|y\|_D \leq 2\gamma + \|x_0\|_D$ , where*

$$\gamma = \left\{ \left(1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt}\right)(a^* + Mb^* + \eta_0^*M_1 + \eta_1^*M_1) + \left(1 + \frac{\int_0^\infty tg(t)dt}{1 - \int_0^\infty g(t)dt}\right)\|x_\infty\| \right\},$$

and there exists a monotone iterative sequence  $\{x_m(t)\}$  such that  $x_m^{(i)}(t) \rightarrow y^{(i)}(t)$  as  $m \rightarrow \infty$  ( $i = 0, 1$ ) uniformly on  $J$  and  $x_m''(t) \rightarrow y''(t)$  as  $m \rightarrow \infty$  for any  $t \in J_+$ , where

$$\begin{aligned}
 x_0(t) &= \frac{1}{1 - \int_0^\infty g(t)dt}\left\{x_\infty \int_0^\infty tg(t)dt + \int_0^\infty g(t)\left[\int_0^\infty G(t, s)f(s, \lambda^*x_0^*, \lambda^*x_0^*)ds\right.\right. \\
 &\quad \left.+\sum_{k=1}^\infty G(t, t_k)I_{1k}(\lambda^*x_0^*, \lambda^*x_0^*) + \sum_{k=1}^\infty G'_s(t, t_k)I_{0k}(\lambda^*x_0^*, \lambda^*x_0^*)\right]dt\left\} + tx_\infty \\
 &\quad + \int_0^\infty G(t, s)f(s, \lambda^*x_0^*, \lambda^*x_0^*)ds + \sum_{k=1}^\infty G(t, t_k)I_{1k}(\lambda^*x_0^*, \lambda^*x_0^*) \tag{43} \\
 &\quad + \sum_{k=1}^\infty G'_s(t, t_k)I_{0k}(\lambda^*x_0^*, \lambda^*x_0^*),
 \end{aligned}$$

and

$$\begin{aligned}
 x_m(t) = & \frac{1}{1 - \int_0^\infty g(t)dt} \left\{ x_\infty \int_0^\infty tg(t)dt + \int_0^\infty g(t) \left[ \int_0^\infty G(t,s)f(s, x_{m-1}(s), x'_{m-1}(s))ds \right. \right. \\
 & + \sum_{k=1}^\infty G(t, t_k)I_{1k}(x_{m-1}(t_k), x'_{m-1}(t_k)) + \sum_{k=1}^\infty G'_s(t, t_k)I_{0k}(x_{m-1}(t_k), x'_{m-1}(t_k)) \left. \left. \right] dt \right\} + tx_\infty \\
 & + \int_0^\infty G(t,s)f(s, x_{m-1}(s), x'_{m-1}(s))ds + \sum_{k=1}^\infty G(t, t_k)I_{1k}(x_{m-1}(t_k), x'_{m-1}(t_k)) \\
 & + \sum_{k=1}^\infty G'_s(t, t_k)I_{0k}(x_{m-1}(t_k), x'_{m-1}(t_k)), \quad \forall t \in J \quad (m = 1, 2, 3, \dots).
 \end{aligned} \tag{44}$$

**Proof.** From (43), we can see that  $x_0 \in C[J, E]$  and

$$x'_0(t) = \int_t^\infty f(s, \lambda^* x_0^*, \lambda^* x_0^*)ds + \sum_{t_k \geq t} I_{1k}(\lambda^* x_0^*, \lambda^* x_0^*) + x_\infty. \tag{45}$$

By (43) and (45), we have that  $x_0^{(i)} \geq \lambda^* x_\infty \geq \lambda^* x_0^*$  ( $i = 0, 1$ ) and

$$\begin{aligned}
 \|x_0(t)\| \leq & \frac{1}{1 - \int_0^\infty g(t)dt} \left\{ \|x_\infty\| \int_0^\infty tg(t)dt + \int_0^\infty g(t) \left[ \int_0^\infty \|f(s, \lambda^* x_0^*, \lambda^* x_0^*)\|ds \right. \right. \\
 & + \sum_{k=1}^\infty \|I_{1k}(\lambda^* x_0^*, \lambda^* x_0^*)\| + \sum_{k=1}^\infty \|I_{0k}(\lambda^* x_0^*, \lambda^* x_0^*)\| \left. \left. \right] dt \right\} + \|x_\infty\| \\
 & + \int_0^\infty \|f(s, \lambda^* x_0^*, \lambda^* x_0^*)\|ds + \sum_{k=1}^\infty \|I_{1k}(\lambda^* x_0^*, \lambda^* x_0^*)\| \\
 & + \sum_{k=1}^\infty \|I_{0k}(\lambda^* x_0^*, \lambda^* x_0^*)\| \\
 \leq & \left( 1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt} \right) \int_0^\infty a(s) + b(s)z(\|\lambda^* x_0^*\|, \|\lambda^* x_0^*\|)ds \\
 & + \left( 1 + \frac{\int_0^\infty g(t)dt}{1 - \int_0^\infty g(t)dt} \right) \sum_{i=0}^1 \sum_{k=1}^\infty \eta_{ik} F_i(\|\lambda^* x_0^*\|, \|\lambda^* x_0^*\|) \\
 & + \left( 1 + \frac{\int_0^\infty tg(t)dt}{1 - \int_0^\infty g(t)dt} \right) \|x_\infty\|,
 \end{aligned}$$

and

$$\begin{aligned}
 \|x'_0(t)\| \leq & \int_t^\infty \|f(s, \lambda^* x_0^*, \lambda^* x_0^*)\|ds + \sum_{t_k \geq t} \|I_{1k}(\lambda^* x_0^*, \lambda^* x_0^*)\| + \|x_\infty\| \\
 \leq & \int_0^\infty a(s) + b(s)z(\|\lambda^* x_0^*\|, \|\lambda^* x_0^*\|)ds + \sum_{k=1}^\infty \eta_{ik} F_i(\|\lambda^* x_0^*\|, \|\lambda^* x_0^*\|) + \|x_\infty\|,
 \end{aligned}$$

which imply that  $\|x_0\|_D < \infty$ . Thus,  $x_0 \in DPC^1[J, E]$ . It follows from (2) and (44) that

$$x_m(t) = (Ax_{m-1})(t), \quad \forall t \in J, \quad m = 1, 2, 3, \dots. \tag{46}$$

By Lemma 1, we have  $x_m \in Q$  and

$$\|x_m\|_D = \|Ax_{m-1}\|_D \leq \frac{1}{2} \|x_{m-1}\|_D + \gamma. \tag{47}$$

By Lemma 6 and (46), we get

$$\lambda^* x_0^* \leq x_0^{(i)}(t) \leq x_1^{(i)}(t) \leq \dots \leq x_m^{(i)}(t) \leq \dots, \quad \forall t \in J \quad (i = 0, 1). \quad (48)$$

It follows from (47), by induction, that

$$\begin{aligned} \|x_m\|_D &\leq \gamma + \frac{1}{2}\gamma + \dots + \left(\frac{1}{2}\right)^{m-1} \gamma + \left(\frac{1}{2}\right)^m \|x_0\|_D \leq \frac{\gamma[1 - (\frac{1}{2})^m]}{1 - \frac{1}{2}} + \|x_0\|_D \\ &\leq 2\gamma + \|x_0\|_D \quad (m = 1, 2, 3, \dots). \end{aligned} \quad (49)$$

Let  $K = \{x \in Q : \|x\|_D \leq 2\gamma + \|x_0\|_D\}$ . Then,  $K$  is a bounded closed convex set in space  $DPC^1[J, E]$  and operator  $A$  maps  $K$  into  $K$ . Clearly,  $K$  is not empty since  $x_0 \in K$ . Let  $W = \{x_m : m = 0, 1, 2, \dots\}$ ,  $AW = \{Ax_m : m = 0, 1, 2, \dots\}$ . Obviously,  $W \subset K$  and  $W = \{x_0\} \cup A(W)$ . Similar to above proof of Theorem 1, we can obtain  $\alpha_D(AW) = 0$ , i.e.,  $W$  is relatively compact in  $DPC^1[J, E]$ . So, there exists a  $y \in DPC^1[J, E]$  and a subsequence  $\{x_{m_j} : j = 1, 2, 3, \dots\} \subset W$  such that  $\{x_{m_j}^{(i)}(t) : j = 1, 2, 3, \dots\}$  converges to  $y^{(i)}(t)$  uniformly on  $J$  ( $i = 0, 1$ ). Since that  $P$  is normal and  $\{x_m^{(i)}(t) : m = 1, 2, 3, \dots\}$  is nondecreasing, it is easy to see that the entire sequence  $\{x_m^{(i)}(t) : m = 1, 2, 3, \dots\}$  converges to  $y^{(i)}(t)$  uniformly on  $J$  ( $i = 0, 1$ ). Since  $x_m \in K$  and  $K$  is a closed convex set in space  $DPC^1[J, E]$ , we have  $y \in K$ . It is clear,

$$f(s, x_m(s), x'_m(s)) \rightarrow f(s, y(s), y'(s)), \quad \text{as } m \rightarrow \infty, \quad \forall s \in J_+. \quad (50)$$

By  $(H_1)$  and (49), we have

$$\begin{aligned} \|f(s, x_m(s), x'_m(s)) - f(s, y(s), y'(s))\| &\leq 4\varepsilon_0 c(s)(1+s)\|x_m\|_D + 2a(s) + 2Mb(s) \\ &\leq 4\varepsilon_0 c(s)(1+s)(2\gamma + \|x_0\|_D) + 2a(s) + 2Mb(s). \end{aligned} \quad (51)$$

Noticing (50) and (51) and taking limit as  $m \rightarrow \infty$  in (44), we obtain

$$\begin{aligned} y(t) &= \frac{1}{1 - \int_0^\infty g(t)dt} \left\{ x_\infty \int_0^\infty tg(t)dt + \int_0^\infty g(t) \left[ \int_0^\infty G(t, s)f(s, y(s), y'(s))ds \right. \right. \\ &\quad \left. \left. + \sum_{k=1}^\infty G(t, t_k)I_{1k}(y(t_k), y'(t_k)) + \sum_{k=1}^\infty G'_s(t, t_k)I_{0k}(y(t_k), y'(t_k)) \right] dt \right\} + tx_\infty \\ &\quad + \int_0^\infty G(t, s)f(s, y(s), y'(s))ds + \sum_{k=1}^\infty G(t, t_k)I_{1k}(y(t_k), y'(t_k)) \\ &\quad + \sum_{k=1}^\infty G'_s(t, t_k)I_{0k}(y(t_k), y'(t_k)), \end{aligned} \quad (52)$$

which implies by Lemma 2 that  $y \in K \cap C^2[J_+, E]$  and  $y(t)$  is a positive solution of BVP (1). Differentiating (44) twice, we get

$$x''_m(t) = -f(t, x_{m-1}(t), x'_{m-1}(t)), \quad \forall t \in J_+, \quad m = 1, 2, 3, \dots.$$

Hence, by (50), we obtain

$$\lim_{m \rightarrow \infty} x''_m(t) = -f(t, y(t), y'(t)) = y''(t), \quad \forall t \in J_+.$$

Let  $u(t)$  be any positive solution of BVP (1). By Lemma 2, we have  $u \in Q$  and  $u(t) = (Au)(t)$ , for  $t \in J$ . It is clear that  $u^{(i)}(t) \geq \lambda^* x_0^* > \theta$  for any  $t \in J$  ( $i = 0, 1$ ). So, by Lemma 6, we have

$u^{(i)}(t) \geq x_0^{(i)}(t)$  for any  $t \in J$  ( $i = 0, 1$ ). Assume that  $u^{(i)}(t) \geq x_{m-1}^{(i)}(t)$  for  $t \in J, m \geq 1$  ( $i = 0, 1$ ). Then, it follows from Lemma 6 that  $(Au)^{(i)}(t) \geq Ax_{m-1}^{(i)}(t)$  for  $t \in J$  ( $i = 0, 1$ ), i.e.  $u^{(i)}(t) \geq x_m^{(i)}(t)$  for  $t \in J$  ( $i = 0, 1$ ). Hence, by induction, we get

$$u^{(i)}(t) \geq x_m^{(i)}(t) \quad \forall t \in J \quad (i = 0, 1; m = 0, 1, 2, \dots). \quad (53)$$

Now, taking limits in (53), we get  $u^{(i)}(t) \geq y^{(i)}(t)$  for  $t \in J$  ( $i = 0, 1$ ). The proof is proved.

**Theorem 3** *Let cone  $P$  be fully regular and conditions  $(H_1)$ ,  $(H_2)$  and  $(H_4)$  be satisfied. Then the conclusion of Theorem 2 holds.*

**Proof.** The proof is almost the same as that of Theorem 2. The only difference is that, instead of using condition  $(H_3)$ , the conclusion  $\alpha_D(W) = 0$  is implied directly by (48) and (49), the full regularity of  $P$  and Lemma 4.

## 4 An example

Consider the infinite system of scalar second order impulsive singular integro-differential equations

$$\left\{ \begin{array}{l} -x_n''(t) = \frac{1}{4n^3 \sqrt[3]{e^{2t}(2+5t)^9}} \left( 5 + x_n(t) + x'_{2n}(t) + \frac{2}{3n^2 x_n(t)} + \frac{4}{7n^5 x'_{2n}(t)} \right)^{\frac{1}{2}} \\ \quad + \frac{1}{4\sqrt[6]{t}(1+3t)^2} \ln[(1+3t)x_n(t)], \\ \Delta x|_{t=t_k} = \frac{1}{n^3} \cdot \frac{k}{2^{k+1}} \left( \frac{1}{x_n(t_k)} + x'_{2n}(t_k) \right)^{\frac{1}{5}}, \quad k = 1, 2, \dots, \\ \Delta x'|_{t=t_k} = \frac{1}{n^4} \cdot \frac{1}{(k+1)^3} \left( \frac{1}{x_{n+1}(t_k)} + x'_{n+2}(t_k) \right)^{\frac{1}{7}}, \quad k = 1, 2, \dots, \\ x_n(0) = \int_0^\infty e^{-t^2} x_n(t) dt, \quad x'_n(\infty) = \frac{1}{n}, \quad n = 1, 2, \dots \end{array} \right. \quad (54)$$

**Proposition 1** *Infinite system (54) has a minimal positive solution  $x_n(t)$  satisfying  $x_n(t), x'_n(t) \geq \frac{1}{n}$  for  $0 \leq t < +\infty$  ( $n = 1, 2, 3, \dots$ ), and this minimal solution can be obtained by taking limits from some iterative sequences.*

**Proof.** Let  $E = c_0 = \{x = (x_1, \dots, x_n, \dots) : x_n \rightarrow 0\}$  with the norm  $\|x\| = \sup_n |x_n|$ . Obviously,  $(E, \|\cdot\|)$  is a real Banach space. Choose  $P = \{x = (x_n) \in c_0 : x_n \geq 0, n = 1, 2, 3, \dots\}$ . It is easy to verify that  $P$  is a normal cone in  $E$  with normal constant 1. Now we consider infinite system (54), which can be regarded as a BVP of form (1) in  $E$  with  $x_\infty = (1, \frac{1}{2}, \frac{1}{3}, \dots)$ . In this situation,  $x = (x_1, \dots, x_n, \dots), y = (y_1, \dots, y_n, \dots), f = (f_1, \dots, f_n, \dots)$ , and  $I_{ik} = (I_{ik1}, \dots, I_{ikn}, \dots)$  ( $i = 0, 1$ ), in which

$$\begin{aligned} f_n(t, x, y) = & \frac{1}{4n^3 \sqrt[3]{e^{2t}(2+5t)^9}} \left( 5 + x_n + y_{2n} + \frac{2}{3n^2 x_n} + \frac{4}{7n^5 y_{2n}} \right)^{\frac{1}{2}} \\ & + \frac{1}{4\sqrt[6]{t}(1+3t)^2} \ln[(1+3t)x_n], \end{aligned} \quad (55)$$

and

$$I_{0kn} = \frac{1}{n^3} \cdot \frac{k}{2^{k+1}} \left( \frac{1}{x_n} + y_{2n} \right)^{\frac{1}{5}}, \quad I_{1kn} = \frac{1}{n^4} \cdot \frac{1}{(k+1)^3} \left( x_{n+1} + \frac{1}{y_{n+2}} \right)^{\frac{1}{7}}. \quad (56)$$

Let  $x_0^* = x_\infty = (1, \frac{1}{2}, \frac{1}{3}, \dots)$ . Then  $P_{0\lambda} = \{x = (x_1, x_2, \dots, x_n, \dots) : x_n \geq \frac{\lambda}{n}, n = 1, 2, 3, \dots\}$ , for  $\lambda > 0$ . It is clear,  $f \in C[J_+ \times P_{0\lambda} \times P_{0\lambda}, P]$  for any  $\lambda > 0$ . Noticing that  $\sqrt[3]{e^{2t}} > \sqrt[6]{t}$  for  $t > 0$ , by (55), we get

$$\|f(t, x, y)\| \leq \frac{1}{4\sqrt[6]{t}(1+3t)^2} \left\{ \left( \frac{143}{21} + \|x\| + \|y\| \right)^{\frac{1}{2}} + \ln[(1+3t)\|x\|] \right\}, \quad (57)$$

which imply (H<sub>1</sub>) is satisfied for  $a(t) = 0$ ,  $b(t) = c(t) = \frac{1}{4\sqrt[6]{t}(1+3t)^2}$ , and

$$z(u_0, u_1) = \left( \frac{143}{21} + u_0 + u_1 \right)^{\frac{1}{2}} + \ln[(1+3t)u_0].$$

On the other hand, for  $x \in P_{0\lambda^*}, y \in P_{0\lambda^*}$ , we have, by (56)

$$\|I_{0k}(x, y)\| \leq \frac{k}{2^{k+1}} (1 + \|y\|)^{\frac{1}{5}}, \quad \|I_{1k}(x, y)\| \leq \frac{1}{(k+1)^3} (\|x\| + 1)^{\frac{1}{7}},$$

which imply (H<sub>2</sub>) is satisfied for

$$F_0(u_0, u_1) = (1 + u_1)^{\frac{1}{5}}, \quad F_1(u_0, u_1) = (1 + u_0)^{\frac{1}{7}},$$

and

$$\eta_{0k} = \frac{k}{2^{k+1}}, \quad \eta_{1k} = \frac{1}{(k+1)^3}, \quad \gamma_{0k} = \frac{k}{2^{k+1}(1+t_k)}, \quad \gamma_{1k} = \frac{1}{(k+1)^3(1+t_k)}.$$

Let  $f^1 = \{f_1^1, f_2^1, \dots, f_n^1, \dots\}$ ,  $f^2 = \{f_1^2, f_2^2, \dots, f_n^2, \dots\}$ , where

$$f_n^1(t, x, y) = \frac{1}{4n^3 \sqrt[3]{e^{2t}}(2+5t)^9} \left( 5 + x_n + y_{2n} + \frac{2}{3n^2 x_n} + \frac{4}{7n^5 y_{2n}} \right)^{\frac{1}{2}}, \quad (58)$$

$$f_n^2(t, x, y) = \frac{1}{4\sqrt[6]{t}(1+3t)^2} \ln[(1+3t)x_n]. \quad (59)$$

Let  $t \in J_+$ , and  $R > 0$  be given and  $\{z^{(m)}\}$  be any sequence in  $f^1(t \times P_{0\lambda^*R}, P_{0\lambda^*R})$ , where  $z^{(m)} = (z_1^{(m)}, \dots, z_n^{(m)}, \dots)$ . By (58), we have

$$0 \leq z_n^{(m)} \leq \frac{1}{4n^3 \sqrt[3]{e^{2t}}(2+5t)^9} \left( \frac{143}{21} + 2R \right)^{\frac{1}{2}} \quad (n, m = 1, 2, 3, \dots). \quad (60)$$

So,  $\{z_n^{(m)}\}$  is bounded and by the diagonal method together with the method of constructing subsequence, we can choose a subsequence  $\{m_i\} \subset \{m\}$  such that

$$\{z_n^{(m_i)}\} \rightarrow \bar{z}_n \quad \text{as } i \rightarrow \infty \quad (n = 1, 2, 3, \dots), \quad (61)$$

which implies by (60)

$$0 \leq \bar{z}_n \leq \frac{1}{4n^3 \sqrt[3]{e^{2t}}(2+5t)^9} \left( \frac{143}{21} + 2R \right)^{\frac{1}{2}} \quad (n = 1, 2, 3, \dots). \quad (62)$$



Hence  $\bar{z} = (\bar{z}_1, \dots, \bar{z}_n, \dots) \in c_0$ . It is easy to see from (60)-(62) that

$$\|z^{(m_i)} - \bar{z}\| = \sup_n |z_n^{(m_i)} - \bar{z}_n| \rightarrow 0 \text{ as } i \rightarrow \infty.$$

Thus, we have proved that  $f^1(t \times P_{0\lambda^*R}, P_{0\lambda^*R})$  is relatively compact in  $c_0$ .

For any  $t \in J_+$ ,  $R > 0$ ,  $x, y, \bar{x}, \bar{y} \in D \subset P_{0\lambda^*R}$ , we have by (59)

$$\begin{aligned} |f_n^2(t, x, y) - f_n^2(t, \bar{x}, \bar{y})| &= \frac{1}{4\sqrt[6]{t}(1+3t)^2} |\ln[(1+3t)x_n] - \ln[(1+3t)\bar{x}_n]| \\ &\leq \frac{1}{4\sqrt[6]{t}(1+3t)} \frac{|x_n - \bar{x}_n|}{(1+3t)\xi_n}, \end{aligned} \quad (63)$$

where  $\xi_n$  is between  $x_n$  and  $\bar{x}_n$ . By (63), we get

$$\|f^2(t, x, y) - f^2(t, \bar{x}, \bar{y})\| \leq \frac{1}{4\sqrt[6]{t}(1+3t)} \|x - \bar{x}\|, \quad x, y, \bar{x}, \bar{y} \in D. \quad (64)$$

Thus, by (64), it is easy to see that  $(H_3)$  holds for  $h_0(t) = \frac{1}{4\sqrt[6]{t}(1+3t)}$ . Our conclusion follows from Theorem 2. This completes the proof.

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