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Asymptotic representation of intermediate solutions to a cyclic systems of second-order difference equations with regularly varying coefficients

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Abstract. The cyclic system of second-order difference equations

$$\Delta(p_i(n)|\Delta x_i(n)|^{\alpha_i-1}\Delta x_i(n)) = q_i(n)|x_{i+1}(n+1)|^{\beta_i-1}x_{i+1}(n+1),$$

for $i = \overline{1,N}$ where $x_{N+1} = x_1$, is analysed in the framework of discrete regular variation. Under the assumption that α_i and β_i are positive constants such that $\alpha_1\alpha_2\cdots\alpha_N > \beta_1\beta_2\cdots\beta_N$ and p_i and q_i are regularly varying sequences it is shown that the situation in which this system possesses regularly varying intermediate solutions can be completely characterized. Besides, precise information can be acquired about the asymptotic behavior at infinity of these solutions.

Keywords: system of difference equations, Emden–Fowler type difference equation, nonlinear difference equations, intermediate solutions, asymptotic behavior, regularly varying sequence, discrete regular variation.

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1 Introduction

There has been some recent interest in studying of various systems of difference equations. Since the mid of nineties there has been a considerable interest in symmetric systems (see, e.g., [3, 38–40, 44] and the references therein). If some parameters in symmetric systems are modified then are obtained more general systems which are now frequently called close-to-symmetric systems (see [49] and [50]); for some other systems of the type see, e.g. [37,41,47,48] and the references therein. Multidimensional extensions of symmetric and close-to-symmetric systems are called cyclic systems of difference equations. Systems of the type were studied, for example, in [13,45,46].

The system of nonlinear difference equations which will be studied in this paper is the following cyclic one:

$$\Delta(p_i(n)|\Delta x_i(n)|^{\alpha_i-1}\Delta x_i(n)) + q_i(n)|x_{i+1}(n+1)|^{\beta_i-1}x_{i+1}(n+1) = 0, \tag{E}$$

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where $i = \overline{1,N}$, $x_{N+1} = x_1$, $n \in \mathbb{N}$, and following conditions hold:

(a) α_i and β_i, $i = \overline{1,N}$ are positive constants such that $\alpha_1\alpha_2 \cdot \ldots \cdot \alpha_N > \beta_1\beta_2 \cdot \ldots \cdot \beta_N$;

(b) $p_i = \{p_i(n)\}$, $q_i = \{q_i(n)\}$ are positive real sequences;

(c) All p_i, $i = \overline{1,N}$ simultaneously satisfy either

$$(I) \qquad S_i = \sum_{n=1}^{\infty} \frac{1}{p_i(n)^{1/\alpha_i}} = \infty,$$

or

$$(II) \qquad S_i = \sum_{n=1}^{\infty} \frac{1}{p_i(n)^{1/\alpha_i}} < \infty.$$

In our case, when $\alpha_1\alpha_2 \cdots \alpha_N > \beta_1\beta_2 \cdots \beta_N$, the system (E) is said to be sub-half-linear. If opposite inequality hold then the system is super-half-linear and if equality hold then system is called half-linear.

We say that a solution $\mathbf{x} = \{\mathbf{x}(n)\} = \{(x_1(n), x_2(n), \ldots, x_N(n))\} \in {}^{\mathbb{N}}\mathbb{R} \times \cdots \times {}^{\mathbb{N}}\mathbb{R}$, ${}^{\mathbb{N}}\mathbb{R} = \{f \mid f : \mathbb{N} \to \mathbb{R}\}$, of (E) *oscillates* if for some $i \in \{1, 2, \ldots, N\}$, and for every integer $n_0 > 0$ there exists $n > n_0$ such that $x_i(n)x_i(n+1) < 0$. A solution $\mathbf{x}(n)$ of (E) is said to be *nonoscillatory* if there exists an integer $n_0 > 0$ such that for each $i = \overline{1,N}$, $x_i(n) \neq 0$ for $n \geq n_0$.

It can easily be seen that all components of positive solution $\mathbf{x}$ of system (E) satisfy

$$c_i \leq x_i(n) \leq C_i \cdot P_i(n), \quad i = \overline{1,N} \quad \text{if} \quad (I) \quad \text{holds} \tag{1.1}$$

or

$$k_i\pi_i(n) \leq x_i(n) \leq K_i, \quad i = \overline{1,N} \quad \text{if} \quad (II) \quad \text{holds} \tag{1.2}$$

where $P_i(n) = \sum_{k=1}^{n-1} \frac{1}{p_i(k)^{1/\alpha_i}}$ and $\pi_i(n) = \sum_{k=n}^{\infty} \frac{1}{p_i(k)^{1/\alpha_i}}$ for $i = \overline{1,N}$, and c_i, C_i, k_i and K_i are positive real constants.

Indeed, if (I) holds, then is easy to see that $\Delta x_i(n) > 0$, for $i = \overline{1,N}$. This implies left inequality in (1.1). Also, $p_i(n)\Delta x_i(n)^{\alpha_i}$, $i = \overline{1,N}$ are decreasing, so there exist positive constants b_i such that $p_i(n)\Delta x_i(n)^{\alpha_i} \leq b_i$. From previous, follows that

$$x_i(n) \leq x_1(n) + b_i^{\frac{1}{\alpha_i}} P_i(n), \quad i = \overline{1,N}.$$

How all $P_i(n)$ are increasing and divergent we get right inequality in (1.1).

Similarly, if (II) holds, then $\Delta x_i(n) < 0$, for $i = \overline{1,N}$, so right inequality in (1.2) holds. On the other side, how $-p_i(n)\left(-\Delta x_i(n)\right)^{\alpha_i} \leq 0$, and decrease for $i = \overline{1,N}$ it follows that $-p_i(n)\left(-\Delta x_i(n)\right)^{\alpha_i} \leq h_i$, $h_i \in \mathbb{R}^+$. Since, x_i, $i = \overline{1,N}$ are decreasing and positive, all $x_i(\infty) = \lim_{n\to\infty} x_i(n)$ are finite and

$$x_i(n) \geq x_i(\infty) + h_i^{\frac{1}{\alpha_i}} \pi_i(n).$$

Using that $\pi_i(n)$, $i = \overline{1,N}$ are decreasing and tends to zero we get left inequality in (1.2).

In the case (I) for each component x_i of solution $\mathbf{x}$ only one of next three possibilities holds:

$(S1)$ $\lim_{n\to\infty} \frac{x_i(n)}{P_i(n)} = const > 0$,

$(IM1)$ $\lim_{n\to\infty} x_i(n) = \infty$, $\lim_{n\to\infty} \frac{x_i(n)}{P_i(n)} = 0$,

(AC) $\lim_{n\to\infty} x_i(n) = const > 0.$

In the case (II) for each component x_i of solution $\mathbf{x}$ only one of next three possibilities holds:

(AC) $\lim_{n\to\infty} x_i(n) = const > 0,$

$(IM2)$ $\lim_{n\to\infty} x_i(n) = 0,\ \lim_{n\to\infty} \frac{x_i(n)}{\pi_i(n)} = \infty,$

$(S2)$ $\lim_{n\to\infty} \frac{x_i(n)}{\pi_i(n)} = const > 0.$

We consider a solutions whose all components are the same type. Solutions where the components are different types have not yet been considered in the existing literature. How we see, in the case (I) we have only increasing solutions. Solution is increasing if all its components are increasing sequences. Solutions of type $(S1)$ are asymptotically equivalent to constant times $P_i(n)$ and solutions of type (AC) are asymptotically equivalent to constant. Solutions of type $(IM1)$ are called intermediate solutions. In the case (II) we have only decreasing solutions. In this case, we again have solutions which are asymptotically equivalent to constant and solutions which are asymptotically equivalent to constant times $\pi_i(n)$, $n \to \infty$, marked as $(S2)$. Here, also, we have intermediate solutions $(IM2)$ but different kind then in the first case.

Our aim in this paper is to observe intermediate solutions $(IM1)$ and $(IM2)$, and to answer to following two questions:

1. Is it possible to establish sufficient and necessary conditions for the existence of intermediate solution of system (E)?
2. Is it possible to establish the unique explicit asymptotic formula for this solutions?

The results obtained in this paper can be extended to a system of N equations of the first order only when N is even.

The asymptotic behavior of nonoscillatory solutions for second-order difference equation has been studied in many papers, see, e.g. [2, 6–10, 19, 34–43], the monograph [1] and references therein. Oscillation and existence criteria for positive solutions of discrete systems were considered in [28–30], however, in the existing literature, there are no results for cyclic system of difference equations of second order.

The recent development of asymptotic analysis of ordinary differential equations and cyclic system of differential equations by means of regular varying functions (see [24–27, 32–42] and monograph [31] for results up to 2000.), suggests investigating the discrete problem in the framework of regularly varying sequences. Thus, we limit ourselves to the system (E) with coefficients $p_i = \{p_i(n)\}$, $q_i = \{q_i(n)\}$ which are regularly varying sequences and we establish necessary and sufficient conditions for the existence of intermediate regularly varying solutions of (E) and obtain precise asymptotic representation of such solutions.

The theory of regularly varying sequences, sometimes called Karamata sequences, was initiated in 1930 by Karamata [20] and further developed in seventies by Galambos, Seneta and Bojanić in [5, 12] and recently in [11]. However, until the paper of Matucci and Řehák [35], relation between regularly varying sequences and difference equations has never been under consideration. In [35], as well as in succeeding papers [34, 36, 43], theory of regularly varying sequences has been further developed and applied in asymptotic analysis of second-order linear and half-linear difference equations. However, to our knowledge, theory of regularly varying sequences has not been used for asymptotic analysis of any other type of second-order nonlinear difference equation, except by Agarwal and Manojlović in [2], Kapešić and Manojlović in [19] and Kharkov in [21] and [22]. Also, Kharkov in [23] give asymptotic representation of solutions of k-th order difference equations of Emden–Fowler type. Assuming that coefficients are normalized regularly varying sequences (introduced by Matucci and Rehak in [34]), asymptotic forms of positive intermediate solutions of Emden–Fowler second-order difference equation has been established in [2]. Thus, the purpose of this paper is to proceed further in this direction and to establish results which can be considered as a discrete analogue of results in the continuous case (see e.g. [17, 18]).

Throughout this paper extensive use is made of the symbol $\sim$ to denote the asymptotic equivalence of two positive sequences, i.e.

$$x_n \sim y_n,\ n \to \infty \quad \Leftrightarrow \quad \lim_{n\to\infty} \frac{y_n}{x_n} = 1.$$

The main results are given in Section 4. Intermediate solution of system (E) are solution of system of equations

$$x_i(n) = c_i + \sum_{k=n_0}^{n-1} \left(\frac{1}{p_i(k)} \sum_{s=k}^{\infty} q_i(s) x_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}}, \qquad i = \overline{1,N} \quad \text{if } (I) \text{ holds}, \tag{1.3}$$

$$x_i(n) = \sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \sum_{s=n_0}^{k-1} q_i(s) x_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}}, \qquad i = \overline{1,N} \quad \text{if } (II) \text{ holds}, \tag{1.4}$$

for some constants $n_0 \geq 1$ and $c_i > 0$. It follows therefore that intermediate solution of (E) satisfies asymptotic relations

$$x_i(n) \sim \sum_{k=n_0}^{n-1} \left(\frac{1}{p_i(k)} \sum_{s=k}^{\infty} q_i(s) x_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}}, \qquad n \to \infty, \quad i = \overline{1,N}, \tag{1.5}$$

or

$$x_i(n) \sim \sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \sum_{s=n_0}^{k-1} q_i(s) x_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}}, \qquad n \to \infty, \quad i = \overline{1,N} \tag{1.6}$$

in case (I) or (II) respectively.

The proof of our main results are essentially based on the fact that a through knowledge of the existence and asymptotic behavior of regularly varying solution of (1.5) and (1.6) can be acquired. In fact, one direction of proof of main theorems is an immediate consequence of manipulation of (1.5) and (1.6) by means of regular variation. The other direction is proved in two steps – first showing the existence of solution of the system of relations (1.5) and (1.6) with the help of fixed point techniques and in the next step using Stolz–Cesarò theorem showing that such solution is regularly varying.

Our main tools are, besides theory of regularly varying sequences presented in Section 2, the fixed point technique and Stolz–Cesarò theorem. Thus, we recall two variants of Stolz–Cesarò theorem as well as Knaster's fixed point theorem [1, Theorem 5.2.1].

Lemma 1.1. *If $f = \{f_n\}$ is a strictly increasing sequence of positive real numbers, such that $\lim_{n\to\infty} f_n = \infty$, then for any sequence $g = \{g_n\}$ of positive real numbers one has the inequalities:*

$$\liminf_{n\to\infty} \frac{\Delta f_n}{\Delta g_n} \leq \liminf_{n\to\infty} \frac{f_n}{g_n} \leq \limsup_{n\to\infty} \frac{f_n}{g_n} \leq \limsup_{n\to\infty} \frac{\Delta f_n}{\Delta g_n}.$$

In particular, if the sequence $\{\Delta f_n / \Delta g_n\}$ *has a limit then*

$$\lim_{n\to\infty} \frac{f_n}{g_n} = \lim_{n\to\infty} \frac{\Delta f_n}{\Delta g_n}. \tag{1.7}$$

Lemma 1.2. *Let* $f = \{f_n\}$, $g = \{g_n\}$ *be sequences of positive real numbers, such that*

(i) $\lim_{n\to\infty} f_n = \lim_{n\to\infty} g_n = 0$;

(ii) the sequence g *is strictly monotone;*

(iii) the sequence $\{\Delta f_n / \Delta g_n\}$ *has a limit.*

Then, a sequence $\{f_n / g_n\}$ *is convergent and* (1.7) *hold.*

Lemma 1.3 (Knaster's fixed point theorem)**.** *Let* X *be a partially ordered Banach space with ordering* $\leq$. *Let* M *be a subset of* X *with the following properties: the infimum of* M *belongs to* M *and every nonempty subset of* M *has a supremum which belongs to* M. *Let* $\mathcal{F} : M \to M$ *be an increasing mapping, i.e.* $x \leq y$ *implies* $\mathcal{F}x \leq \mathcal{F}y$. *Then* $\mathcal{F}$ *has a fixed point in* M.

2 Regularly varying sequences

We state here definitions and some basic properties of regularly varying sequences which will be essential in establishing our main results on the asymptotic behavior of nonoscillatory solutions stated and proved in the next section. For a comprehensive treatise on regular variation the reader is referred to Bingham et al. [4].

Two main approaches are known in the basic theory of regularly varying sequences: the approach due to Karamata [20], based on a definition that can be understood as a direct discrete counterpart of simple and elegant continuous definition (see Definition 2.3), and the approach due to Galambos and Seneta, based on purely sequential definition.

Definition 2.1 (Karamata [20])**.** A positive sequence $y = \{y(k)\}, k \in \mathbb{N}$ is said to be *regularly varying of index* $\rho \in \mathbb{R}$ if

$$\lim_{k\to\infty} \frac{y([\lambda k])}{y(k)} = \lambda^{\rho} \quad \text{for } \forall \lambda > 0,$$

where $[u]$ denotes the integer part of u.

Definition 2.2 (Galambos and Seneta [12])**.** A positive sequence $y = \{y(k)\}, k \in \mathbb{N}$ is said to be *regularly varying of index* $\rho \in \mathbb{R}$ if there exists a positive sequence $\{\alpha(k)\}$ satisfying

$$\lim_{k\to\infty} \frac{y(k)}{\alpha(k)} = C,\ 0 < C < \infty, \qquad \lim_{k\to\infty} k\,\frac{\Delta\alpha(k-1)}{\alpha(k)} = \rho\,.$$

Definition 2.3. A measurable function $f : (a,\infty) \to (0,\infty)$ for some $a > 0$ is said to be regularly varying at infinity of index $\rho \in \mathbb{R}$ if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^{\rho} \quad \text{for all } \lambda > 0.$$

If $\rho = 0$, then y is said to be *slowly varying*. The totality of regularly varying sequences of index ρ and slowly varying sequences denoted, respectively, by $\mathcal{RV}(\rho)$ and $\mathcal{SV}$.

Bojanić and Seneta have shown in [5] that Definition 2.1 and Definition 2.2 are equivalent.

The concept of normalized regularly varying sequences were introduced by Matucci and Rehak in [35], where they also offered a modification of Definition 2.2, i.e. they proved that second limit in Definition 2.2 can be replaced with

$$\lim_{k\to\infty} k\,\frac{\Delta\alpha(k)}{\alpha(k)} = \rho\,.$$

Definition 2.4. A positive sequence $y = \{y(k)\}, k \in \mathbb{N}$ is said to be *normalized regularly varying of index* $\rho \in \mathbb{R}$ if it satisfies

$$\lim_{k\to\infty} \frac{k\Delta y(k)}{y(k)} = \rho.$$

If $\rho = 0$, then y is called a *normalized slowly varying sequence.*

In what follows, $\mathcal{NRV}(\rho)$ and $\mathcal{NSV}$ will be used to denote the set of all normalized regularly varying sequences of index ρ and the set of all normalized slowly varying sequences.

Typical examples are:

$$\{\log k\} \in \mathcal{NSV}, \quad \{k^\rho \log k\} \in \mathcal{NRV}(\rho), \quad \{1 + (-1)^k/k\} \in \mathcal{SV} \setminus \mathcal{NSV}\,.$$

There exist various necessary and sufficient conditions for a sequence of positive numbers to be regularly varying (see [5,12,34,35]) and consequently each one of them could be used to define regularly varying sequence. The one that is the most important is the following Representation theorem (see [5, Theorem 3]), while some other representation formula for regularly varying sequences were established in [35, Lemma 1].

Theorem 2.5 (Representation theorem)**.** *A positive sequence* $\{y(k)\}, k \in \mathbb{N}$ *is said to be regularly varying of index* $\rho \in \mathbb{R}$ *if and only if there exists sequences* $\{c(k)\}$ *and* $\{\delta(k)\}$ *such that*

$$\lim_{k\to\infty} c(k) = c_0 \in (0,\infty) \quad \textit{and} \quad \lim_{k\to\infty} \delta(k) = 0,$$

and

$$y(k) = c(k)\,k^\rho\,\exp\left(\sum_{i=1}^{k} \frac{\delta(i)}{i}\right).$$

In [5] very useful imbedding theorem was proved, which gives possibility of using the continuous theory in developing a theory of regularly varying sequences. However, as noticed in [5], such development in not generally close and sometimes far from a simple imitation of arguments for regularly varying functions.

Theorem 2.6 (Imbedding theorem)**.** *If* $y = \{y(n)\}$ *is regularly varying sequence of index* $\rho \in \mathbb{R}$*, then function* $Y(t)$ *defined on* $[0,\infty)$ *by* $Y(t) = y([t])$ *is a regularly varying function of index* ρ*. Conversely, if* $Y(t)$ *is a regularly varying function on* $[0,\infty)$ *of index* ρ*, then a sequence* $\{y(k)\}$*,* $y(k) = Y(k)$*,* $k \in \mathbb{N}$ *is regularly varying of index* ρ*.*

Next, we state some important properties of $\mathcal{RV}$ sequences useful for the development of asymptotic behavior of solutions of (E) in the subsequent section (for more properties and proofs see [5,34]).

Theorem 2.7. *Following properties hold:*

(i) $y \in \mathcal{RV}(\rho)$ *if and only if* $y(k) = k^\rho\, l_k$*, where* $l = \{l(k)\} \in \mathcal{SV}$.

(ii) *Let* $x \in \mathcal{RV}(\rho_1)$ *and* $y \in \mathcal{RV}(\rho_2)$. *Then,* $xy \in \mathcal{RV}(\rho_1 + \rho_2)$*,* $x + y \in \mathcal{RV}(\rho)$*,* $\rho = \max\{\rho_1, \rho_2\}$ *and* $1/x \in \mathcal{RV}(-\rho_1)$.

(iii) *If* $y \in \mathcal{RV}(\rho)$*, then* $\lim_{k\to\infty} \frac{y(k+1)}{y(k)} = 1$.

(iv) *If* $l \in \mathcal{SV}$ *and* $l(k) \sim L(k)$*,* $k \to \infty$*, then* $L \in \mathcal{SV}$.

(v) *If* $y \in \mathcal{NRV}(\rho)$*, then* $\{n^{-\sigma}y(n)\}$ *is eventually increasing for each* $\sigma < \rho$ *and* $\{n^{-\mu}y(n)\}$ *is eventually decreasing for each* $\mu > \rho$.

In view of the statement (i) of the previous theorem, if for $y \in \mathcal{RV}(\rho)$

$$\lim_{k\to\infty} \frac{y(k)}{k^\rho} = \lim_{k\to\infty} l(k) = const > 0,$$

then $y = \{y(n)\}$ is said to be a *trivial regularly varying sequence of index* ρ and is denoted by $y \in tr - \mathcal{RV}(\rho)$. Otherwise y is said to be a *nontrivial regularly varying sequence of index* ρ, denoted by $y \in ntr - \mathcal{NRV}(\rho)$.

Next Theorem can be found in [2] for normalized regularly varying sequences, but it clearly hold for all regularly varying sequences because its proof is based on the Mean Value Theorem and property (iii) from Theorem 2.7 which holds for all $\mathcal{RV}$ sequences (not only for $\mathcal{NRV}$).

Theorem 2.8. *If* $f = \{f(n)\} \in \mathcal{RV}$ *is a strictly decreasing sequence, such that* $\lim_{n\to\infty} f(n) = 0$*, then for each* $\gamma \in \mathbb{R}$

$$\lim_{n\to\infty} f(n)^{-\gamma} \sum_{k=n}^{\infty} f(k)^{\gamma-1}\big(-\Delta f(k)\big) = \frac{1}{\gamma}. \tag{2.1}$$

If $g = \{g(n)\} \in \mathcal{RV}$ *is a strictly increasing sequence such that* $\lim_{n\to\infty} g(n) = \infty$*, then*

$$\lim_{n\to\infty} g(n)^{-\gamma} \sum_{k=1}^{n-1} g(k)^{\gamma-1}\Delta g(k) = \frac{1}{\gamma}. \tag{2.2}$$

The following theorem can be concerned as *the discrete analog of the Karamata's integration theorem* and plays a central role in the proof of our main results in the Section 3. Proof of this Theorem can be found in [5] and [19] and partially in [43].

Theorem 2.9. *Let* $l = \{l(n)\} \in \mathcal{SV}$.

(i) *If* $\alpha > -1$*, then* $\lim_{n\to\infty} \frac{1}{n^{\alpha+1}l(n)} \sum_{k=1}^{n} k^\alpha l(k) = \frac{1}{1+\alpha}$;

(ii) *If* $\alpha < -1$*, then* $\lim_{n\to\infty} \frac{1}{n^{\alpha+1}l(n)} \sum_{k=n}^{\infty} k^\alpha l(k) = -\frac{1}{1+\alpha}$;

(iii) *If* $\sum_{k=1}^{\infty} \frac{l(k)}{k} < \infty$*, then* $\sum_{k=n}^{\infty} \frac{l(k)}{k} \in \mathcal{SV}$ *and* $\lim_{n\to\infty} \frac{1}{l(n)} \sum_{k=n}^{\infty} \frac{l(k)}{k} = \infty$;

(iv) *If* $\sum_{k=1}^{\infty} \frac{l(k)}{k} = \infty$*, then* $\sum_{k=1}^{n} \frac{l(k)}{k} \in \mathcal{SV}$ *and* $\lim_{n\to\infty} \frac{1}{l(n)} \sum_{k=1}^{n} \frac{l(k)}{k} = \infty$.

Remark 2.10. It is easy to see, in view od Theorem 2.7 (iii) and Theorem 2.9 (i), that for $l \in \mathcal{SV}$, if $\alpha > -1$, we have

$$\sum_{k=1}^{n-1} k^\alpha l(k) \sim \frac{(n-1)^{\alpha+1} l(n-1)}{\alpha+1} \sim \frac{n^{\alpha+1} l(n)}{\alpha+1} \sim \sum_{k=1}^{n} k^\alpha l(k), \qquad n \to \infty,$$

and since $\lim_{n\to\infty} \sum_{k=1}^{n-1} k^\alpha l(k) = \infty$, we also get

$$\sum_{k=n_0}^{n} k^\alpha l(k) \sim \sum_{k=1}^{n} k^\alpha l(k), \qquad n \to \infty.$$

If $\lim_{n\to\infty} \sum_{k=1}^{n} k^{-1} l(k) = \infty$, we have

$$\sum_{k=n_0}^{n} k^{-1} l(k) \sim \sum_{k=1}^{n} k^{-1} l(k), \qquad n \to \infty.$$

Definition 2.11. A vector $\mathbf{x} \in {}^{\mathbb{N}}\mathbb{R} \times \ldots \times {}^{\mathbb{N}}\mathbb{R}$ is said to be regularly varying of index $(\rho_1, \rho_2, \ldots, \rho_N)$ if $x_i \in \mathcal{RV}(\rho_i)$ for $i = \overline{1, N}$. If all ρ_i are positive (or negative), then $\mathbf{x}$ is called regularly varying of positive (or negative) index $(\rho_1, \rho_2, \ldots, \rho_N)$. The set of all regularly varying vectors of index $(\rho_1, \rho_2, \ldots, \rho_N)$ is denoted by $\mathcal{RV}(\rho_1, \rho_2, \ldots, \rho_N)$.

3 Preliminaries and preparation results

In this section assume that $p_i \in \mathcal{RV}(\lambda_i)$ and $q_i \in \mathcal{RV}(\mu_i)$ and represent them with

$$p_i(n) = n^{\lambda_i} l_i(n), \quad q_i(n) = n^{\mu_i} m_i(n), \quad l_i, m_i \in \mathcal{SV}, \quad i = \overline{1, N}. \tag{3.1}$$

We also assume that all sequences p_i, $i = \overline{1, N}$ satisfy either (I) or (II). Condition (I) (resp. (II)) holds if and only if

$$\lambda_i < \alpha_i \quad \text{or} \quad \lambda_i = \alpha_i \quad \text{and} \quad \sum_{n=1}^{\infty} n^{-1} l_i(n)^{-\frac{1}{\alpha_i}} = \infty, \tag{3.2}$$

resp.

$$\lambda_i > \alpha_i \quad \text{or} \quad \lambda_i = \alpha_i \quad \text{and} \quad \sum_{n=1}^{\infty} n^{-1} l_i(n)^{-\frac{1}{\alpha_i}} < \infty. \tag{3.3}$$

In this paper we do not consider cases when $\lambda_i = \alpha_i$ for one or all i (these cases lead to $\rho_i = 0$) because of computational difficulty. Therefore, we have requirements of positivity or negativity for the regularity indices of solutions.

Therefore, if the case (I) is satisfied, then $\lambda_i < \alpha_i$ and for the sequences $P_i = \{P_i(n)\}$ given by $P_i(n) = \sum_{k=1}^{n-1} p_i(k)^{-\frac{1}{\alpha_i}}$, $i = \overline{1, N}$ we have

$$P_i(n) \sim \frac{\alpha_i}{\alpha_i - \lambda_i} n^{\frac{\alpha_i - \lambda_i}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}}. \tag{3.4}$$

In the case (II), when $\lambda_i > \alpha_i$ for the sequences $\pi_i = \{\pi_i(n)\}$ given by $\pi_i(n) = \sum_{k=n}^{\infty} p_i(k)^{-\frac{1}{\alpha_i}}$, $i = \overline{1, N}$ we have

$$\pi_i(n) \sim \frac{\alpha_i}{\lambda_i - \alpha_i} n^{\frac{\alpha_i - \lambda_i}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}}. \tag{3.5}$$

In what follows to simplfy notation we denote $A_N = \alpha_1\alpha_2 \cdot \ldots \cdot \alpha_N$, $B_N = \beta_1\beta_2 \cdot \ldots \cdot \beta_N$ and use matrix

$$M = \begin{pmatrix} 1 & \frac{\beta_1}{\alpha_1} & \frac{\beta_1\beta_2}{\alpha_1\alpha_2} & \cdots & \cdots & \frac{\beta_1\beta_2\cdots\beta_{N-1}}{\alpha_1\alpha_2\cdots\alpha_{N-1}} \\ & 1 & \frac{\beta_2}{\alpha_2} & \frac{\beta_2\beta_3}{\alpha_2\alpha_3} & \cdots & \frac{\beta_2\beta_3\cdots\beta_{N-1}}{\alpha_2\alpha_3\cdots\alpha_{N-1}} \\ & & 1 & \frac{\beta_3}{\alpha_3} & \cdots & \frac{\beta_3\cdots\beta_{N-1}}{\alpha_3\cdots\alpha_{N-1}} \\ & & & \ddots & \ddots & \vdots \\ & & & & 1 & \frac{\beta_{N-1}}{\alpha_{N-1}} \\ & & & & & 1 \end{pmatrix}, \tag{3.6}$$

whose elements will be denoted by $M = (M_{ij})$, where the lower triangular elements are omitted for economy of notation. In fact, the $i-$th row of (M_{ij}) is obtained by shifting the vector

$$\left(1, \frac{\beta_i}{\alpha_i}, \frac{\beta_i\beta_{i+1}}{\alpha_i\alpha_{i+1}}, \ldots, \frac{\beta_i\beta_{i+1}\cdots\beta_{i+(N-2)}}{\alpha_i\alpha_{i+1}\cdots\alpha_{i+(N-2)}}\right), \quad \alpha_{N+j} = \alpha_j,\ \beta_{N+j} = \beta_j,\ j = \overline{1, N-2}$$

$(i-1)$-times to the right cyclically, so that the lower triangular elements $M_{ij}, i > j$, satisfy the relation

$$M_{ij}M_{ji} = \frac{\beta_1\beta_2\cdots\beta_N}{\alpha_1\alpha_2\cdots\alpha_N}, \quad i > j, \quad i = \overline{2, N}.$$

The following theorem gives us necessary and sufficient condition for the existence of regularly varying solution $\mathbf{x}$ of positive index $(\rho_1, \rho_2, \ldots, \rho_N)$ of the system of asymptotic relations (1.5).

Theorem 3.1. *Let $p_i \in \mathcal{RV}(\lambda_i)$, $q_i \in \mathcal{RV}(\mu_i)$ and suppose that $\lambda_i < \alpha_i$, $i = \overline{1, N}$. The system of asymptotic relations (1.5) has regularly varying solution $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \ldots, \rho_N)$ with $\rho_i \in \left(0, \frac{\alpha_i - \lambda_i}{\alpha_i}\right)$, $i = \overline{1, N}$ if and only if*

$$0 < \sum_{j=1}^{N} M_{ij} \frac{\alpha_j - \lambda_j + \mu_j + 1}{\alpha_j} < \frac{\alpha_i - \lambda_i}{\alpha_i}\left(1 - \frac{B_N}{A_N}\right), \qquad i = \overline{1, N} \tag{3.7}$$

holds, in which case ρ_i are uniquely determined by

$$\rho_i = \frac{A_N}{A_N - B_N} \sum_{j=1}^{N} M_{ij} \frac{\alpha_j - \lambda_j + \mu_j + 1}{\alpha_j}, \qquad i = \overline{1, N} \tag{3.8}$$

and the asymptotic behavior of any such solution is governed by the unique formula

$$x_i(n) \sim n^{\rho_i} \left[\prod_{j=1}^{N} \left(\frac{l_j(n)^{-\frac{1}{\alpha_j}} m_j(n)^{\frac{1}{\alpha_j}}}{D_j} \right)^{M_{ij}} \right]^{\frac{A_N}{A_N - B_N}}, \qquad n \to \infty, \quad i = \overline{1, N}. \tag{3.9}$$

Proof. Let $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \ldots, \rho_N)$ with all $\rho_i > 0$ be a solution of (1.5) expressed in the form

$$x_i(n) = n^{\rho_i}\xi_i(n), \qquad \xi_i \in \mathcal{SV}, \quad i = \overline{1, N}. \tag{3.10}$$

From (3.4) and (1.1) each ρ_i must satisfy $\rho_i < \frac{\alpha_i - \lambda_i}{\alpha_i}$, $i = \overline{1, N}$. Using (3.1) and (3.10), we have

$$\sum_{k=n}^{\infty} q_i(k) x_{i+1}(k+1)^{\beta_i} \sim \sum_{k=n}^{\infty} k^{\mu_i + \beta_i\rho_{i+1}} m_i(k) \xi_{i+1}(k)^{\beta_i}, \qquad n \geq n_0, \quad i = \overline{1, N}. \tag{3.11}$$

The convergence of (3.11) as $n \to \infty$ implies that $\mu_i + \beta_i \rho_{i+1} \leq -1$, $i = \overline{1,N}$. If for some i equality holds, then since

$$\left(\frac{1}{p_i(n)}\sum_{k=n}^{\infty} q_i(k)x_{i+1}(k+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} \sim n^{-\frac{\lambda_i}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}} \left(\sum_{k=n}^{\infty} k^{-1} m_i(k)\xi_{i+1}(k)^{\beta_i}\right)^{\frac{1}{\alpha_i}}, \tag{3.12}$$

from (1.5) and Theorem 2.9 we find that

$$x_i(n) \sim \frac{\alpha_i}{\alpha_i - \lambda_i} n^{\frac{\alpha_i - \lambda_i}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}} \left(\sum_{k=n}^{\infty} k^{-1} m_i(k)\xi_{i+1}(k+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} \in \mathcal{RV}\left(\frac{\alpha_i - \lambda_i}{\alpha_i}\right)$$

as $n \to \infty$. This implies that $\rho_i = \frac{\alpha_i - \lambda_i}{\alpha_i}$ which is a contradiction with our assumption. It follows that $\mu_i + \beta_i \rho_{i+1} < -1$ for $i = \overline{1,N}$. Application of Theorem 2.9 to (3.11) gives

$$\left(\frac{1}{p_i(n)}\sum_{k=n}^{\infty} q_i(k)x_{i+1}(k+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} \sim \frac{n^{\frac{-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}} m_i(n)^{\frac{1}{\alpha_i}} \xi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{(-(\mu_i + \beta_i \rho_{i+1} + 1))^{\frac{1}{\alpha_i}}}, \tag{3.13}$$

when $n \to \infty$, $i = \overline{1,N}$. Because $x_i(n) \to \infty$, $n \to \infty$, we conclude from (3.13) that it must be $(-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1)/\alpha_i \geq -1$, $i = \overline{1,N}$. Here, also, the equality should be ruled out. If the equality holds for some i, then summing (3.13) from n_0 to $n-1$ yields

$$x_i(n) \sim \left(\frac{1}{\alpha_i - \lambda_i}\right) \sum_{k=n_0}^{n-1} k^{-1} l_i(k)^{-\frac{1}{\alpha_i}} m_i(k)^{\frac{1}{\alpha_i}} \xi_{i+1}(k)^{\frac{\beta_i}{\alpha_i}} \in \mathcal{SV}, \qquad n \to \infty,$$

which is impossible. Therefore, $(-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1)/\alpha_i > -1$, $i = \overline{1,N}$. Summing (3.13) from n_0 to $n-1$ and applying Theorem 2.9 we conclude that

$$x_i(n) \sim \frac{n^{\frac{-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1}{\alpha_i} + 1} l_i(n)^{-\frac{1}{\alpha_i}} m_i(n)^{\frac{1}{\alpha_i}} \xi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{(-(\mu_i + \beta_i \rho_{i+1} + 1))^{\frac{1}{\alpha_i}} \left(\frac{-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1}{\alpha_i} + 1\right)}, \qquad n \to \infty, \quad i = \overline{1,N}. \tag{3.14}$$

From previous relation we see that

$$\rho_i = \frac{-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1}{\alpha_i} + 1, \qquad i = \overline{1,N}, \quad \rho_{N+1} = \rho_1$$

which is equivalent with

$$\rho_i - \frac{\beta_i}{\alpha_i} \rho_{i+1} = \frac{\alpha_i - \lambda_i + \mu_i + 1}{\alpha_i}, \qquad i = \overline{1,N}, \quad \rho_{N+1} = \rho_1. \tag{3.15}$$

We now have a linear cyclic system of equations in which ρ_i are unknown. To find ρ_i, let we denote with A the coefficient matrix of system (3.15), i.e.

$$A = A\left(\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \dots, \frac{\beta_N}{\alpha_N}\right) = \begin{pmatrix} 1 & -\frac{\beta_1}{\alpha_1} & 0 & \dots & 0 & 0 \\ 0 & 1 & -\frac{\beta_2}{\alpha_2} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -\frac{\beta_{N-1}}{\alpha_{N-1}} \\ -\frac{\beta_N}{\alpha_N} & 0 & 0 & \dots & 0 & 1 \end{pmatrix}. \tag{3.16}$$

This matrix is nonsingular because, according to condition (a),

$$\det(A) = 1 - \frac{\beta_1\beta_2\cdots\beta_N}{\alpha_1\alpha_2\cdots\alpha_N} > 0, \tag{3.17}$$

so system (3.15) has an unique solution and matrix A is invertible

$$A^{-1} = \frac{A_N}{A_N - B_N} M \tag{3.18}$$

where matrix M is given by (3.6).

Solving the system (3.15) we get the unique solution ρ_i, $i = \overline{1,N}$ which is given explicitly by (3.8). From (3.8) we can see that all ρ_i satisfy $0 < \rho_i < \frac{\alpha_i - \lambda_i}{\alpha_i}$, $i = \overline{1,N}$ if and only if (3.7) holds.

Using (3.1) and (3.10) we can transform (3.12) in the following form

$$x_i(n) \sim \frac{n^{\frac{\alpha_i+1}{\alpha_i}} p_i(n)^{-\frac{1}{\alpha_i}} q_i(n)^{\frac{1}{\alpha_i}} x_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{D_i}, \qquad n \to \infty, \tag{3.19}$$

where

$$D_i = (\alpha_i - \lambda_i - \alpha_i\rho_i)^{\frac{1}{\alpha_i}} \rho_i, \tag{3.20}$$

for $i = \overline{1,N}$. Without difficulty we can obtain following explicit formula for each x_i from the cyclic system of asymptotic relations (3.19)

$$x_i(n) \sim \left[\prod_{j=1}^{N} \left(\frac{n^{\frac{\alpha_j+1}{\alpha_j}} p_j(n)^{-\frac{1}{\alpha_j}} q_j(n)^{\frac{1}{\alpha_j}}}{D_j} \right)^{M_{ij}} \right]^{\frac{A_N}{A_N - B_N}}, \qquad n \to \infty, \quad i = \overline{1,N}. \tag{3.21}$$

Previous relation can be rewritten in the following form

$$x_i(n) \sim n^{\rho_i} \left[\prod_{j=1}^{N} \left(\frac{l_j(n)^{-\frac{1}{\alpha_j}} m_j(n)^{\frac{1}{\alpha_j}}}{D_j} \right)^{M_{ij}} \right]^{\frac{A_N}{A_N - B_N}}, \qquad n \to \infty, \quad i = \overline{1,N}.$$

implying that regularity index of x_i is exactly ρ_i.

Suppose now that (3.7) holds and define ρ_i with (3.8) and D_i with (3.20). Denote

$$X_i(n) = \left[\prod_{j=1}^{N} \left(\frac{n^{\frac{\alpha_j+1}{\alpha_j}} p_j(n)^{-\frac{1}{\alpha_j}} q_j(n)^{\frac{1}{\alpha_j}}}{D_j} \right)^{M_{ij}} \right]^{\frac{A_N}{A_N - B_N}}, \qquad i = \overline{1,N}. \tag{3.22}$$

Clearly, $X_i \in \mathcal{RV}(\rho_i)$, $i = \overline{1,N}$ and X_i's satisfy the system of asymptotic relations (1.5), i.e.

$$\sum_{k=n_1}^{n-1} \left(\frac{1}{p_i(k)} \sum_{s=k}^{\infty} q_i(s) X_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \sim X_i(n), \qquad n \to \infty, \quad i = \overline{1,N}, \tag{3.23}$$

for any $n > n_1$, where $X_{N+1}(n) = X_1(n)$. Indeed, $X_i(n)$ can be expressed as

$$X_i(n) = n^{\rho_i} \chi_i(n), \qquad \chi_i(n) = \left[\prod_{j=1}^{N} \left(\frac{l_j(n)^{-\frac{1}{\alpha_j}} m_j(n)^{\frac{1}{\alpha_j}}}{D_j} \right)^{M_{ij}} \right]^{\frac{A_N}{A_N - B_N}}, \tag{3.24}$$

and using Theorem 2.9, we obtain

$$\left(\frac{1}{p_i(n)}\sum_{k=n}^{\infty} q_i(k)X_{i+1}(k+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} \sim \frac{n^{\rho_i-1}l_i(n)^{-\frac{1}{\alpha_i}}m_i(n)^{\frac{1}{\alpha_i}}\chi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{(\alpha_i-\lambda_i-\alpha_i\rho_i)^{\frac{1}{\alpha_i}}},$$

and

$$\sum_{k=n_1}^{n-1}\left(\frac{1}{p_i(k)}\sum_{s=k}^{\infty} q_i(s)X_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} \sim \frac{n^{\rho_i}l_i(n)^{-\frac{1}{\alpha_i}}m_i(n)^{\frac{1}{\alpha_i}}\chi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{D_i}, \tag{3.25}$$

as $n\to\infty$. Since for the elements of matrix M hold

$$M_{i+1,i}\frac{\beta_i}{\alpha_i}=\frac{B_N}{A_N}, \qquad M_{i+1,j}\frac{\beta_i}{\alpha_i}=M_{ij}, \quad \text{for } j\neq i, \tag{3.26}$$

where $M_{N+1,j}=M_{1,j}$, $j=\overline{1,N}$, relation (3.25) can be transformed as

$$\frac{l_i(n)^{-\frac{1}{\alpha_i}}m_i(n)^{\frac{1}{\alpha_i}}}{D_i}\chi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}} = \frac{l_i(n)^{-\frac{1}{\alpha_i}}m_i(n)^{\frac{1}{\alpha_i}}}{D_i}\left[\prod_{j=1}^{N}\left(\frac{l_j(n)^{-\frac{1}{\alpha_j}}m_j(n)^{\frac{1}{\alpha_j}}}{D_j}\right)^{M_{i+1,j}\frac{\beta_i}{\alpha_i}}\right]^{\frac{A_N}{A_N-B_N}}$$

$$=\left[\prod_{j=1}^{N}\left(\frac{l_j(n)^{-\frac{1}{\alpha_j}}m_j(n)^{\frac{1}{\alpha_j}}}{D_j}\right)^{M_{ij}}\right]^{\frac{A_N}{A_N-B_N}} = \chi_i(n),$$

so from (3.25) we obtain that X_i, $i=\overline{1,N}$ satisfy (3.23). □

In the same way, we can solve the second problem. Assuming that (II) holds, we are in position to find necessary and sufficient condition that system of asymptotic relations (1.6) possesses regularly varying solution $\mathbf{x}$ of negative index $(\rho_1,\rho_2,\ldots,\rho_N)$.

Theorem 3.2. *Let $p_i\in\mathcal{RV}(\lambda_i)$, $q_i\in\mathcal{RV}(\mu_i)$ and suppose that $\lambda_i>\alpha_i$, $i=\overline{1,N}$. System of asymptotic relations (1.6) has regularly varying solution $\mathbf{x}\in\mathcal{RV}(\rho_1,\rho_2,\ldots,\rho_N)$ with $\rho_i\in\left(\frac{\alpha_i-\lambda_i}{\alpha_i},0\right)$, $i=\overline{1,N}$ if and only if*

$$\frac{\alpha_i-\lambda_i}{\alpha_i}\left(1-\frac{B_N}{A_N}\right) < \sum_{j=1}^{N} M_{ij}\frac{\alpha_j-\lambda_j+\mu_j+1}{\alpha_j} < 0 \tag{3.27}$$

in which case ρ_i are given by (3.8) and the asymptotic behavior of any such solution is governed by the unique formula

$$x_i(n) \sim \left[\prod_{j=1}^{N}\left(\frac{n^{\frac{\alpha_j+1}{\alpha_j}}p_j(n)^{-\frac{1}{\alpha_j}}q_j(n)^{\frac{1}{\alpha_j}}}{W_j}\right)^{M_{ij}}\right]^{\frac{A_N}{A_N-B_N}}, \qquad n\to\infty, \quad i=\overline{1,N}, \tag{3.28}$$

where

$$W_i=(\lambda_i-\alpha_i+\alpha_i\rho_i)^{\frac{1}{\alpha_i}}(-\rho_i), \qquad i=\overline{1,N}. \tag{3.29}$$

Proof. Let $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \ldots, \rho_N)$ with all $\rho_i < 0$ be a solution of (1.6) where its components are given by (3.10). Using (3.1) and (3.10) we obtain

$$\sum_{k=n_0}^{n-1} q_i(k) x_{i+1}(k+1)^{\beta_i} \sim \sum_{k=n_0}^{n-1} k^{\mu_i+\beta_i\rho_{i+1}} m_i(k) \xi_{i+1}(k)^{\beta_i}, \qquad n \geq n_0,\ i = \overline{1,N}, \tag{3.30}$$

as $n \to \infty$. Both sums are diveregent, so $\mu_i + \beta_i \rho_{i+1} \geq -1$ for all i. If the equality holds for some i, then noting that

$$\left(\frac{1}{p_i(n)} \sum_{k=n_0}^{n-1} q_i(k) x_{i+1}(k+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \sim n^{-\frac{\lambda_i}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}} \left(\sum_{k=n_0}^{n-1} k^{-1} m_i(k) \xi_{i+1}(k)^{\beta_i} \right),$$

and summing this from n to ∞, using Theorem 2.9 we get

$$x_i(n) \sim \frac{\alpha_i}{\alpha_i - \lambda_i} n^{\frac{\alpha_i-\lambda_i}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}} \left(\sum_{k=n_0}^{n-1} k^{-1} m_i(k) \xi_{i+1}(k)^{\beta_i} \right) \in \mathcal{RV}\left(\frac{\alpha_i - \lambda_i}{\alpha_i} \right)$$

which is impossible, due to $\rho_i > \frac{\alpha_i-\lambda_i}{\alpha_i}$. It follows that $\mu_i + \beta_i\rho_{i+1} > -1$ for all i. Applying Theorem 2.9 to (3.30), we get

$$\left(\frac{1}{p_i(n)} \sum_{k=n_0}^{n-1} q_i(k) x_{i+1}(k+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \sim \frac{n^{\frac{-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}} m_i(n)^{\frac{1}{\alpha_i}} \xi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{(\mu_i + \beta_i\rho_{i+1} + 1)^{\frac{1}{\alpha_i}}}, \tag{3.31}$$

as $n \to \infty$. Since $x_i(n) \to 0$, we see that $\frac{-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i} \leq -1$ for all i. All inequalities should be strict, because if the equality holds for some i, then summing (3.31) from n to ∞, we get

$$x_i(n) \sim (\lambda_i - \alpha_i)^{-\frac{1}{\alpha_i}} \sum_{k=n}^{\infty} k^{-1} l_i(k)^{-\frac{1}{\alpha_i}} m_i(k)^{\frac{1}{\alpha_i}} \xi_{i+1}(k)^{\frac{\beta_i}{\alpha_i}} \in \mathcal{SV},$$

which is also impossible, due to $\rho_i < 0$. Therefore, we see that $\frac{-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i} < -1$ for all i, in which case summing (3.31) from n to ∞, using Theorem 2.9 yields

$$x_i(n) \sim \frac{n^{\frac{-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i}+1} l_i(n)^{-\frac{1}{\alpha_i}} m_i(n)^{\frac{1}{\alpha_i}} \xi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{(\mu_i + \beta_i\rho_{i+1} + 1)^{\frac{1}{\alpha_i}} \left[-\left(\frac{-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i} + 1 \right) \right]}, \qquad n \to \infty, \quad i = \overline{1,N}. \tag{3.32}$$

This implies that

$$\rho_i = \frac{-\lambda_i + \mu_i + \beta_i\rho_{i+1} + 1}{\alpha_i} + 1, \qquad i = \overline{1,N}, \quad \rho_{N+1} = \rho_1,$$

which is equivalent to the linear algebraic system (3.15) in ρ_i. Proceeding exactly like in the proof of previous theorem we get that system (1.6) has regularly varying solution of negative indices $\rho_i \in \left(\frac{\alpha_i-\lambda_i}{\alpha_i}, 0\right)$ only if (3.27) is fulfilled.

Now assume that (3.27) holds. Define $\rho_i \in \left(\frac{\alpha_i-\lambda_i}{\alpha_i}, 0\right)$ by (3.8) and let $X_i \in \mathcal{RV}(\rho_i)$ denote the sequences

$$X_i(n) = \left[\prod_{j=1}^{N} \left(\frac{n^{\frac{\alpha_j+1}{\alpha_j}} p_j(n)^{-\frac{1}{\alpha_j}} q_j(n)^{\frac{1}{\alpha_j}}}{W_j} \right)^{M_{ij}} \right]^{\frac{A_N}{A_N-B_N}}, \qquad i = \overline{1,N}. \tag{3.33}$$

Sequences X_i satisfy the system of asymptotic relations (1.6) (with $n_0 = n_1$), i.e.

$$\sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\sum_{s=n_1}^{k-1} q_i(s)X_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} \sim X_i(n), \qquad n\to\infty, \quad i=\overline{1,N}, \tag{3.34}$$

for any $n_1 \geq 1$, where $X_{N+1}(n) = X_1(n)$. To prove this, we will use the following expression for $X_i(n)$

$$X_i(n) = n^{\rho_i}\chi_i(n), \quad \chi_i(n) = \left[\prod_{j=1}^{N}\left(\frac{l_j(n)^{-\frac{1}{\alpha_j}} m_j(n)^{\frac{1}{\alpha_j}}}{W_j}\right)^{M_{ij}}\right]^{\frac{A_N}{A_N-B_N}},$$

to obtain the asymptotic relation

$$\sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\sum_{s=n_1}^{k-1} q_i(s)X_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} \sim \frac{n^{\rho_i} l_i(n)^{-\frac{1}{\alpha_j}} m_i(n)^{\frac{1}{\alpha_j}} \chi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{W_i}, \tag{3.35}$$

as $n\to\infty$. Like in proof of previous theorem, with help of (3.26), it can be verified that

$$\frac{l_i(n)^{-\frac{1}{\alpha_j}} m_i(n)^{\frac{1}{\alpha_j}} \chi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{W_i} = \chi_i(n)$$

and the desired relation (3.34) immediately follows from (3.35). This completes the proof of the theorem. □

4 Regularly varying intermediate solutions of (*E*)

We are now in position to state and prove our main results on the existence and the precise asymptotic behavior of regularly varying intermediate solutions of system (*E*) with regularly varying coefficients p_i and q_i. Use is made of the notation and properties of the matrix (3.6), (3.16) and (3.18).

Theorem 4.1. *Let $p_i \in \mathcal{RV}(\lambda_i)$ and $q_i \in \mathcal{RV}(\mu_i)$, $i=\overline{1,N}$. Suppose that $\lambda_i < \alpha_i$ for all $i=\overline{1,N}$. System (E) possesses intermediate solutions in $\mathcal{RV}(\rho_1,\rho_2,\dots,\rho_N)$ with $\rho_i \in \left(0, \frac{\alpha_i-\lambda_i}{\alpha_i}\right)$, $i=\overline{1,N}$, if and only if (3.7) holds, in which case ρ_i are given by (3.8) and the asymptotic behavior of any such solution* **x** *is governed by the unique formula (3.21).*

Theorem 4.2. *Let $p_i \in \mathcal{RV}(\lambda_i)$ and $q_i \in \mathcal{RV}(\mu_i)$, $i=\overline{1,N}$. Suppose that $\lambda_i > \alpha_i$ for all $i=\overline{1,N}$. System(E) possesses intermediate solutions in $\mathcal{RV}(\rho_1,\rho_2,\dots,\rho_N)$ with $\rho_i \in \left(\frac{\alpha_i-\lambda_i}{\alpha_i}, 0\right)$, $i=\overline{1,N}$, if and only if (3.27) holds, in which case ρ_i are given by (3.8) and the asymptotic behavior of any such solution* **x** *is governed by the unique formula (3.28).*

We remark that the "only if" parts of these theorems follow immediately from the corresponding parts of Theorem 3.1 and Theorem 3.2 because any solution **x** of (E) with the indicated property satisfies the asymptotic relations (1.5) and $(IM1)$ or (1.6) and $(IM2)$.

Proof of the "if" part of Theorem 4.1. Using (3.1) let we define the sequences $X_i = \{X_i(n)\} \in \mathcal{RV}(\rho_i)$ by

$$X_i(n) = n^{\rho_i}\left[\prod_{j=1}^{N}\left(\frac{l_j(n)^{-\frac{1}{\alpha_j}} m_j(n)^{\frac{1}{\alpha_j}}}{D_j}\right)^{M_{ij}}\right]^{\frac{A_N}{A_N-B_N}}, \qquad n\geq 1, \quad i=\overline{1,N}, \tag{4.1}$$

where D_j for $j = \overline{1, N}$ are given by (3.20). It is known that

$$\sum_{k=1}^{n-1} \left(\frac{1}{p_i(k)} \sum_{s=k}^{\infty} q_i(s) X_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \sim X_i(n), \qquad n \to \infty, \quad i = \overline{1, N}, \tag{4.2}$$

from which it follows that there exists $n_0 > 1$ such that

$$\sum_{k=n_0}^{n-1} \left(\frac{1}{p_i(k)} \sum_{s=k}^{\infty} q_i(s) X_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \leq 2X_i(n), \qquad n \geq n_0, \quad i = \overline{1, N}. \tag{4.3}$$

Without loss of generality we may assume that each $X_i(n)$ is increasing for $n \geq n_0$ because it is known that any regularly varying sequence of positive index is asymptotically equivalent to an increasing $\mathcal{RV}$ sequence of the same index. Since (4.2) holds for n_0 it is possible to choose $n_1 > n_0$ so large that

$$\sum_{k=n_0}^{n-1} \left(\frac{1}{p_i(k)} \sum_{s=k}^{\infty} q_i(s) X_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \geq \frac{1}{2} X_i(n), \qquad n \geq n_1, \quad i = \overline{1, N}. \tag{4.4}$$

Let we choose positive constants c_i and C_i so that

$$c_i \leq \frac{1}{2} c_{i+1}^{\frac{\beta_i}{\alpha_i}}, \quad C_i \geq 4C_{i+1}^{\frac{\beta_i}{\alpha_i}}, \quad i = \overline{1, N}, \quad c_{N+1} = c_1, \quad C_{N+1} = C_1.$$

An example of such choices is

$$c_i = \left(\frac{1}{2} \right)^{\frac{A_N}{A_N - B_N} \sum_{j=1}^{N} M_{ij}}, \qquad C_i = 4^{\frac{A_N}{A_N - B_N} \sum_{j=1}^{N} M_{ij}} \tag{4.5}$$

for $i = \overline{1, N}$. Clearly $c_i \leq 1 \leq C_i$.

The constants c_i and C_i can be chosen so that $C_i / c_i \geq 2X_i(n_1)/X_i(n_0)$, that is

$$2c_i X_i(n_1) \leq C_i X_i(n_0), \quad i = \overline{1, N}, \tag{4.6}$$

because these constants are independent of $X_i(n)$ and the choice of n_0 and n_1.

Consider the Banach space Y_{n_0} of vectors $\mathbf{x} = (x_1, x_2, \ldots, x_N) \in {}^{\mathbb{N}}\mathbb{R} \times \cdots \times {}^{\mathbb{N}}\mathbb{R}$, with norm

$$\|\mathbf{x}\| = \max_{1 \leq i \leq N} \left\{ \sup_{n \geq n_0} \frac{x_i(n)}{X_i(n)} \right\}$$

endowed with the usual pointwise ordering $\leq$: For $\mathbf{x}, \mathbf{y} \in \mathrm{Y}_{n_0}$, $\mathbf{x} \leq \mathbf{y}$ means $x_i(n) \leq y_i(n)$ for all $n \geq n_0$ and $i = \overline{1, N}$. Then Y_{n_0} is partially ordered. Define the subset $\mathcal{X} \subset \mathrm{Y}_{n_0}$ with

$$\mathcal{X} = \left\{ \mathbf{x} \mid c_i X_i(n) \leq x_i(n) \leq C_i X_i(n), \qquad n \geq n_0, \quad i = \overline{1, N} \right\}. \tag{4.7}$$

It is easy to see that for any $\mathbf{x} \in \mathcal{X}$ the norm of $\mathbf{x}$ is finite. Also, for any subset $B \subset \mathcal{X}$, it is obvious that $\inf B \in \mathcal{X}$ and $\sup B \in \mathcal{X}$. We will define the operators $\mathcal{F}_i$ by

$$\mathcal{F}_i x(n) = b_i + \sum_{k=n_0}^{n-1} \left(\frac{1}{p_i(k)} \sum_{s=k}^{\infty} q_i(s) x(s+1)^{\beta_i} \right)^{\frac{1}{\alpha i}}, \qquad n \geq n_0, \quad i = \overline{1, N}, \tag{4.8}$$

where b_i are positive constants such that

$$c_iX_i(n_1) \le b_i \le \frac{1}{2}C_iX_i(n_0), \qquad i = \overline{1,N}, \tag{4.9}$$

and define the mapping $\Phi : \mathcal{X} \to Y_{n_0}$ by

$$\Phi(x_1, x_2, \ldots, x_N)(n) = (\mathcal{F}_1x_2(n), \mathcal{F}_2x_3(n), \ldots, \mathcal{F}_Nx_{N+1}(n)), \qquad n \ge n_0, \tag{4.10}$$

where $x_{N+1} = x_1$. We will show that Φ has a fixed point by using Lemma 1.3. Namely, the operator Φ has the following properties:

(i) Φ *maps* $\mathcal{X}$ *into itself*: Let $\mathbf{x} \in \mathcal{X}$. Then, using (4.3)–(4.10), we see that

$$\begin{aligned}\mathcal{F}_ix_{i+1}(n) &\le \frac{1}{2}C_iX_i(n_0) + C_{i+1}^{\frac{\beta_i}{\alpha_i}} \sum_{k=n_0}^{n-1} \left(\frac{1}{p_i(k)} \sum_{s=k}^{\infty} q_i(s)X_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \\ &\le \frac{1}{2}C_iX_i(n_0) + 2C_{i+1}^{\frac{\beta_i}{\alpha_i}}X_i(n) \le \frac{1}{2}C_iX_i(n) + \frac{1}{2}C_iX_i(n) = C_iX_i(n)\end{aligned}$$

for $n \ge n_0$ and

$$\mathcal{F}_ix_{i+1}(n) \ge b_i \ge c_iX_i(n_1) \ge c_iX_i(n), \quad \text{for } n_0 \le n \le n_1,$$

$$\begin{aligned}\mathcal{F}_ix_{i+1}(n) &\ge c_{i+1}^{\frac{\beta_i}{\alpha_i}} \sum_{k=n_0}^{n-1} \left(\frac{1}{p_i(k)} \sum_{s=k}^{\infty} q_i(s)X_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \\ &\ge \frac{1}{2}c_{i+1}^{\frac{\beta_i}{\alpha_i}}X_i(n) \ge c_iX_i(n), \qquad n \ge n_1.\end{aligned}$$

This shows that $\Phi(\mathbf{x}) \in \mathcal{X}$, that is, Φ is a self-map on $\mathcal{X}$.

(ii) Φ *is increasing*, i.e. for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, $\mathbf{x} \le \mathbf{y}$ implies $\Phi(\mathbf{x}) \le \Phi(\mathbf{y})$.

Thus all the hypotheses of Lemma 1.3 are fulfilled implying the existence of a fixed point $\mathbf{x} \in \mathcal{X}$ of Φ, which satisfies

$$x_i(n) = \mathcal{F}_ix_{i+1}(n) = b_i + \sum_{k=n_0}^{n-1} \left(\frac{1}{p_i(k)} \sum_{s=k}^{\infty} q_i(s)x_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha i}}, \qquad n \ge n_0, \quad i = \overline{1,N}.$$

This shows that $\mathbf{x} \in \mathcal{X}$ is a solution of system (E) and it is easy to see that it is an intermediate solution. Notice that $x_i(n)$ satisfy the system of relations (3.19).

It remains to verify that $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \ldots, \rho_N)$. We define

$$u_i(n) = \sum_{k=n_0}^{n-1} \left(\frac{1}{p_i(k)} \sum_{s=k}^{\infty} q_i(s)X_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}}, \qquad i = \overline{1,N}, \tag{4.11}$$

and put

$$r_i = \liminf_{n\to\infty} \frac{x_i(n)}{u_i(n)}, \qquad R_i = \limsup_{n\to\infty} \frac{x_i(n)}{u_i(n)}.$$

Since $c_iX_i(n) \le x_i(n) \le C_iX_i(n)$, $n \ge n_0$, $i = \overline{1,N}$ and

$$u_i(n) \sim X_i(n), \quad n \to \infty, \quad i = \overline{1,N}, \tag{4.12}$$

it follows that $0 < r_i \le R_i < \infty$, $i = \overline{1,N}$. Using Lemma 1.1 we obtain

$$r_i \ge \liminf_{n\to\infty} \frac{\Delta x_i(n)}{\Delta u_i(n)} = \liminf_{n\to\infty} \frac{\left(\frac{1}{p_i(n)} \sum_{k=n}^{\infty} q_i(k) x_{i+1}(k+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}}}{\left(\frac{1}{p_i(n)} \sum_{k=n}^{\infty} q_i(k) X_{i+1}(k+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}}}$$
$$= \liminf_{n\to\infty} \left(\frac{\sum_{k=n}^{\infty} q_i(k) x_{i+1}(k+1)^{\beta_i}}{\sum_{k=n}^{\infty} q_i(k) X_{i+1}(k+1)^{\beta_i}}\right)^{\frac{1}{\alpha_i}} = \left(\liminf_{n\to\infty} \frac{\sum_{k=n}^{\infty} q_i(k) x_{i+1}(k+1)^{\beta_i}}{\sum_{k=n}^{\infty} q_i(k) X_{i+1}(k+1)^{\beta_i}}\right)^{\frac{1}{\alpha_i}}$$
$$\ge \left(\liminf_{n\to\infty} \frac{q_i(n) x_{i+1}(n+1)^{\beta_i}}{q_i(n) X_{i+1}(n+1)^{\beta_i}}\right)^{\frac{1}{\alpha_i}} = \liminf_{n\to\infty} \left(\frac{x_{i+1}(n+1)}{X_{i+1}(n+1)}\right)^{\frac{\beta_i}{\alpha_i}} = r_{i+1}^{\frac{\beta_i}{\alpha_i}}$$

where (4.12) has been used in the last step. Thus, r_i satisfy the cyclic system of inequalities

$$r_i \ge r_{i+1}^{\frac{\beta_i}{\alpha_i}}, \quad i = \overline{1,N}, \quad r_{N+1} = r_1. \tag{4.13}$$

Likewise, by taking the upper limits instead of the lower limits we are led to the cyclic inequalities

$$R_i \le R_{i+1}^{\frac{\beta_i}{\alpha_i}}, \quad i = \overline{1,N}, \quad R_{N+1} = R_1. \tag{4.14}$$

From (4.13) and (4.14) we easily see that

$$r_i \ge r_i^{\frac{\beta_1\beta_2\cdots\beta_N}{\alpha_1\alpha_2\cdots\alpha_N}}, \qquad R_i \le R_i^{\frac{\beta_1\beta_2\cdots\beta_N}{\alpha_1\alpha_2\cdots\alpha_N}},$$

whence, because of the hypothesis $\beta_1\beta_2\cdots\beta_N/\alpha_1\alpha_2\cdots\alpha_N < 1$, we find that $r_i \ge 1$ and $R_i \le 1$, $i = \overline{1,N}$. It follows therefore that $r_i = R_i = 1$ i.e. $\lim_{n\to\infty} x_i(n)/u_i(n) = 1$ for $i = \overline{1,N}$. Combined this with (4.12) implies that $x_i(n) \sim u_i(n) \sim X_i(n)$ as $n \to \infty$, which shows that each $x_i(n)$ is a regularly varying sequence of index ρ_i. Thus the proof of the "if" part of Theorem 4.1. is completed. □

Proof of the "if" part of Theorem 4.2. Assume that $p_i(n)$ and $q_i(n)$ are expressed in the form (3.1). Suppose that (3.27) holds. Define ρ_i and W_i by (3.8) and (3.29), respectively, and consider the regularly varying sequences of indices ρ_i

$$Y_i(n) = n^{\rho_i} \left[\prod_{j=1}^{N} \left(\frac{l_j(n)^{-\frac{1}{\alpha_j}} m_j(n)^{\frac{1}{\alpha_j}}}{W_j}\right)^{M_{ij}}\right]^{\frac{A_N}{A_N - B_N}}, \quad i = \overline{1,N}.$$

Since Y_i, $i = \overline{1,N}$ satisfy the asymptotic relations

$$\begin{aligned} Y_i(n) &\sim \sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \sum_{s=1}^{k-1} q_i(s) Y_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} \\ &\sim \sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \left(h_i + \sum_{s=1}^{k-1} q_i(s) Y_{i+1}(s+1)^{\beta_i}\right)\right)^{\frac{1}{\alpha_i}}, \end{aligned} \tag{4.15}$$

as $n \to \infty$ for $i = \overline{1,N}$, where $h_i > 0$ are arbitrary fixed real constants for $i = \overline{1,N}$, one can choose $n_1 > n_0 > 1$ so that

$$\sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \left(h_i + \sum_{s=n_0}^{k-1} q_i(s) Y_{i+1}(s+1)^{\beta_i}\right)\right)^{\frac{1}{\alpha_i}} \le 2Y_i(n), \qquad n \ge n_0, \quad i = \overline{1,N} \tag{4.16}$$

and

$$\frac{1}{2}Y_i(n) \le \sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\left(h_i + \sum_{s=n_0}^{k-1} q_i(s)Y_{i+1}(s+1)^{\beta_i}\right)\right)^{\frac{1}{\alpha_i}}, \qquad n \ge n_1, \quad i = \overline{1,N}. \tag{4.17}$$

Let we choose the positive constants ω_i and Ω_i which satisfy the cyclic system of inequalities

$$\omega_i \le \frac{1}{2}\omega_{i+1}^{\frac{\beta_i}{\alpha_i}}, \quad \Omega_i \ge 2\,\Omega_{i+1}^{\frac{\beta_i}{\alpha_i}}, \quad i = \overline{1,N}, \quad \omega_{N+1} = \omega_1, \quad \Omega_{N+1} = \Omega_1,$$

and $\omega_i \le h_i^{\frac{1}{\alpha_i}} \cdot \min_{n_0 \le k \le n_1}\left\{\frac{\pi_i(k)}{Y_i(k)}\right\}$. An example of such choices is

$$\omega_i = \min\left\{c_i, \gamma_i\, h_i^{\frac{1}{\alpha_i}}\right\}, \quad \Omega_i = 2^{\frac{A_N}{A_N - B_N}\sum_{j=1}^{N} M_{ij}}, \quad i = \overline{1,N}$$

where

$$\gamma_i = \min\left\{\min_{n_0 \le k \le n_1}\left\{\frac{\pi_i(k)}{Y_i(k)}\right\}, h_i^{-\frac{1}{\alpha_i}}c_i\right\}$$

and

$$c_i = \left(\frac{1}{2}\right)^{\frac{A_N}{A_N - B_N}\sum_{j=1}^{N} M_{ij}}.$$

It is easy to see that for such constants $\omega_i \le 1 \le \Omega_i$.

Consider the Banach space Y_{n_0} of vectors $\mathbf{x} = (x_1, x_2, \dots, x_N) \in {}^{\mathbb{N}}\mathbb{R} \times \dots \times {}^{\mathbb{N}}\mathbb{R}$, with norm

$$\|\mathbf{x}\| = \max_{1 \le i \le N}\left\{\sup_{n \ge n_0}\frac{x_i(n)}{Y_i(n)}\right\}$$

endowed with the usual pointwise ordering $\le$: For $\mathbf{x}, \mathbf{y} \in Y_{n_0}$, $\mathbf{x} \le \mathbf{y}$ means $x_i(n) \le y_i(n)$ for all $n \ge n_0$ and $i = \overline{1,N}$. Then Y_{n_0} is partially ordered. Define the subset $\mathcal{Y} \subset Y_{n_0}$ like

$$\mathcal{Y} = \left\{\mathbf{x} \in Y_{n_0} \mid \omega_i Y_i(n) \le x_i(n) \le \Omega_i Y_i(n), \quad n \ge n_0, \quad i = \overline{1,N}\right\}. \tag{4.18}$$

For any subset $B \subset \mathcal{Y}$, it is obvious that $\inf B \in \mathcal{Y}$ and $\sup B \in \mathcal{Y}$. We consider the mapping $\Psi : \mathcal{Y} \to Y_{n_0}$ defined by

$$\Psi(x_1, x_2, \dots, x_N)(n) = (\mathcal{G}_1 x_2(n), \mathcal{G}_2 x_3(n), \dots, \mathcal{G}_N x_{N+1}(n)), \qquad n \ge n_0, \quad x_{N+1} = x_1, \tag{4.19}$$

where

$$\mathcal{G}_i x(n) = \sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\left(h_i + \sum_{s=n_0}^{k-1} q_i(s)x(s+1)^{\beta_i}\right)\right)^{\frac{1}{\alpha_i}}, \qquad n \ge n_0, \quad i = \overline{1,N}. \tag{4.20}$$

We will show that Ψ has a fixed point by using Lemma 1.3. Namely, the operator Ψ has the following properties:

(i) Ψ *maps* $\mathcal{Y}$ *into itself*: Let $\mathbf{x} \in \mathcal{Y}$. Then, using (4.16)–(4.20), we see that

$$\begin{aligned}
\mathcal{G}_i x_{i+1}(n) &\le \sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \left(h_i + \Omega_{i+1}^{\beta_i} \sum_{s=n_0}^{k-1} q_i(s) Y_{i+1}(s+1)^{\beta_i} \right) \right)^{\frac{1}{\alpha_i}} \\
&= \sum_{k=n}^{\infty} \left(\frac{\Omega_{i+1}^{\beta_i}}{p_i(k)} \left(\frac{h_i}{\Omega_{i+1}^{\beta_i}} + \sum_{s=n_0}^{k-1} q_i(s) Y_{i+1}(s+1)^{\beta_i} \right) \right)^{\frac{1}{\alpha_i}} \\
&\le \Omega_{i+1}^{\frac{\beta_i}{\alpha_i}} \sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \left(h_i + \sum_{s=n_0}^{k-1} q_i(s) Y_{i+1}(s+1)^{\beta_i} \right) \right)^{\frac{1}{\alpha_i}} \\
&\le \frac{1}{2} \Omega_i \, 2 Y_i(n) = \Omega_i Y_i(n)
\end{aligned}$$

for $n \ge n_0$ and

$$\mathcal{G}_i x_{i+1}(n) \ge \sum_{k=n}^{\infty} \left(\frac{h_i}{p_i(k)} \right)^{\frac{1}{\alpha_i}} = h_i^{\frac{1}{\alpha_i}} \pi_i(n) \ge h_i^{\frac{1}{\alpha_i}} \gamma_i \, Y_i(n) \ge \omega_i \, Y_i(n) \quad \text{for } n_0 \le n \le n_1,$$

$$\begin{aligned}
\mathcal{G}_i x_{i+1}(n) &\ge \sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \left(h_i + \omega_{i+1}^{\beta_i} \sum_{s=n_0}^{k-1} q_i(s) Y_{i+1}(s+1)^{\beta_i} \right) \right)^{\frac{1}{\alpha_i}} \\
&= \sum_{k=n}^{\infty} \left(\frac{\omega_{i+1}^{\beta_i}}{p_i(k)} \left(\frac{h_i}{\omega_{i+1}^{\beta_i}} + \sum_{s=n_0}^{k-1} q_i(s) Y_{i+1}(s+1)^{\beta_i} \right) \right)^{\frac{1}{\alpha_i}} \\
&\ge \omega_{i+1}^{\frac{\beta_i}{\alpha_i}} \sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \left(h_i + \sum_{s=n_0}^{k-1} q_i(s) Y_{i+1}(s+1)^{\beta_i} \right) \right)^{\frac{1}{\alpha_i}} \\
&\ge 2\omega_i \, \frac{1}{2} Y_i(n) = \omega_i Y_i(n), \qquad n \ge n_1.
\end{aligned}$$

This shows that $\Psi(\mathbf{x}) \in \mathcal{Y}$, that is, Ψ is a self-map on $\mathcal{Y}$.

(ii) Ψ *is increasing*, i.e. for any $\mathbf{x}, \mathbf{y} \in \mathcal{Y}$, $\mathbf{x} \le \mathbf{y}$ implies $\Psi(\mathbf{x}) \le \Psi(\mathbf{y})$.

Thus all the hypotheses of Lemma 1.3 are fulfilled implying the existence of a fixed point $\mathbf{x} \in \mathcal{Y}$ of Ψ, which satisfies

$$x_i(n) = \mathcal{G}_i x_{i+1}(n) = \sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \left(h_i + \sum_{s=n_0}^{k-1} q_i(s) x_{i+1}(s+1)^{\beta_i} \right) \right)^{\frac{1}{\alpha_i}}, \qquad n \ge n_0, \quad i = \overline{1, N}.$$

This shows that $\mathbf{x} \in \mathcal{Y}$ is a solution of system (E) and it is easy to see that it is an intermediate solution. In the essentially the same way as in proof of previous theorem, we obtain that $\mathbf{x}$ is regularly varying solution of (E) of index $(\rho_1, \rho_2, \dots, \rho_N)$. □

Our main results can be applied to the well-known second order difference equation of Emden–Fowler type

$$\Delta(p(n)|\Delta x(n)|^{\alpha-1}\Delta x(n)) + q(n)|x(n+1)|^{\beta-1}x(n+1) = 0, \tag{4.21}$$

to produce new results on the existence and the asymptotic behavior of its intermediate solution. As an immediate consequence of Theorem 4.1 and Theorem 4.2 we have the next two results for the equation (4.21).

Theorem 4.3. *Let $\{p(n)\} \in \mathcal{RV}(\lambda)$ and $\{q(n)\} \in \mathcal{RV}(\mu)$. Suppose that $\lambda < \alpha$. Equation (4.21) possesses regularly varying intermediate solutions of index $\rho \in \left(0, \frac{\alpha-\lambda}{\alpha}\right)$ if and only if*

$$\lambda - \alpha - 1 < \mu < \lambda\frac{\beta}{\alpha} - \beta - 1$$

in which case ρ is given by

$$\rho = \frac{\alpha - \lambda + \mu + 1}{\alpha - \beta},$$

and the asymptotic behavior of any such solution x is governed by the unique formula

$$x_i(n) \sim \left[\frac{n^{\alpha+1}p(n)^{-1}q(n)}{(\alpha - \lambda - \alpha\rho)\rho^{\alpha}}\right]^{\frac{1}{\alpha-\beta}}, \qquad n \to \infty.$$

Theorem 4.4. *Let $\{p(n)\} \in \mathcal{RV}(\lambda)$ and $\{q(n)\} \in \mathcal{RV}(\mu)$. Suppose that $\lambda > \alpha$. Equation (4.21) possesses regularly varying intermediate solutions of index $\rho \in \left(\frac{\alpha-\lambda}{\alpha}, 0\right)$ if and only if*

$$\lambda\frac{\beta}{\alpha} - \beta - 1 < \mu < \lambda - \alpha - 1$$

in which case ρ is given by

$$\rho = \frac{\alpha - \lambda + \mu + 1}{\alpha - \beta},$$

and the asymptotic behavior of any such solution x is governed by the unique formula

$$x_i(n) \sim \left[\frac{n^{\alpha+1}p(n)^{-1}q(n)}{(\lambda - \alpha + \alpha\rho)(-\rho)^{\alpha}}\right]^{\frac{1}{\alpha-\beta}}, \qquad n \to \infty.$$

A special case of the equation (4.21) with $p(n) = 1, n \in \mathbb{N}$ and with the coefficient q belonging to the set of normalized $\mathcal{RV}$ sequences was considered in [2]. Here, we obtained necessary and sufficient conditions for the existence of $\mathcal{RV}$ solutions and their precise asymptotic behavior, for the equation (4.21) with coefficients p, q belonging to larger set of $\mathcal{RV}$ sequences. Thus, Theorem 4.3 and Theorem 4.4 greatly improve results in [2].

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