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Sharp estimation for the solutions of inhomogeneous delay differential and Halanay-type inequalities

Dedicated to Professor László Hatvani on the occasion of his 75th birthday

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Abstract. This paper is devoted to inhomogeneous Halanay-type inequalities together with inhomogeneous linear delay differential inequalities and equations. Based on the the variation of constants formula and some results borrowed from a recent paper of the authors, sharp conditions for the boundedness and the existence of the limit of the nonnegative solutions are established. The sharpness of the results are illustrated by examples and by comparison of results in some earlier works.

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1 Introduction

Halanay [11] proved an upper estimation for the nonnegative solutions of an autonomous continuous time delay differential inequality with maxima. This, so called Halanay inequality, and its generalizations became a powerful tool in the stability theory of delay differential equations (see for instance [5,6,9,14,18,22]).

Halanay-type inequalities are also studied in the theory of difference inequalities and equations (see [2,3,10,17,20]), and in the theory of time scales (see [1,4,13,15]).

Motivated by the original result of Halanay, the study of the asymptotic behavior of nonnegative solutions of the homogeneous Halanay-type inequality

$$y'(t) \leq -\alpha(t)\, y(t) + \beta(t) \sup_{t-\tau(t)\leq s\leq t} y(s)\,, \quad t \geq t_0$$

has received a lot of attention by many authors (see [5,6,9,12,18,19,21,23]).

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However, there are almost no papers (see [6] and [12]) which have been devoted to the asymptotic analysis of the nonnegative solutions of the inhomogeneous Halanay-type differential inequality

$$x'(t) \leq -\alpha(t)\,x(t) + \beta(t) \sup_{t-\tau(t)\leq s\leq t} x(s) + \varrho(t), \quad t \geq t_0, \tag{1.1}$$

its newly introduced counterpart

$$x'(t) \geq -\alpha(t)\,x(t) + \beta(t) \inf_{t-\tau(t)\leq s\leq t} x(s) + \varrho(t), \quad t \geq t_0, \tag{1.2}$$

together with the inhomogeneous linear delay differential inequalities

$$x'(t) \leq -\alpha(t)\,x(t) + \beta(t)\,x(t-\tau(t)) + \varrho(t), \quad t \geq t_0, \tag{1.3}$$

and

$$x'(t) \geq -\alpha(t)\,x(t) + \beta(t)\,x(t-\tau(t)) + \varrho(t), \quad t \geq t_0. \tag{1.4}$$

In this paper we study these inequalities and the inhomogeneous linear delay differential equation

$$x'(t) = -\alpha(t)\,x(t) + \beta(t)\,x(t-\tau(t)) + \varrho(t), \quad t \geq t_0. \tag{1.5}$$

under the following conditions:

(A_1) $t_0 \in \mathbb{R}$ is fixed, the functions $\alpha : [t_0, \infty[\to \mathbb{R}$, $\beta : [t_0, \infty[\to \mathbb{R}_+$ and $\varrho : [t_0, \infty[\to \mathbb{R}_+$ are locally integrable,

(A_2) $\tau : [t_0, \infty[\to \mathbb{R}_+$ is measurable and it obeys the inequality

$$t_0 - \tau_0 \leq t - \tau(t), \quad t \geq t_0$$

with a constant $\tau_0 \geq 0$.

By $\mathbb{R}_+$ we mean the set of nonnegative numbers. A function $p : [t_0, \infty[\to \mathbb{R}$ is called locally integrable, if it is integrable on every compact subset of $[t_0, \infty[$.

Our aim is to give sharp upper bounds for the nonnegative solutions of (1.1), (1.3) and (1.5), and sharp lower bounds for the nonnegative solutions of (1.2), (1.4) and (1.5). We also obtain information on the approach of the nonnegative solutions of (1.5) to a limit. It is worth to note that in the literature and also in our paper just the nonnegative solutions of the Halanay-type inequality (1.1) are investigated, because they give estimation for the norm of the solutions of more complicated systems of delay differential equations. Our investigation is based on the variation of constants formula and some results borrowed from our recent paper [9]. We remind the reader that Lemma 5.3 plays an important role in the proofs. Our approach to the problem is completely different from that of [6] and [12].

The paper is organized as follows. In Section 2 the main results are established. Section 3 is devoted to the discussion. Sections 4 and 5 are collections of some auxiliary results. Section 6 contains the proofs.

2 Main results

We say that a function $x : [t_0 - \tau_0, \infty[\to \mathbb{R}$ is a solution of the differential equation (1.5) or the differential inequalities (1.1)–(1.4) if x is Borel measurable and bounded on $[t_0 - \tau_0, t_0]$, locally absolutely continuous on $[t_0, \infty[$, and x satisfies (1.5) or (1.1)–(1.4) almost everywhere on $[t_0, \infty[$, respectively.

A function $x : [t_0 - \tau_0, \infty[\to \mathbb{R}$ that is absolutely continuous on $[t_0, t]$ for every $t \in [t_0, \infty[$ is said to be locally absolutely continuous on $[t_0, \infty[$.

First we establish sharp conditions for the boundedness and the existence of the limit of the nonnegative solutions of (1.5). There are only few results in this direction.

Theorem 2.1. *Assume (A_1) and (A_2). Assume further that there exists $t_1 \ge t_0$ for which*

$$\alpha(t) - \beta(t) > 0, \quad t \ge t_1, \tag{2.1}$$

and every nonnegative solution of the homogeneous differential equation

$$y'(t) = -\alpha(t)\, y(t) + \beta(t)\, y(t - \tau(t)), \quad t \ge t_0 \tag{2.2}$$

tends to zero at infinity. Then

(a) For every nonnegative solution $x : [t_0 - \tau_0, \infty[\to \mathbb{R}_+$ of (1.5), we have

$$\liminf_{t\to\infty} \frac{\varrho(t)}{\alpha(t) - \beta(t)} \le \liminf_{t\to\infty} x(t) \le \limsup_{t\to\infty} x(t) \le \limsup_{t\to\infty} \frac{\varrho(t)}{\alpha(t) - \beta(t)}. \tag{2.3}$$

(b) If

$$\lim_{t\to\infty} \frac{\varrho(t)}{\alpha(t) - \beta(t)} \in [0, \infty] \tag{2.4}$$

exists, then for every nonnegative solution $x : [t_0 - \tau_0, \infty[\to \mathbb{R}_+$ of (1.5), we have

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{\varrho(t)}{\alpha(t) - \beta(t)}.$$

By using the following result from [9], we can obtain explicit conditions under which every nonnegative solution of (2.2) tends to zero at infinity.

Theorem A (see [9, Theorem 2.8, Theorem 3.3 and Theorem 3.5])**.** *Assume (A_1), (A_2) and*

$$\beta(t) \le \alpha(t), \quad t \ge t_0.$$

Consider the homogeneous Halanay-type differential inequality

$$y'(t) \le -\alpha(t)\, y(t) + \beta(t) \sup_{t-\tau(t) \le s \le t} y(s), \quad t \ge t_0. \tag{2.5}$$

Every solution of (2.5) tends to zero at infinity, if one of the following sets of conditions is satisfied:

(a) There exists a locally integrable function $\delta : [t_0 - r, \infty[\to \mathbb{R}$ such that

$$\delta(t) + \beta(t) \exp\left(\int_{t-\tau(t)}^{t} \delta(s)\, ds \right) \le \alpha(t), \quad t \ge t_0 \tag{2.6}$$

and

$$\lim_{t\to\infty} \int_{t_0}^{t} \delta(s)\, ds = \infty.$$

(b)

$$\lim_{t\to\infty}(t-\tau(t))=\infty, \tag{2.7}$$

and there exists a constant $q\in\,]0,1[$ *such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}(q\alpha(s)-\beta(s))\,ds=\infty.$$

(c) (2.7) *is satisfied,*

$$0\le\beta(t)\le q\alpha(t),\quad t\ge t_0 \text{ with } q\in\,]0,1[, \tag{2.8}$$

and

$$\int_{t_0}^{\infty}\alpha(s)\,ds=\infty. \tag{2.9}$$

Remark 2.2. Assume (A_1), (A_2) and

$$\beta(t)\le\alpha(t),\quad t\ge t_0.$$

(a) By Theorem 2.8 in [9], every solution of (2.5) tends to zero at infinity if and only if the condition (a) in Theorem A holds.

(b) Theorem 2.11 in [9] shows that every solution of (2.5) tends to zero exponentially at infinity if and only if there exists a locally integrable function $\delta:[t_0-r,\infty[\,\to\mathbb{R}$ such that (2.6) is satisfied and

$$\liminf_{t\to\infty}\frac{1}{t-t_0}\int_{t_0}^{t}\delta(s)\,ds>0.$$

It is worth to note that the estimates in Theorem 2.1 are sharp for some equations. This is illustrated by the next example.

Example 2.3. Consider the inhomogeneous linear delay differential equation

$$\begin{aligned}x'(t)=&-(t+1)x(t)+\frac{1}{2}x\left(t-\frac{\pi}{2}\right)\\&+\sin(2t)+(t+1)\left(1+\sin^2(t)\right)-\frac{1}{2}\left(1+\cos^2(t)\right),\quad t\ge0.\end{aligned} \tag{2.10}$$

In this case $t_0=0$, and the functions τ, α, β, $\varrho:[0,\infty[\,\to\mathbb{R}$ are defined by

$$\tau(t)=\frac{\pi}{2},\quad \alpha(t)=t+1,\quad \beta(t)=\frac{1}{2},$$

and

$$\varrho(t)=\sin(2t)+(t+1)\left(1+\sin^2(t)\right)-\frac{1}{2}\left(1+\cos^2(t)\right).$$

Some easy calculation shows that (A_1), (A_2), (2.1), and (2.7), (2.8), (2.9) are satisfied, and

$$\liminf_{t\to\infty}\frac{\varrho(t)}{\alpha(t)-\beta(t)}=1<2=\limsup_{t\to\infty}\frac{\varrho(t)}{\alpha(t)-\beta(t)}.$$

It is also easy to check that the nonnegative function

$$x:\left[-\frac{\pi}{2},\infty\right[\,\to\mathbb{R}_+,\quad x(t)=1+\sin^2(t)$$

is a solution of (2.10), and for this solution

$$\liminf_{t\to\infty}x(t)=1,\quad \limsup_{t\to\infty}x(t)=2.$$

In the following result we investigate the asymptotic behavior of the nonnegative solutions of (1.1)–(1.4).

Theorem 2.4. *Assume (A_1) and (A_2). Assume further that there exists $t_1 \geq t_0$ for which*

$$\alpha(t) - \beta(t) > 0, \quad t \geq t_1,$$

and every nonnegative solution of the Halanay-type inequality

$$y'(t) \leq -\alpha(t)\, y(t) + \beta(t) \sup_{t-\tau(t) \leq s \leq t} y(s), \quad t \geq t_0 \tag{2.11}$$

tends to zero at infinity. Then

(a) For every nonnegative solution $x : [t_0 - \tau_0, \infty[\to \mathbb{R}_+$ of either (1.1) *or* (1.3)*, we have*

$$\limsup_{t\to\infty} x(t) \leq \limsup_{t\to\infty} \frac{\varrho(t)}{\alpha(t) - \beta(t)}. \tag{2.12}$$

(b) For every nonnegative solution $x : [t_0 - \tau_0, \infty[\to \mathbb{R}_+$ of either (1.2) *or* (1.4)*, we have*

$$\liminf_{t\to\infty} x(t) \geq \liminf_{t\to\infty} \frac{\varrho(t)}{\alpha(t) - \beta(t)}.$$

By using Theorem A, we can also give explicit conditions under which every nonnegative solution of (2.11) tends to zero at infinity.

3 Discussion of the results

First, we deal with the necessity of the condition (2.4) in Theorem (2.1) (b).

The following lemma has a preparatory character.

Lemma 3.1. *Assume that (A_1), (A_2) and conditions* (2.1), (2.7), (2.8) and (2.9) are satisfied. If $x : [t_0 - \tau_0, \infty[\to \mathbb{R}_+$ is a nonnegative solution of (1.5) such that

$$x(\infty) := \lim_{t\to\infty} x(t) \in [0, \infty[\tag{3.1}$$

exists, then

$$x(\infty) = \lim_{t\to\infty} \frac{\varrho(t) - x'(t)}{\alpha(t) - \beta(t)}.$$

Proof. It follows from (2.1) and (2.8) that

$$\frac{\beta(t)}{\alpha(t) - \beta(t)} \leq \frac{\beta(t)}{(1-q)\,\alpha(t)} \leq \frac{q}{1-q}, \quad t \geq t_1. \tag{3.2}$$

By rearranging the equation (1.5), we have

$$\frac{\varrho(t) - x'(t)}{\alpha(t) - \beta(t)} = x(t) + \frac{\beta(t)}{\alpha(t) - \beta(t)}\left(x(t) - x(t - \tau(t))\right), \quad t \geq t_0.$$

This implies the result by using (3.2), (2.7) and (3.1).

The proof is complete. □

Remark 3.2. Assume that the conditions of the previous lemma are satisfied.

If

$$\lim_{t\to\infty} \frac{\varrho(t)}{\alpha(t)-\beta(t)} \in [0,\infty[\tag{3.3}$$

exists, then Theorem 2.1 (b), Theorem A (c) and Lemma 3.1 imply that for every nonnegative solution $x : [t_0 - \tau_0, \infty[\to \mathbb{R}_+$ of (1.5) we have

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{\varrho(t)}{\alpha(t)-\beta(t)} \quad \text{and} \quad \lim_{t\to\infty} \frac{x'(t)}{\alpha(t)-\beta(t)} = 0.$$

Conversely, if there exists a nonnegative solution $x_0 : [t_0 - \tau_0, \infty[\to \mathbb{R}_+$ of (1.5) such that $\lim_{t\to\infty} x_0(t)$ exists and finite, and also

$$\lim_{t\to\infty} \frac{x_0'(t)}{\alpha(t)-\beta(t)} = 0,$$

then by Lemma 3.1, (3.3) is satisfied.

Lemma 3.1 suggests that though the existence of the limit

$$\lim_{t\to\infty} \frac{\varrho(t)}{\alpha(t)-\beta(t)}$$

ensures the existence of $x(\infty)$ in Theorem 2.1 (b), but this condition is not necessary in general. The next example illustrates this phenomenon.

Example 3.3. Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function with support $[0,1]$ such that the range of h is $[0,1]$, and $\int_0^1 h = \frac{1}{2}$. Define the functions y, x, $\varrho : [0,\infty[\to \mathbb{R}$ by

$$y(t) := \sum_{n=0}^{\infty} h\left((t-2n)\,2^{2n}\right) - \sum_{n=0}^{\infty} h\left((t-(2n+1))\,2^{2n+1}\right),$$

$$x(t) := 1 + \int_0^t y(s)\,ds$$

and

$$\varrho := x + y.$$

Then y is well defined, since at most one of the summands different from zero at every $t \in [0,\infty[$. The function y is obviously infinitely differentiable, and

$$\limsup_{t\to\infty} y(t) = 1, \quad \liminf_{t\to\infty} y(t) = -1.$$

It is also easy to check that y is integrable and

$$\int_0^{\infty} y(s)\,ds = \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{2^{2n}} - \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{2^{2n+1}} = \frac{1}{3}.$$

It follows from the previous properties of y that the function x is positive, differentiable, and

$$\lim_{t\to\infty} x(t) = \frac{4}{3}.$$

Plainly we get that the function ϱ is positive, continuous, and

$$\limsup_{t\to\infty} \varrho(t) = \frac{7}{3}, \quad \liminf_{t\to\infty} \varrho(t) = \frac{1}{3}.$$

We can see that x is a positive solution of the differential equation

$$x'(t) = -x(t) + \varrho(t), \quad t \geq 0 \tag{3.4}$$

with finite limit, and every solution of (3.4) tends to the same limit. Equation (3.4) has the form (1.5) with α, β, $\tau : [0, \infty[\to \mathbb{R}$,

$$\alpha(t) = 1, \quad \beta(t) = \tau(t) = 0,$$

but

$$\limsup_{t\to\infty} \frac{\varrho(t)}{\alpha(t) - \beta(t)} = \frac{7}{3}, \quad \liminf_{t\to\infty} \frac{\varrho(t)}{\alpha(t) - \beta(t)} = \frac{1}{3}.$$

Remark 3.4. Assume $0 < K_1 < K_2 < \infty$, and choose $L \in [K_1, K_2]$. By using the method in the previous example, we can construct an equation of the form (1.5) such that

$$\limsup_{t\to\infty} \frac{\varrho(t)}{\alpha(t) - \beta(t)} = K_2, \quad \liminf_{t\to\infty} \frac{\varrho(t)}{\alpha(t) - \beta(t)} = K_1,$$

and every positive solution of the constructed equation tends to L.

Now we compare our estimates with some known ones.

The paper of Backer [6] considers inhomogeneous Halanay-type inequalities (1.1), among others. Corollary 3.4 there implies the following statement.

Proposition B. *Consider the inhomogeneous Halanay-type differential inequality* (1.1), *and suppose that*

$$\alpha(t) \geq \alpha_* > 0, \quad \beta(t) \geq \beta_* \geq 0, \quad t \geq t_0,$$

where α, β and ϱ are bounded and continuous on $[t_0, \infty[$,

$$t - \tau(t) \leq t, \quad t_* = \inf_{t\in[t_0,\infty[} (t - \tau(t)) \quad \textit{and} \quad \lim_{t\to\infty} (t - \tau(t)) = \infty$$

and x is nonnegative, bounded and continuous on $[t_*, \infty[$. Suppose also that there exists a value $\varsigma > 0$ such that

$$0 < \varsigma \leq \alpha(t) - \beta(t), \quad t \geq t_0.$$

Then every nonnegative solution of the homogeneous Halanay-type inequality (2.11) *tends to zero at infinity, and*

$$\limsup_{t\to\infty} x(t) \leq \sup_{t\in[t_0,\infty[} \frac{\varrho(t)}{\alpha(t) - \beta(t)}. \tag{3.5}$$

An interesting result was proved by Hien, Phat and Trinh for the inhomogeneous Halanay-type differential inequality (1.1) (see Theorem 3.2 in [12]), which gives that

Proposition C. *Consider the inhomogeneous Halanay-type differential inequality* (1.1)*, and suppose that*

$$\alpha(t) > 0, \quad \beta(t) \geq 0, \quad \varrho(t) \geq 0, \quad t \geq t_0, \tag{3.6}$$

where α*,* β *and* ϱ *are continuous on* $[t_0, \infty[$*,*

$$t - \tau(t) \leq t, \quad \text{and} \quad \lim_{t\to\infty} (t - \tau(t)) = \infty$$

and x *is nonnegative and continuous on* $]-\infty, \infty[$*, and bounded on* $]-\infty, t_0]$*.*

If

$$\lim_{t\to\infty} \int_{t_0}^{t} \alpha(s)\, ds = \infty,$$

$$M := \sup_{t \geq t_0} \int_{\max(t-\tau(t), t_0)}^{t} \alpha(s)\, ds < \infty.$$

and

$$\sup_{t \geq t_0} \frac{\beta(t)}{\alpha(t)} < 1 \tag{3.7}$$

hold, then for every nonnegative solution of (1.1) *we have*

$$\limsup_{t\to\infty} x(t) \leq \frac{\varrho_\alpha}{1 - \delta_\infty^0}, \tag{3.8}$$

where $\delta_\infty^0 = \sup_{t \geq t_0} \frac{\beta(t)}{\alpha(t)}$ *and* $\varrho_\alpha = \sup_{t \geq t_0} \frac{\varrho(t)}{\alpha(t)}$*.*

The next proposition and example show that our estimate (2.12) is much better than either of the estimates (3.5) and (3.8) in general. It is also true that the conditions in Theorem 2.4 are less restrictive than the conditions in either Proposition B or Proposition C.

Proposition 3.5. *If* (A_1)*,* (3.6) *and* (3.7) *are satisfied, then*

$$\limsup_{t\to\infty} \frac{\varrho(t)}{\alpha(t) - \beta(t)} \leq \sup_{t \in [t_0, \infty[} \frac{\varrho(t)}{\alpha(t) - \beta(t)} \leq \frac{\varrho_\alpha}{1 - \delta_\infty^0}.$$

Proof. For all $t_1 \geq t_0$ we have

$$\sup_{t \geq t_1} \frac{\varrho(t)}{\alpha(t) - \beta(t)} = \sup_{t \geq t_1} \frac{\frac{\varrho(t)}{\alpha(t)}}{1 - \frac{\beta(t)}{\alpha(t)}} \leq \frac{\sup\limits_{t \geq t_1} \frac{\varrho(t)}{\alpha(t)}}{\inf\limits_{t \geq t_1} \left(1 - \frac{\beta(t)}{\alpha(t)}\right)} = \frac{\sup\limits_{t \geq t_1} \frac{\varrho(t)}{\alpha(t)}}{1 - \sup\limits_{t \geq t_1} \frac{\beta(t)}{\alpha(t)}} \leq \frac{\varrho_\alpha}{1 - \delta_\infty^0},$$

which implies the result.

The proof is complete. □

Example 3.6. Let $K > 0$ be fixed, and consider the Halanay-type differential inequality

$$x'(t) \leq -\frac{1}{t} x(t) + \frac{1}{2t} \sup_{t-\tau(t) \leq s \leq t} x(s) + \frac{K}{t^2}, \quad t \geq 1. \tag{3.9}$$

Here, $t_0 := 1$, $\varrho(t) := \frac{K}{t^2}$, $\alpha(t) := \frac{1}{t}$, $\beta(t) := \frac{1}{2t}$ $(t \geq 1)$, and $\tau : [t_0, \infty[\to \mathbb{R}_+$ is measurable satisfying the inequality

$$1 - \tau_0 \leq t - \tau(t), \quad t \geq 1$$

with a constant $\tau_0 \geq 0$ and

$$\lim_{t\to\infty} (t - \tau(t)) = \infty.$$

Since

$$\lim_{t\to\infty} \frac{\varrho(t)}{\alpha(t) - \beta(t)} = \lim_{t\to\infty} \frac{\frac{K}{t^2}}{\frac{1}{t} - \frac{1}{2t}} = \lim_{t\to\infty} \frac{2K}{t} = 0,$$

our result Theorem 2.4 (a) yields that every nonnegative solution of (3.9) tends to zero at infinity.

Because

$$\sup_{t\in[1,\infty[} \frac{\varrho(t)}{\alpha(t) - \beta(t)} = \frac{\varrho_\alpha}{1 - \delta_\infty^0} = \frac{\sup\limits_{t\geq 1} \frac{\varrho(t)}{\alpha(t)}}{1 - \sup\limits_{t\geq 1} \frac{\beta(t)}{\alpha(t)}} = \frac{\sup\limits_{t\geq 1} \frac{K}{t}}{1 - \sup\limits_{t\geq 1} \frac{1}{2}} = 2K,$$

the results Proposition B and C give estimates that all nonnegative solutions of (3.9) are only bounded by a positive constant.

4 General framework

Assume (A_1) and (A_2), and let $\varphi : [t_0 - \tau_0, t_0] \to \mathbb{R}$ be Borel measurable and bounded. Denote $y(\varphi)$ the unique solution of the initial value problem

$$\left.\begin{aligned} &y'(t) = -\alpha(t)\, y(t) + \beta(t)\, y(t - \tau(t)), \quad t \geq t_0, \\ &y(t) = \varphi(t), \quad t_0 - \tau_0 \leq t \leq t_0 \end{aligned}\right\}, \tag{4.1}$$

and $v : [t_0 - \tau_0, \infty[\times [t_0, \infty[\to \mathbb{R}$ the so called fundamental solution of the homogeneous linear delay differential equation in (4.1), that is

$$\left.\begin{aligned} &\frac{\partial v(t,s)}{\partial t} = -\alpha(t)\, v(t,s) + \beta(t)\, v(t - \tau(t), s), \quad t_0 \leq s \leq t \\ &v(t,s) = \begin{cases} 1, & t = s \\ 0, & t < s \end{cases} \end{aligned}\right\}.$$

The initial value problem

$$\left.\begin{aligned} &x'(t) = -\alpha(t)\, x(t) + \beta(t)\, x(t - \tau(t)) + \varrho(t), \quad t \geq t_0, \\ &x(t) = \varphi(t), \quad t_0 - \tau_0 \leq t \leq t_0 \end{aligned}\right\} \tag{4.2}$$

has also a unique solution $x(\varphi)$. It is known that this solution can be obtained by

$$x(\varphi)(t) = y(\varphi)(t) + \int_{t_0}^{t} v(t,s)\, \varrho(s)\, ds, \quad t \geq t_0. \tag{4.3}$$

Assume further that φ and ϱ are nonnegative functions. Then

(a) if $\varphi(t_0) > 0$, then $x(\varphi)(t) > 0$ for every $t \in [t_0, \infty[$;

(b) if $\varphi(t_0) = 0$, then $x(t) \geq 0$ for every $t \in [t_0, T[$;

(c) if $\psi : [t_0 - \tau_0, t_0] \to \mathbb{R}$ is Borel measurable, bounded, and $\psi(t) \geq \varphi(t) \geq 0$ $(t_0 - \tau_0 \leq t \leq t_0)$, then
$$x(\psi)(t) \geq x(\varphi)(t), \quad t \geq t_0.$$

The following result is analogous to Theorem 2.2 in [9]. It shows that there is a close connection between inequalities (1.1) and (1.3).

Theorem 4.1. *Assume (A_1) and (A_2).*

(a) If $\eta : [t_0, \infty[\to \mathbb{R}_+$ is a measurable function such that
$$\eta(t) \leq \tau(t), \quad t \geq t_0, \tag{4.4}$$
then every solution of
$$x'(t) \leq -\alpha(t)x(t) + \beta(t)x(t - \eta(t)) + \varrho(t), \quad t \geq t_0, \tag{4.5}$$
is a solution of (1.1) too.

(b) Conversely, if the function $x : [t_0 - \tau_0, \infty[\to \mathbb{R}_+$ is a solution of (1.1), then there exist a measurable function $\eta : [t_0, \infty[\to \mathbb{R}_+$ depending on x such that η satisfies (4.4), and there exists a solution $\hat{x} : [t_0 - \tau_0, \infty[\to \mathbb{R}_+$ of (4.5) such that
$$\hat{x}(t) = x(t), \quad t \geq t_0 \quad \text{and} \quad \sup_{t_0 - \tau_0 \leq s \leq t_0} \hat{x}(s) = \sup_{t_0 - \tau_0 \leq s \leq t_0} x(s). \tag{4.6}$$

The next result explains the correspondence between inequalities (1.2) and (1.4).

Theorem 4.2. *Assume (A_1) and (A_2).*

(a) If $\eta : [t_0, \infty[\to \mathbb{R}_+$ is a measurable function such that (4.4) is satisfied, then every solution of
$$x'(t) \geq -\alpha(t)x(t) + \beta(t)x(t - \eta(t)) + \varrho(t), \quad t \geq t_0, \tag{4.7}$$
is a solution of (1.2) too.

(b) Conversely, if the function $x : [t_0 - \tau_0, \infty[\to \mathbb{R}_+$ is a solution of (1.2), then there exist a measurable function $\eta : [t_0, \infty[\to \mathbb{R}_+$ depending on x such that η satisfies (4.4), and there exists a solution $\hat{x} : [t_0 - \tau_0, \infty[\to \mathbb{R}_+$ of (4.7) such that
$$\hat{x}(t) = x(t), \quad t \geq t_0 \quad \text{and} \quad \inf_{t_0 - \tau_0 \leq s \leq t_0} \hat{x}(s) = \inf_{t_0 - \tau_0 \leq s \leq t_0} x(s). \tag{4.8}$$

5 Auxiliary results

The next two results are slight modifications of Lemma 5.3 and Lemma 5.4 in [9], respectively.

Lemma 5.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, and define the function $\chi : \{(t,s) \in \mathbb{R}^2 \mid s \leq t\} \to \mathbb{R}$ by*
$$\chi(t,s) = \min\left\{u \in [s,t] \mid f(u) = \min_{s \leq v \leq t} f(v)\right\}. \tag{5.1}$$
Then χ is lower semi-continuous.

Proof. For every $(t,s) \in \mathbb{R}^2$ with $s \le t$ we have

$$\chi(t,s) = \min\left\{u \in [s,t] \mid -f(u) = \max_{s \le v \le t}(-f(v))\right\},$$

and hence Lemma 5.3 in [9] can be applied to the function $-f$.

The proof is complete. □

Lemma 5.2. *Let $t_0 \in \mathbb{R}$ and $\tau_0 \ge 0$ be fixed, and $f : [t_0 - \tau_0, \infty[\to \mathbb{R}$ be continuous. Assume $\tau : [t_0, \infty[\to \mathbb{R}_+$ is measurable such that*

$$t_0 - \tau_0 \le t - \tau(t), \quad t \ge t_0.$$

(a) Then the function

$$\vartheta : [t_0, \infty[\to \mathbb{R}, \quad \vartheta(t) = \min\left\{u \in [t - \tau(t), t] \mid f(u) = \min_{t-\tau(t) \le v \le t} f(v)\right\}$$

is measurable.

(b) Define the function

$$\eta : [t_0, \infty[\to \mathbb{R}, \quad \eta(t) = t - \vartheta(t).$$

Then η is measurable and $t - \tau(t) \le t - \eta(t) \le t$ $(t \ge t_0)$.

Proof. We can copy the proof of Lemma 5.4 in [9], by using Lemma 5.1. □

The following lemma is needed in the proofs of the main results.

Lemma 5.3. *Assume (A_1) and (A_2). Assume further that every nonnegative solution of the differential equation*

$$x'(t) = -\alpha(t)x(t) + \beta(t)x(t - \tau(t)), \quad t \ge t_0 \tag{5.2}$$

tends to zero at infinity.

If $T \ge t_0$, then

(a) Every nonnegative solution of the differential equation

$$x'(t) = -\alpha(t)x(t) + \beta(t)x(t - \tau(t)), \quad t \ge T \tag{5.3}$$

tends to zero at infinity too.

(b) If $\vartheta : [t_0, \infty[\to \mathbb{R}$ is locally integrable, then

$$\lim_{t \to \infty} \int_{t_0}^{T} v(t,s)\vartheta(s)\,ds = 0. \tag{5.4}$$

(c)

$$\lim_{t \to \infty} \int_{T}^{t} v(t,s)(\alpha(s) - \beta(s))\,ds = 1. \tag{5.5}$$

Proof. (a) Let $x_T : [t_0 - \tau_0, \infty[\to \mathbb{R}$ be a nonnegative solution of (5.3), and let $x : [t_0 - \tau_0, \infty[\to \mathbb{R}$ be the solution of the initial value problem

$$\left.\begin{array}{l} x'(t) = -\alpha(t)x(t) + \beta(t)x(t-\tau(t)), \quad t \ge t_0, \\ x(t) = 1, \quad t_0 - \tau_0 \le t \le t_0 \end{array}\right\}.$$

Then $x(t) > 0$ for all $t \ge t_0 - \tau_0$, and there exists $c > 0$ such that

$$x(t) \ge c, \quad t_0 - \tau_0 \le t \le T.$$

This inequality and the fact that x_T is nonnegative and bounded on $[t_0 - \tau_0, T]$ mean that there exists $k > 0$ such that

$$x_T(t) \le kx(t), \quad t_0 - \tau_0 \le t \le T. \tag{5.6}$$

Since equations (5.2) and (5.3) are homogeneous, kx is a solution of both equations. Therefore, recalling that every nonnegative solution of (5.2) tends to zero at infinity, we have

$$\lim_{t\to\infty} kx(t) = 0. \tag{5.7}$$

From (5.6) it follows that

$$x_T(t) \le kx(t), \quad t_0 - \tau_0 \le t \le \infty. \tag{5.8}$$

Now, (5.7) and (5.8) imply the result.

(b) Let $\widehat{\vartheta} : [t_0, \infty[\to \mathbb{R}$ be defined by

$$\widehat{\vartheta}(t) = \begin{cases} \vartheta(t), & t_0 \le t < T, \\ 0, & t \ge T, \end{cases}$$

and let $z : [t_0 - \tau_0, \infty[\to \mathbb{R}$ be given by

$$z(t) = \begin{cases} 0, & t_0 - \tau_0 \le t \le t_0, \\ \displaystyle\int_{t_0}^{t} v(t,s)\,\widehat{\vartheta}(s)\,ds, & t \ge t_0. \end{cases}$$

By using (4.3), we have that z is the solution of the initial value problem

$$\left.\begin{array}{l} x'(t) = -\alpha(t)x(t) + \beta(t)x(t-\eta(t)) + \widehat{\vartheta}(t), \quad t \ge t_0, \\ x(t) = 0, \quad t_0 - \tau_0 \le t \le t_0 \end{array}\right\}.$$

The definition of $\widehat{\vartheta}$ shows that z is a solution of the differential equation (5.3), and therefore by (a),

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} \int_{t_0}^{T} v(t,s)\,\vartheta(s)\,ds = 0$$

which gives (5.4).

(c) It is obvious that the function $x : [t_0 - \tau_0, \infty[\to \mathbb{R}$, $x(t) = 1$ is a solution of the initial value problem

$$\left.\begin{array}{l} x'(t) = -\alpha(t)x(t) + \beta(t)x(t-\eta(t)) + \alpha(t) - \beta(t), \quad t \ge t_0, \\ x(t) = 1, \quad t_0 - \tau_0 \le t \le t_0 \end{array}\right\},$$

and hence by using (4.3), and the condition that every nonnegative solution of (5.2) tends to zero at infinity, we obtain

$$\lim_{t\to\infty}\int_{t_0}^{t} v\left(t,s\right)\left(\alpha\left(s\right)-\beta\left(s\right)\right)ds = 1. \tag{5.9}$$

Since

$$\int_{T}^{t} v\left(t,s\right)\left(\alpha\left(s\right)-\beta\left(s\right)\right)ds = \int_{t_0}^{t} v\left(t,s\right)\left(\alpha\left(s\right)-\beta\left(s\right)\right)ds - \int_{t_0}^{T} v\left(t,s\right)\left(\alpha\left(s\right)-\beta\left(s\right)\right)ds, \quad t\geq T,$$

the result follows from (5.9) and (5.4).

The proof is complete. □

6 Proofs

Proof of Theorem 4.1. We can copy the proof of Theorem 2.2 in [9]. □

Proof of Theorem 4.2. (a) Let $x : [t_0-\tau_0,\infty[\ \to \mathbb{R}$ be a solution of (4.7). Since β is nonnegative and (4.4) is satisfied,

$$\begin{aligned} x'\left(t\right) &\geq -\alpha\left(t\right)x\left(t\right)+\beta\left(t\right)x\left(t-\eta\left(t\right)\right)+\varrho\left(t\right)\\ &\geq -\alpha\left(t\right)x\left(t\right)+\beta\left(t\right)\inf_{t-\eta(t)\leq s\leq t} x\left(s\right)+\varrho\left(t\right)\\ &\geq -\alpha\left(t\right)x\left(t\right)+\beta\left(t\right)\inf_{t-\tau(t)\leq s\leq t} x\left(s\right)+\varrho\left(t\right), \quad \text{a.e. on } [t_0,\infty[, \end{aligned}$$

and therefore x is also a solution of (1.2).

(b) Suppose $x : [t_0-\tau_0,\infty[\ \to \mathbb{R}_+$ is a solution of (1.2).

Let $m := \inf_{t_0-\tau_0\leq s\leq t_0} x\left(s\right)$, define the number

$$t_1 := \begin{cases} \infty, \text{ if } x\left(t\right) > m, & \text{for all } t\geq t_0,\\ \min\left\{t\geq t_0 \mid x\left(t\right)=m\right\}, & \text{otherwise,} \end{cases}$$

and introduce the measurable sets

$$\begin{aligned} A_1 &:= \left\{t\in [t_0,t_1[\ \mid t-\tau\left(t\right) < t_0\right\},\\ A_2 &:= \left\{t\in [t_1,\infty[\ \mid t-\tau\left(t\right) < t_0\right\} \end{aligned}$$

and

$$B := [t_0,\infty[\ \setminus \left(A_1\cup A_2\right).$$

If $\tau_0 > 0$, choose a strictly increasing sequence $\left(a_n\right)_{n\geq 1}$ from $[t_0-\tau_0,t_0[$ such that $a_1 := t_0-\tau_0$ and $a_n\to t_0$, and define

$$\hat{x} : [t_0-\tau_0,\infty[\ \to \mathbb{R}_+, \quad \hat{x}\left(t\right) = \begin{cases} m, & \text{if } t=a_n,\ n\geq 1,\\ x\left(t\right), & \text{otherwise.} \end{cases}$$

If $\tau_0 = 0$, then $t_1 = t_0$, $A_1 = A_2 = \varnothing$ and $B = [t_0,\infty[$, and let $\hat{x} := x$.

Since β is nonnegative, and

$$\inf_{t-\tau(t)\leq s\leq t} x\left(s\right) \geq \inf_{t-\tau(t)\leq s\leq t} \hat{x}\left(s\right), \quad t\geq t_0,$$

$\hat{x}$ is also a solution of (1.2). It is easy to check that

$$\inf_{t-\tau(t)\le s\le t} \hat{x}(s) = \begin{cases} m, & \text{if } t \in A_1, \\ \min\limits_{t_0\le s\le t} x(s), & \text{if } t \in A_2, \\ \min\limits_{t-\tau(t)\le s\le t} x(s), & \text{if } t \in B. \end{cases}$$

Introduce the functions $\vartheta : [t_0, \infty[\to \mathbb{R}$,

$$\vartheta(t) = \begin{cases} a_{n+1}, & \text{if } t \in A_1 \text{ and } t-\tau(t) \in [a_n, a_{n+1}[, \\ \min\left\{u \in [t_0, t] \mid x(u) = \min\limits_{t_0\le s\le t} x(s)\right\}, & \text{if } t \in A_2 \\ \min\left\{u \in [t-\tau(t), t] \mid x(u) = \min\limits_{t-\tau(t)\le s\le t} x(s)\right\}, & \text{if } t \in B, \end{cases}$$

and

$$\eta : [t_0, \infty[\to \mathbb{R}_+, \quad \eta(t) = t - \vartheta(t).$$

It is obvious that η is measurable on A_1, and it satisfies (4.4). As we have seen in Lemma 5.2, the function η is measurable on $A_2 \cup B$ and

$$t - \tau(t) \le t - \eta(t) \le t \quad (t \ge t_0),$$

and hence (4.4) holds.

It follows from the definition of η that $\hat{x}$ is a solution of (4.5) with this η.

The proof is complete. □

Proof of Theorem 2.1. Fix a nonnegative solution $x : [t_0 - \tau_0, \infty[\to \mathbb{R}_+$ of (1.5).

(a) We can obviously suppose that

$$\limsup_{t\to\infty} \frac{\varrho(t)}{\alpha(t) - \beta(t)} < \infty. \tag{6.1}$$

According to (2.1) and (6.1), for every $c > \limsup_{t\to\infty} \frac{\varrho(t)}{\alpha(t)-\beta(t)}$ there exists $T > \max(t_0, t_1)$ (depends on c) such that

$$\frac{\varrho(t)}{\alpha(t) - \beta(t)} \le c, \quad t \ge T. \tag{6.2}$$

If $y(x) : [t_0 - \tau_0, \infty[\to \mathbb{R}_+$ is the solution of the initial value problem

$$\begin{aligned} y'(t) &= -\alpha(t) y(t) + \beta(t) y(t - \tau(t)), \quad t \ge t_0, \\ y(t) &= x(t), \quad t_0 - \tau_0 \le t \le t_0, \end{aligned}$$

then by using (4.3), we have

$$x(t) = y(x)(t) + \int_{t_0}^{t} v(t,s)\, \varrho(s)\, ds, \quad t \ge t_0. \tag{6.3}$$

By the assumption,

$$\lim_{t\to\infty} y(x)(t) = 0,$$

and hence (6.3) shows that

$$\limsup_{t\to\infty} x(t) = \limsup_{t\to\infty} \int_{t_0}^{t} v(t,s)\,\varrho(s)\,ds. \tag{6.4}$$

By using Lemma 5.3 (b) with $\vartheta = \varrho$, we have from (6.5) that

$$\limsup_{t\to\infty} x(t) = \limsup_{t\to\infty} \int_{T}^{t} v(t,s)\,\varrho(s)\,ds. \tag{6.5}$$

By (6.2) and (2.1),

$$\begin{aligned}\int_{T}^{t} v(t,s)\,\varrho(s)\,ds &= \int_{T}^{t} v(t,s)\,(\alpha(s)-\beta(s))\,\frac{\varrho(s)}{\alpha(s)-\beta(s)}ds \\ &\le c\int_{T}^{t} v(t,s)\,(\alpha(s)-\beta(s))\,ds, \quad t\ge T,\end{aligned}$$

and therefore Lemma 5.3 (c) yields that

$$\limsup_{t\to\infty} \int_{T}^{t} v(t,s)\,\varrho(s)\,ds \le c\limsup_{t\to\infty} \int_{T}^{t} v(t,s)\,(\alpha(s)-\beta(s))\,ds = c. \tag{6.6}$$

Combining this with (6.5), the third inequality in (2.3) follows.

Now we continue the proof of the first inequality in (2.3).

If

$$\liminf_{t\to\infty} \frac{\varrho(t)}{\alpha(t)-\beta(t)} = 0,$$

then nothing to prove, so we can suppose that

$$\liminf_{t\to\infty} \frac{\varrho(t)}{\alpha(t)-\beta(t)} > 0. \tag{6.7}$$

We can follow as in (a).

It comes from (2.1) and (6.7) that for every $0 < c < \liminf_{t\to\infty} \frac{\varrho(t)}{\alpha(t)-\beta(t)}$ there exists $T > \max(t_0, t_1)$ (depends on c) such that

$$\frac{\varrho(t)}{\alpha(t)-\beta(t)} \ge c, \quad t \ge T. \tag{6.8}$$

The formula (6.5) can be written now as

$$\liminf_{t\to\infty} x(t) = \liminf_{t\to\infty} \int_{T}^{t} v(t,s)\,\varrho(s)\,ds. \tag{6.9}$$

By using (6.8) and Lemma 5.3 (b), we have

$$\begin{aligned}\liminf_{t\to\infty} \int_{T}^{t} v(t,s)\,\varrho(s)\,ds &= \liminf_{t\to\infty} \int_{T}^{t} v(t,s)\,(\alpha(s)-\beta(s))\,\frac{\varrho(s)}{\alpha(s)-\beta(s)}ds \\ &\ge c\liminf_{t\to\infty} \int_{T}^{t} v(t,s)\,(\alpha(s)-\beta(s))\,ds = c.\end{aligned}$$

This gives the result by (6.9).

(b) It is an easy consequence of (a).

The proof is complete. □

Proof of Theorem 2.4. (a) Since every solution of (1.3) is also a solution of (1.1), it is enough to consider (1.1).

Fix a nonnegative solution $x : [t_0 - \tau_0, \infty[\to \mathbb{R}_+$ of (1.1).

By Theorem 4.1 (b), there exists a measurable function $\eta : [t_0, \infty[\to \mathbb{R}_+$ satisfying (4.4), and a nonnegative solution $\hat{x} : [t_0 - \tau_0, \infty[\to \mathbb{R}_+$ of (4.5) such that (4.6) holds. It follows that there exists a locally integrable function $\vartheta : [t_0, \infty[\to \mathbb{R}$ such that

$$\vartheta(t) \le \varrho(t), \quad t \ge t_0 \tag{6.10}$$

and

$$\hat{x}'(t) = -\alpha(t)x(t) + \beta(t)\hat{x}(t - \eta(t)) + \vartheta(t), \quad \text{a.e. on } [t_0, \infty[,$$

which shows that $\hat{x}$ is a nonnegative solution of the differential equation

$$x'(t) = -\alpha(t)x(t) + \beta(t)x(t - \eta(t)) + \vartheta(t), \quad t \ge t_0. \tag{6.11}$$

Since every nonnegative solution of the differential equation

$$y'(t) = -\alpha(t)y(t) + \beta(t)y(t - \eta(t)), \quad t \ge t_0 \tag{6.12}$$

is a solution of (2.11), we have that every nonnegative solution of (6.12) tends to zero at infinity.

By (4.3) and (4.6),

$$x(t) = y(x)(t) + \int_{t_0}^{t} v_\eta(t,s)\vartheta(s)\,ds, \quad t \ge t_0,$$

where v_η is the fundamental solution of (6.12) and $y(x)$ is the solution of (6.12) with the initial value

$$y(x)(t) = x(t), \quad t_0 - \tau_0 \le t \le t_0.$$

On the other hand, (6.10) and the nonnegativity of v_η imply

$$x(t) \le y(x)(t) + \int_{t_0}^{t} v_\eta(t,s)\varrho(s)\,ds, \quad t \ge t_0,$$

and therefore

$$\limsup_{t\to\infty} x(t) \le \limsup_{t\to\infty} \int_{t_0}^{t} v_\eta(t,s)\varrho(s)\,ds.$$

Now, we can proceed as between (6.4) and (6.6) in the proof of Theorem 2.1.

(b) We can prove similarly to (a), by using Theorem 4.2 instead of Theorem 4.1.

The proof is complete. □

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