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Boundary layer analysis for nonlinear singularly perturbed differential equations

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Abstract

This paper focuses on the boundary layer phenomenon arising in the study of singularly perturbed differential equations. Our tools include the method of lower and upper solutions combined with analysis of the integral equation associated with the class of nonlinear equations under consideration.

Key words and phrases: singularly perturbed systems, three–point boundary value problem, method of lower and upper solutions.

AMS Subject Classifications: 34E15, 34A34, 34A40, 34B10

1 Introduction

This paper is devoted to study the second-order semilinear singularly perturbed differential equation

$$\epsilon y'' + ky = f(t, y), \quad t \in [a, b], \quad k < 0 \tag{1}$$

subject to the three–point boundary value conditions

$$y'_\epsilon(a) = 0, \quad y_\epsilon(b) = y_\epsilon(c), \quad a < c < b, \tag{2}$$

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where ϵ is a small perturbation parameter ($0 < \epsilon << 1$).

In the past few years, much attention has been paid to the study of nonlocal boundary value problems, whose study for ordinary differential equations has been initiated by the work of Il'in and Moiseev [11, 12].

In particular, existence of solutions for differential equation

$$y'' + g(t)f(y(t)) = 0, \quad 0 < t < 1$$

under one of the m-point boundary conditions

$$y'(0) = 0, \quad y(1) = \sum_{i=1}^{m-2} \alpha_i y(\eta_i), \quad 0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$$

or

$$y(0) = 0, \quad y(1) = \sum_{i=1}^{m-2} \alpha_i y(\eta_i), \quad 0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1,$$

as an important subclass of nonlocal boundary conditions has been thoroughly studied by Gupta *et al.*, see, for example, [5, 6, 7, 8, 9]. Eloe and Gao [3] discussed the quasilinearization method for a three-point semilinear boundary value problem which provides an iterative scheme for approximating the solutions.

The subject of multi-point nonlocal boundary value problems for singularly perturbed differential equations has been also addressed by many authors, see e.g. [1, 2], and the references therein. For example, Du *et al.* [1] have studied a third-order multi-point singularly perturbed boundary value problem

$$\epsilon y''' + f(t, y, y', y'', \epsilon) = 0, \ 0 \leq t \leq 1, \ 0 < \epsilon << 1,$$

$$y_\epsilon(0) = 0,$$

$$a y'_\epsilon(0) - b y''_\epsilon(0) + \sum_{i=1}^{n-2} \alpha_i y_\epsilon(\xi_i) = A,$$

$$c y'_\epsilon(1) + d y''_\epsilon(1) + \sum_{i=1}^{n-2} \beta_i y_\epsilon(\eta_i) = B,$$

where $0 < \xi_1 < \xi_2 < \cdots < \xi_{n-2} < 1$ and $0 < \eta_1 < \eta_2 < \cdots < \eta_{n-2} < 1$, applying differential inequalities technique (method of lower and upper solutions) and Leray–Schauder degree theory. This paper contains a large amount of material and can serve as an introduction to some of principles and methods of singular perturbation theory, not only for third-order nonlinear differential equations.

Singular perturbation problems can also arise in heat transfer problem with large Peclet numbers [14], Navier-Stokes flows with large Reynolds numbers, chemical reactor theory, aerodynamics, reaction-diffusion processes, quantum mechanics, optimal control [15], for example.

As far as we know, there is no paper related to the boundary layer analysis for nonlinear multi-point nonlocal singularly perturbed boundary value problems.

Let $D(u)$ denotes the set

$$\{(t,y)| \quad a \le t \le b,\ |y-u(t)| \le d(t)\},$$

where $d(t)$ is the positive continuous function on the interval $[a,b]$ such that

$$d(t) = \begin{cases} \delta & \text{for } a \le t \le b-\delta, \\ |u(b)-u(c)|+\delta & \text{for } b-\frac{\delta}{2} \le t \le b, \end{cases}$$

where δ is a small positive constant.

Recently in [16], we have shown that for every $\epsilon > 0$ sufficiently small ($\epsilon \in (0, \epsilon_0]$) there is a unique solution y_ϵ of BVP (1), (2) such that $\{(t, y_\epsilon(t))| \quad a \le t \le b\} \subset D(u)$ and y_ϵ converges uniformly to the solution u of reduced problem $ku = f(t,u)$ for $\epsilon \to 0^+$ on every compact subset $K \subset [a,b)$. Consequently, $y_\epsilon(b) = y_\epsilon(c) \to u(c)$ for $\epsilon \to 0^+$.

In the present paper, we focus our attention on the detailed analysis of the behavior of the solutions y_ϵ for (1), (2) in the point $t = b$ when a small parameter ϵ tends to zero. We show that the solutions y_ϵ of (1), (2) remain close to u on K with an arising fast transient of y_ϵ to $y_\epsilon(b)$ ($|y'_\epsilon(b)| \to \infty$ for $u(b) \ne u(c)$ and $\epsilon \to 0^+$), which is the so-called boundary layer phenomenon ([4, 13]). Boundary layers are formed due to the nonuniform convergence of the exact solution y_ϵ to the solution u of reduced problem in the neighborhood of the right end b.

We will assume that the following conditions are satisfied throughout this paper:

(H1) The solution u of a reduced problem $ku = f(t,u)$ is a C^3 function defined on the interval $[a,b]$.

(H2) $f(c, u(c)) \ne f(b, u(c))$

It is instructive for the future to keep in mind that this assumption implies that $u(c) \ne u(b)$ and $f(c, y_\epsilon(c)) \ne f(b, y_\epsilon(b))$ for every sufficiently small ϵ, say $0 < \epsilon < \epsilon_0$.

(H3) $f \in C^1(D(u))$ and there exists a positive constant w such that

$$\left|\frac{\partial f(t,y)}{\partial y}\right| \leq w < -k \quad \text{for every} \quad (t,y) \in D(u).$$

Notation.

$g_{1,\epsilon}(t) = k - \frac{\partial f(t,y_\epsilon(t))}{\partial y}$

$g_{2,\epsilon}(t) = \frac{\partial f(t,y_\epsilon(t))}{\partial t}$

$m = -k - w$

$\gamma_\epsilon(t) = \frac{1}{m}\left|\epsilon u'''(t) + g_{1,\epsilon}(t)u'(t) - g_{2,\epsilon}(t)\right|.$

Obviously, $\gamma_\epsilon(t) \geq 0$ and $\lim\limits_{\epsilon\to 0^+} \gamma_\epsilon(t) = 0$ for $t \in [a,b)$. Further, $\lim\limits_{\epsilon\to 0^+} \gamma_\epsilon(b) \neq 0$ for $u'(b) \neq \frac{\partial f(b,u(c))}{\partial t}\left(k - \frac{\partial f(b,u(c))}{\partial y}\right)^{-1}$. The equality $u(b) = u(c)$ implies $\lim\limits_{\epsilon\to 0^+} \gamma_\epsilon(b) = 0$.

2 Boundary layer phenomenon at $t = b$

For an illustrative example we consider (1), (2) with $f(t,y) = t^2$, $a = 0$, $b = 2$, $c = 1$ and its solution

$$\begin{aligned} y_\epsilon(t) = &-\frac{3}{k} \cdot \frac{\mathrm{e}^{2\sqrt{-\frac{k}{\epsilon}}}}{\mathrm{e}^{4\sqrt{-\frac{k}{\epsilon}}} - \mathrm{e}^{3\sqrt{-\frac{k}{\epsilon}}} - \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}} + 1} \cdot \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}t} \\ &- \frac{3}{k} \cdot \frac{\mathrm{e}^{2\sqrt{-\frac{k}{\epsilon}}}}{\mathrm{e}^{4\sqrt{-\frac{k}{\epsilon}}} - \mathrm{e}^{3\sqrt{-\frac{k}{\epsilon}}} - \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}} + 1} \cdot \mathrm{e}^{-\sqrt{-\frac{k}{\epsilon}}t} + \frac{t^2}{k} - \frac{2\epsilon}{k^2}. \end{aligned}$$

Hence we have

1. $\lim\limits_{\epsilon\to 0^+} y_\epsilon(t_0) = \frac{f(t_0)}{k} = u(t_0)$ for every $t_0 \in [0,2)$
2. $\lim\limits_{\epsilon\to 0^+} y_\epsilon(2) = \frac{f(1)}{k} = u(1)$
3. $\lim\limits_{\epsilon\to 0^+} |y'_\epsilon(2)| = \infty$ (a boundary layer phenomenon).

We precede the main result of this article with the following important lemmas.

Lemma 2.1 *Let the assumptions (H1) and (H3) hold. Let* $[t, y_\epsilon(t)] \subset D(u)$ *for* $\epsilon \in (0, \epsilon_0]$ *and* $t \in [a,b]$ *where* y_ϵ *is the solution of (1), (2). Then we have on* $[a,b]$ *the estimate*

$$\left|y'_\epsilon(t) - u'(t)\right| \leq v_{L,\epsilon}(t) + v_{R,\epsilon}(t) + \gamma_{\epsilon,\max} \tag{3}$$

where

$$v_{L,\epsilon}(t) = \left|u'(a)\right| \mathrm{e}^{\sqrt{\frac{m}{\epsilon}}(a-t)}$$
$$v_{R,\epsilon}(t) = \left|u'(b) - y_\epsilon'(b)\right| \mathrm{e}^{\sqrt{\frac{m}{\epsilon}}(t-b)}$$
$$\gamma_{\epsilon,\max} = \max\left\{\gamma_\epsilon(t);\ t \in [a,b]\right\}.$$

Proof. Differentiating (1) with respect to the variable t we obtain for y_ϵ', $\epsilon \in (0, \epsilon_0]$ linear differential equation

$$\epsilon z'' + g_{1,\epsilon}(t) z = g_{2,\epsilon}(t) \tag{4}$$

with the Dirichlet boundary condition

$$z_\epsilon(a) = 0,\ z_\epsilon(b) = y_\epsilon'(b). \tag{5}$$

First we show that $z_\epsilon = y_\epsilon'$ is an unique solution of Dirichlet BVP (4), (5) for y_ϵ, $\epsilon \in (0, \epsilon_0]$. Assume to the contrary, that Z_1, Z_2 are two solutions of (4), (5) for $\epsilon \in (0, \epsilon_0]$ fixed. Denote $Z(t) = Z_1(t) - Z_2(t)$. Then Z is a solution of the homogeneous Dirichlet problem

$$\epsilon z'' + g_{1,\epsilon}(t) z = 0,$$

$$z_\epsilon(a) = 0,\ z_\epsilon(b) = 0.$$

Thus there is $t_0 \in (a,b)$ such that $Z(t_0) \neq 0$, $Z'(t_0) = 0$ and $Z(t_0)\, Z''(t_0) \leq 0$ which contradicts to the assumption (H3). To prove Lemma 2.1 it is sufficient to show that for every y_ϵ, $\epsilon \in (0, \epsilon_0]$ there is a solution z_ϵ of (4), (5) satisfying (3). We apply the method of lower and upper solutions ([10]). As usual, a function α_ϵ is called a lower solution of the Dirichlet BVP (4), (5) if $\alpha_\epsilon \in C^2([a,b])$ and satisfies

$$\epsilon \alpha_\epsilon''(t) + g_{1,\epsilon}(t)\alpha_\epsilon \geq g_{2,\epsilon}(t) \tag{6}$$

$$\alpha_\epsilon(a) \leq 0,\ \alpha_\epsilon(b) \leq y_\epsilon'(b).$$

An upper solution $\beta_\epsilon \in C^2([a,b])$ of the problem (4), (5) is defined similarly by reversing the inequalities. If $\alpha_\epsilon \leq \beta_\epsilon$ on $[a,b]$ then there exists a solution z_ϵ with $\alpha_\epsilon \leq z_\epsilon \leq \beta_\epsilon$ on $[a,b]$.

Define

$$\alpha_\epsilon(t) = u'(t) - v_{L,\epsilon}(t) - v_{R,\epsilon}(t) - \gamma_{\epsilon,\max}$$

and

$$\beta_\epsilon(t) = u'(t) + v_{L,\epsilon}(t) + v_{R,\epsilon}(t) + \gamma_{\epsilon,\max}.$$

It is easy to check that $\alpha_\epsilon(a) \leq 0 \leq \beta_\epsilon(a)$, $\alpha_\epsilon(b) \leq y_\epsilon'(b) \leq \beta_\epsilon(b)$ and $\alpha_\epsilon(t) \leq \beta_\epsilon(t)$ for $t \in [a,b]$. Now we show that the inequality (6) holds.

For β_ϵ we proceed analogously.

$$\begin{aligned}
&\epsilon\alpha_\epsilon''(t) + g_{1,\epsilon}(t)\alpha_\epsilon(t) - g_{2,\epsilon}(t)\\
&\qquad = \epsilon u'''(t) - \epsilon v_{L,\epsilon}''(t) - \epsilon v_{R,\epsilon}''(t)\\
&\qquad + g_{1,\epsilon}(t)\left(u'(t) - v_{L,\epsilon}(t) - v_{R,\epsilon}(t) - \gamma_{\epsilon,\max}\right) - g_{2,\epsilon}(t)\\
&\qquad \geq \epsilon u'''(t) - \epsilon v_{L,\epsilon}''(t) - \epsilon v_{R,\epsilon}''(t)\\
&\qquad + g_{1,\epsilon}(t)u'(t) + m v_{L,\epsilon}(t) + m v_{R,\epsilon}(t) + m\gamma_{\epsilon,\max} - g_{2,\epsilon}(t)\\
&\qquad = \epsilon u'''(t) + g_{1,\epsilon}(t)u'(t) - g_{2,\epsilon}(t) + m\gamma_{\epsilon,\max} \geq 0.
\end{aligned}$$

The Lemma 2.1 is proven. □

Lemma 2.2 *Let the assumptions (H1) and (H3) hold. Then the set*

$$\left\{\epsilon\left|y_\epsilon'(b)\right|;\ \epsilon\in(0,\epsilon_0]\right\}$$

is bounded.

Proof. By Lagrange's Theorem and from Diff. Eq. (1) we obtain

$$\begin{aligned}
\left|y_\epsilon'(b) - y_\epsilon'(a)\right| &= \left|y_\epsilon''(\tau_\epsilon)\right|(b-a) = \frac{1}{\epsilon}\left|f\left(\tau_\epsilon, y_\epsilon(\tau_\epsilon)\right) - k y_\epsilon(\tau_\epsilon)\right|(b-a)\\
&\leq \frac{C_\delta^*}{\epsilon}(b-a)
\end{aligned}$$

where $\tau_\epsilon\in(a,b)$ and $C_\delta^* = \max\left\{|f(t,y)-ky|;\ (t,y)\in D(u)\right\}$.
Hence $\epsilon\left|y_\epsilon'(b)\right| \leq C_\delta^*(b-a)$ for $\epsilon\in(0,\epsilon_0]$. □

3 Main result

Our main result is the following.

Theorem 3.1 *Under the assumptions (H1)-(H3) the problem (1), (2) has for every ϵ, $\epsilon\in(0,\epsilon_0]$ the unique solution y_ϵ in $D(u)$ which converges uniformly to the solution u of reduced problem for $\epsilon\to0^+$ on an arbitrary compact subset K of $[a,b)$ and the set*

$$\left\{\left|y_\epsilon'(t)\right|;\ t\in[a,b],\epsilon\in(0,\epsilon_0]\right\}$$

is unbounded.

More precisely,

$$\left|y_\epsilon'(b)\right| = O\left(\frac{1}{\sqrt{-k\epsilon}}\right) \quad \text{i. e. } \left|y_\epsilon'(b)\right|\to\infty \text{ for } \epsilon\to0^+. \tag{7}$$

Proof. The existence, uniqueness in $D(u)$ and asymptotic behavior of the solutions for (1), (2) on the compact subset $K \subset [a, b)$ has been proven in [16]. It remains to prove (7), a boundary layer phenomenon at $t = b$.

Assume to the contrary that the set

$$\left\{ \left| y'_\epsilon(t) \right| ;\ t \in [a, b], \epsilon \in (0, \epsilon_0] \right\}$$

is bounded. Consequently,

$$\left| \frac{\mathrm{d} f\left(t, y_\epsilon(t)\right)}{\mathrm{d}t} \right| = \left| \frac{\partial f\left(t, y_\epsilon(t)\right)}{\partial t} + \frac{\partial f\left(t, y_\epsilon(t)\right)}{\partial y} y'_\epsilon \right| \le \tilde{C}_\delta, \tag{8}$$

on $[a, b]$, $\tilde{C}_\delta > 0$ is constant. The problem (1), (2) is equivalent to the nonlinear integral equation

$$\begin{aligned} y_\epsilon(t) = & \frac{I}{\Lambda} \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(t-a)} + \frac{I}{\Lambda} \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(a-t)} \\ & + \int_a^t \frac{\mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(t-s)} - \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(s-t)}}{2\sqrt{-\frac{k}{\epsilon}}} \cdot \frac{f\left(s, y_\epsilon(s)\right)}{\epsilon} \mathrm{d}s, \end{aligned} \tag{9}$$

where

$$\begin{aligned} I = & \int_a^c \frac{\mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(c-s)} - \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(s-c)}}{2\sqrt{-\frac{k}{\epsilon}}} \cdot \frac{f\left(s, y_\epsilon(s)\right)}{\epsilon} \mathrm{d}s \\ & - \int_a^b \frac{\mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(b-s)} - \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(s-b)}}{2\sqrt{-\frac{k}{\epsilon}}} \cdot \frac{f\left(s, y_\epsilon(s)\right)}{\epsilon} \mathrm{d}s, \\ \Lambda = & \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(b-a)} + \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(a-b)} - \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(c-a)} - \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(a-c)}. \end{aligned}$$

Differentiating the integral equation (9) with respect to the variable t we obtain

$$\begin{aligned} y'_\epsilon(t) = & \frac{I\sqrt{-\frac{k}{\epsilon}}}{\Lambda} \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(t-a)} - \frac{I\sqrt{-\frac{k}{\epsilon}}}{\Lambda} \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(a-t)} \\ & + \int_a^t \frac{\mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(t-s)} + \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(s-t)}}{2} \cdot \frac{f\left(s, y_\epsilon(s)\right)}{\epsilon} \mathrm{d}s. \end{aligned}$$

Hence

$$\begin{aligned} y'_\epsilon(b) = & \frac{I\sqrt{-\frac{k}{\epsilon}}}{\Lambda} \left(\mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(b-a)} - \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(a-b)} \right) \\ & + \frac{1}{2} \int_a^b \left(\mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(b-s)} + \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(s-b)} \right) \frac{f\left(s, y_\epsilon(s)\right)}{\epsilon} \mathrm{d}s. \end{aligned} \tag{10}$$

Integrating all integrals in (10) by parts and after little algebraic arrangement we obtain

$$y_\epsilon'(b) = \frac{\sqrt{-\frac{k}{\epsilon}}}{k}\Bigg[\left(f\left(c, y_\epsilon(c)\right) - f\left(b, y_\epsilon(b)\right)\right)\sigma_\epsilon$$

$$+\frac{\sigma_\epsilon}{2}\left(\int_a^b \left(\mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(b-s)} + \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(s-b)}\right)\frac{\mathrm{d}f\left(s, y_\epsilon(s)\right)}{\mathrm{d}s}\mathrm{d}s\right.$$

$$\left.-\int_a^c \left(\mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(c-s)} + \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(s-c)}\right)\frac{\mathrm{d}f\left(s, y_\epsilon(s)\right)}{\mathrm{d}s}\mathrm{d}s\right)$$

$$+\frac{1}{2}\int_a^b \left(-\mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(b-s)} + \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(s-b)}\right)\frac{\mathrm{d}f\left(s, y_\epsilon(s)\right)}{\mathrm{d}s}\mathrm{d}s\Bigg]$$

where

$$\sigma_\epsilon = \frac{\mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(b-a)} - \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(a-b)}}{\Lambda} \to 1^+ \text{ for } \epsilon \to 0^+. \tag{11}$$

Taking into consideration (8), the integrals

$$\int_a^b \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(s-b)}\frac{\mathrm{d}f\left(s, y_\epsilon(s)\right)}{\mathrm{d}s}\mathrm{d}s, \qquad \int_a^c \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(s-c)}\frac{\mathrm{d}f\left(s, y_\epsilon(s)\right)}{\mathrm{d}s}\mathrm{d}s$$

are $O\left(\sqrt{\epsilon}\right)$ by the mean value theorem for integrals.

Thus we have

$$y_\epsilon'(b) = \frac{\sqrt{-\frac{k}{\epsilon}}}{k}\Bigg[\left(f\left(c, y_\epsilon(c)\right) - f\left(b, y_\epsilon(b)\right)\right)\sigma_\epsilon$$

$$+\frac{1}{2}(\sigma_\epsilon - 1)\int_a^b \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(b-s)}\frac{\mathrm{d}f\left(s, y_\epsilon(s)\right)}{\mathrm{d}s}\mathrm{d}s$$

$$-\frac{1}{2}\sigma_\epsilon\int_a^c \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(c-s)}\frac{\mathrm{d}f\left(s, y_\epsilon(s)\right)}{\mathrm{d}s}\mathrm{d}s + O\left(\sqrt{\epsilon}\right)\Bigg]. \tag{12}$$

From (11) we can write

$$\sigma_\epsilon - 1 = \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(c-b)}\omega_\epsilon \to 0^+ \text{ for } \epsilon \to 0^+$$

where

$$\omega_\epsilon = \frac{1}{\Lambda}\left(\mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(b-a)} + \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(a+b-2c)} - 2\mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(a-c)}\right) \to 1^+ \text{ for } \epsilon \to 0^+.$$

Thus from (12) we have

$$y'_\epsilon(b) = \frac{1}{\sqrt{-k\epsilon}}\left[\left(f\left(c, y_\epsilon(c)\right) - f\left(b, y_\epsilon(b)\right)\right)\sigma_\epsilon \right.$$

$$+ \frac{1}{2}(\omega_\epsilon - \sigma_\epsilon)\int\limits_a^c \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(c-s)} \frac{\mathrm{d}f\left(s, y_\epsilon(s)\right)}{\mathrm{d}s}\mathrm{d}s$$

$$\left. + \frac{1}{2}\omega_\epsilon \int\limits_c^b \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(c-s)} \frac{\mathrm{d}f\left(s, y_\epsilon(s)\right)}{\mathrm{d}s}\mathrm{d}s + O\left(\sqrt{\epsilon}\right)\right].$$

The integral

$$\int\limits_c^b \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(c-s)} \frac{\mathrm{d}f\left(s, y_\epsilon(s)\right)}{\mathrm{d}s}\mathrm{d}s$$

is $O\left(\sqrt{\epsilon}\right)$ by the analogous argument as above and

$$\int\limits_a^c \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(c-s)} \left|\frac{\mathrm{d}f\left(s, y_\epsilon(s)\right)}{\mathrm{d}s}\right| \mathrm{d}s \le (c-a)\tilde{C}_\delta \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(c-a)}. \tag{13}$$

Using (13), we have

$$\left|\frac{1}{2}(\omega_\epsilon - \sigma_\epsilon)\int\limits_a^c \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(c-s)} \frac{\mathrm{d}f\left(s, y_\epsilon(s)\right)}{\mathrm{d}s}\mathrm{d}s\right|$$

$$\le \frac{1}{2}(\omega_\epsilon - \sigma_\epsilon)(c-a)\tilde{C}_\delta \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(c-a)}$$

$$= \frac{1}{2}(c-a)\tilde{C}_\delta \frac{1}{\Lambda}\left(\mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}\frac{(b-c)}{2}} - \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}\frac{(c-b)}{2}}\right)^2 = O\left(\mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(a-c)}\right).$$

Hence

$$y'_\epsilon(b) = \frac{1}{\sqrt{-k\epsilon}}\left[\left(f\left(c, y_\epsilon(c)\right) - f\left(b, y_\epsilon(b)\right)\right)\sigma_\epsilon + O\left(\sqrt{\epsilon}\right)\right] \tag{14}$$

which gives a contradiction. Combining Lemma 2.2 and (3) we obtain the uniform boundedness of y'_ϵ on every compact set $K \subset [a, b)$ and $\epsilon \in (0, \epsilon_0]$. The proof of Theorem 3.1 is complete. □

Remark 3.2 As we can see from (14) the assumption (H2) is essential for an appearance the boundary layer phenomenon for singularly perturbed system (1), (2) at the point $t = b$.

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