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The center problem and the composition condition for a family of quartic differential systems

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Abstract. In this paper we give coefficient conditions for existence of a center in a family of planar quartic polynomial differential systems. We also show that for all considered center cases the Composition Condition is satisfied.

Keywords: quartic system, center condition, composition condition.

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1 Introduction

We consider the planar analytic differential system

$$\begin{aligned} x' &= -y + p(x,y), \\ y' &= x + q(x,y) \end{aligned} \tag{1.1}$$

with p and q being polynomials without constant and linear terms, and seek for conditions under which the origin is a center (that is, the critical point at the origin is surrounded by closed orbits). The derivation of conditions for a center is a difficult and long-standing problem in the theory of nonlinear differential equations, however due to complexity of the problem necessary and sufficient conditions are known only for a very few families of polynomial systems (1.1). The conditions for a center in the quadratic system have been obtained in [11, 15, 16], and in [18, 20] the problem has been solved for systems in which p and q are cubic polynomials without quadratic terms. The problem is also solved for some families of cubic systems and systems in the form of the linear center perturbed by homogeneous quartic and quintic nonlinearities, see e.g. [1, 3, 5, 10, 12, 13, 17, 23] and references given there.

By the Poincaré–Lyapunov theorem [19, 21] system (1.1) has a center at the origin if and only if it admits a first integral of the form

$$\Phi(x,y) = x^2 + y^2 + \sum_{i+j\geq 3} d_{ij} x^i y^j, \tag{1.2}$$

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where the series converge in a neighbourhood of the origin in $\mathbb{R}^2$.

The problem of distinguishing between a center and a focus for polynomial systems (1.1) has an analog for the corresponding periodic differential equations [17,26]. To see this we note that the phase curves of (1.1) near the origin (0,0) in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ are determined by the equation

$$\frac{dr}{d\theta} = \frac{\cos\theta p(r\cos\theta, r\sin\theta) + \sin\theta q(r\cos\theta, r\sin\theta)}{1 + r^{-1}(\cos\theta q(r\cos\theta, r\sin\theta) - \sin\theta p(r\cos\theta, r\sin\theta))}. \tag{1.3}$$

Therefore, the planar vector field (1.1) has a center at $(0,0)$ if and only if all solutions $r(\theta)$ of equation (1.3) near the solution $r \equiv 0$ are periodic, $r(0) = r(2\pi)$. In such case it is said that equation (1.3) has a center at $r = 0$.

If $r^{-1}(\cos\theta q(r\cos\theta, r\sin\theta) - \sin\theta p(r\cos\theta, r\sin\theta)) = f(\theta) \neq -1$, and $p = \sum_{i+j=2}^{n} p_{ij}x^i y^j$ and $q = \sum_{i+j=2}^{n} q_{ij}x^i y^j$, from (1.3) we get the polynomial equation

$$\frac{dr}{d\theta} = r^2 \sum_{i=0}^{n-2} A_i(\theta) r^i, \tag{1.4}$$

where $A_i(\theta)$ $(i = 0, 1, 2, \ldots, n-2)$ are 2π-periodic functions. Thus, finding the conditions for existence of a center at the origin of system (1.1) is equivalent to finding conditions which fulfilment yields 2π-periodicity of all solutions of polynomial equation (1.4) near $r = 0$ [3].

If p and q are homogeneous polynomials of degree n, then the substitution

$$\rho = \frac{r^{n-1}}{1 + r^{n-1}(\cos\theta q(\cos\theta, \sin\theta) - \sin\theta p(\cos\theta, \sin\theta))}$$

transforms equation (1.3) into the Abel equation

$$\frac{d\rho}{d\theta} = \rho^2 (A_1(\theta) + A_2(\theta)\rho), \tag{1.5}$$

where $A_i(\theta)$ $(i = 1, 2)$ are 2π-periodic functions. Thus, finding the center conditions for (1.1) is equivalent to studying when Abel equation (1.5) has a center at $\rho = 0$. This problem has been investigated in [2,10,26] and some other works.

In this paper, we study the quartic differential system

$$\begin{aligned} x' &= -y + x(P_1(x,y) + P_2(x,y) + P_3(x,y)) = P(x,y), \\ y' &= x + y(P_1(x,y) + P_2(x,y) + P_3(x,y)) = Q(x,y), \end{aligned} \tag{1.6}$$

where $P_n(x,y) = \sum_{i+j=n} p_{ij}x^i y^j$ $(n = 1, 2, 3)$, $p_{i,j}$ $(i, j = 0, 1, 2, 3)$ are real constants. We give the necessary and sufficient conditions for the origin of (1.6) to be a center when $p_{10}^2 + p_{01}^2 = 0$. For the case $p_{10}^2 + p_{01}^2 \neq 0$, we obtain conditions which are sufficient and most probably, are necessary for the origin of system (1.6) to be a center. However, we can not prove their necessity due to computational difficulties arising in the investigation of the zero set of the focus quantities of system (1.6). We apply the obtained results to prove that for all obtained center cases the Composition Condition [2] holds for the corresponding periodic differential equation

$$\frac{dr}{d\theta} = r^2 (P_1(\cos\theta, \sin\theta) + P_2(\cos\theta, \sin\theta) r + P_3(\cos\theta, \sin\theta) r^2). \tag{1.7}$$

2 Center conditions for system (1.6)

Alwash and Lloyd [2,3] proved the following statement.

Lemma 2.1 ([2,3]). *If there exists a differentiable function u of period 2π such that*

$$A_1(\theta) = u'(\theta)\hat{A}_1(u(\theta)), \qquad A_2(\theta) = u'(\theta)\hat{A}_2(u(\theta))$$

for some continuous functions $\hat{A}_1$ and $\hat{A}_2$, then the Abel differential equation

$$\frac{dr}{d\theta} = A_1(\theta)r^2 + A_2(\theta)r^3$$

has a center at $r \equiv 0$.

The condition in Lemma 2.1 is called the *Composition Condition*. This is a sufficient but not a necessary condition for $r = 0$ to be a center [1].

The following statement presents a generalization of Lemma 2.1.

Lemma 2.2. *If there exists a differentiable function u of period 2π such that*

$$A_i(\theta) = u'\hat{A}_i(u), \qquad (i = 1, 2, \dots, n) \tag{2.1}$$

for some continuous functions $\hat{A}_i$ $(i = 1, 2, \dots, n)$, then the differential equation

$$\frac{dr}{d\theta} = r\sum_{i=1}^{n} A_i(\theta)r^i \tag{2.2}$$

has a center at $r = 0$.

Proof. For simplicity of notations we present the proof for the case $n = 3$. The proof for the general case goes similarly.

Let $r(\theta, c)$ be the solution of (2.2) such that $r(0, c) = c$ $(0 < c \ll 1)$. We write

$$r(\theta, c) = \sum_{n=1}^{\infty} a_n(\theta)c^n,$$

where $a_1(0) = 1$ and $a_n(0) = 0$ for $n > 1$. The origin is a center if and only if $r(\theta + 2\pi, c) = r(\theta, c)$, i.e., $a_1(2\pi) = 1$, $a_n(2\pi) = 0$ $(n = 2, 3, 4, \dots)$ [2,3].

Substituting $r(\theta, c)$ into (2.2) we obtain

$$\sum_{n=1}^{\infty} a_n'(\theta)c^n = A_1(\theta)\left(\sum_{n=1}^{\infty} a_n(\theta)c^n\right)^2 + A_2(\theta)\left(\sum_{n=1}^{\infty} a_n(\theta)c^n\right)^3 + A_3(\theta)\left(\sum_{n=1}^{\infty} a_n(\theta)c^n\right)^4.$$

Equating the corresponding coefficients of c^n yields

$$\begin{aligned}
&a_1' = 0, && a_1(0) = 1;\\
&a_2' = a_1^2 A_1, && a_2(0) = 0;\\
&a_3' = 2a_1a_2A_1 + a_1^3A_2, && a_3(0) = 0;\\
&a_n' = A_1(\theta)\sum_{i+j=n} a_ia_j + A_2(\theta)\sum_{i+j+k=n} a_ia_ja_k + A_3(\theta)\sum_{i+j+k+l=n} a_ia_ja_ka_l, && a_n(0) = 0,\\
&\qquad (n = 4, 5, 6, \dots).
\end{aligned} \tag{2.3}$$

Solving the first equation of (2.3), we get $a_1(\theta) = 1$. Substituting it into the second equation and solving it we obtain

$$a_2(\theta) = \int_0^\theta A_1(\theta)d\theta = \int_0^\theta u'(\theta)\hat{A}_1(u)d\theta = \int_0^\theta \hat{A}_1(u)du = \hat{a}_2(u),\ a_2(2\pi) = 0.$$

Substituting $a_1 = 1$, $a_2(\theta) = \hat{a}_2(u)$ into the third equation of (2.3) and integrating it we have

$$a_3(\theta) = \int_0^\theta u'(\theta)(2\hat{a}_2(u)\hat{A}_1(u) + \hat{A}_2(u))d\theta = \int_0^\theta (2\hat{a}_2(u)\hat{A}_1(u) + \hat{A}_2(u))du = \hat{a}_3(u),$$
$$a_3(2\pi) = 0.$$

Suppose that when $n = m - 1$, the function $a_n(\theta) = \hat{a}_n(u)$ is a 2π-periodic function and $a_n(2\pi) = 0$. We prove that when $n = m$ the function $a_n(\theta) = \hat{a}_n(u)$ is a 2π-periodic function and $a_n(2\pi) = 0$.

Indeed, substituting $a_1(\theta) = 1$, $a_n(\theta) = \hat{a}_n(u)$ $(n = 1, 2, \ldots, m-1)$ into (2.3) and integrating it we get

$$\begin{aligned} a_m(\theta) &= \int_0^\theta u'(\theta)(\hat{A}_1(u) \sum_{i+j=m} \hat{a}_i(u)\hat{a}_j(u) + \hat{A}_2(u) \sum_{i+j+k=m} \hat{a}_i(u)\hat{a}_j(u)\hat{a}_k(u) \\ &\quad + \hat{A}_3(u) \sum_{i+j+k+l=m} \hat{a}_i(u)\hat{a}_j(u)\hat{a}_k(u)\hat{a}_l(u))d\theta \\ &= \int_0^\theta (\hat{A}_1(u) \sum_{i+j=m} \hat{a}_i(u)\hat{a}_j(u) + \hat{A}_2(u) \sum_{i+j+k=m} \hat{a}_i(u)\hat{a}_j(u)\hat{a}_k(u) \\ &\quad + \hat{A}_3(u) \sum_{i+j+k+l=m} \hat{a}_i(u)\hat{a}_j(u)\hat{a}_k(u)\hat{a}_l(u))du = \hat{a}_n(u). \end{aligned}$$

Since $u(\theta)$ is a 2π-periodic function, $\hat{a}_m(u)$ is a 2π-periodic function and $a_m(2\pi) = 0$. By mathematical induction, the functions $a_n(\theta)$ $(n = 2, 3, 4\ldots)$ are 2π- periodic and $a_n(2\pi) = 0$ $(n = 2, 3, 4, \ldots)$. Thus, the $r = 0$ is a center of (2.2). □

We now consider the center problem for quartic system (1.6).

Theorem 2.3. *If $p_{10}^2 + p_{01}^2 \neq 0$, then the origin is a center for (1.6) if the following condition is satisfied*

$$\begin{aligned} p_{20} + p_{02} &= 0, \\ p_{20}(p_{01}^2 - p_{10}^2) - p_{01}p_{10}p_{11} &= 0, \\ 3(p_{03}p_{10} - p_{30}p_{01}) + p_{10}p_{21} - p_{01}p_{12} &= 0, \\ p_{30}p_{01}^3 - p_{03}p_{10}^3 + p_{10}p_{01}(p_{12}p_{10} - p_{21}p_{01}) &= 0. \end{aligned} \tag{2.4}$$

Proof. In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ system (1.6) has the form

$$\frac{dr}{dt} = r^2(P_1 + P_2 r + P_3 r^2), \qquad \frac{d\theta}{dt} = 1,$$

where

$$P_k = P_k(\cos\theta, \sin\theta) = \sum_{i+j=k} p_{ij}\cos^i\theta\sin^j\theta, \qquad (k = 1, 2, 3).$$

From this system we get equation (1.7). The origin is a center for system (1.6) if and only if $r = 0$ is a center for equation (1.7) [1, 17].

Following [7] to compute the necessary conditions for existence of a first integral of the form (1.2) for system (1.6) we look for a function

$$\Phi_{2m+1}(x,y) = x^2 + y^2 + \sum_{i+j=3}^{2m+1} d_{ij} x^i y^j \tag{2.5}$$

such that

$$\frac{\partial \Phi_{2m+1}}{\partial x} P + \frac{\partial \Phi_{2m+1}}{\partial x} Q = g_1(p) x^4 + g_2(p) x^6 + \dots, \tag{2.6}$$

where p stands for the 9-tuple of the parameters of system (1.6), that is,

$$(p) = (p_{10}, p_{01}, p_{20}, p_{11}, p_{02}, p_{30}, p_{21}, p_{12}, p_{03}).$$

Equating the coefficients of the same monomials on both sides of (2.6) we compute the coefficients $d_{ij}(p)$ in (2.5) and the polynomials $g_1(p)$, $g_2(p), \dots$ in (2.6). We call $g_i(p)$ $(i = 1, 2, 3, \dots)$ the focus quantities of system (1.6). Using the computer algebra system Mathematica we have computed for system (1.6) the first 8 focus quantities, where the first three of them are:

$$\begin{aligned}
g_1 &= p_{20} + p_{02};\\
g_2 &= 169 p_{01}^2 p_{02} - 18 p_{03} p_{10} + 11 p_{02} p_{10}^2 + 4 p_{02} p_{11} + 6 p_{01} p_{10} p_{11} + 6 p_{01} p_{12}\\
&\quad + 163 p_{01}^2 p_{20} + 17 p_{10}^2 p_{20} + 4 p_{11} p_{20} - 6 p_{10} p_{21} + 18 p_{01} p_{30};\\
g_3 &= 88383 p_{01}^4 p_{02} - 1230 p_{02}^3 - 8944 p_{01} p_{02} p_{03} - 6688 p_{01} p_{02}^2 p_{10} - 10098 p_{01}^2 p_{03} p_{10}\\
&\quad + 8118 p_{01}^2 p_{02} p_{10}^2 - 414 p_{03} p_{10}^3 + 243 p_{02} p_{10}^4 + 6497 p_{01}^2 p_{02} p_{11} + 3366 p_{01}^3 p_{10} p_{11}\\
&\quad + 198 p_{03} p_{10} p_{11} - 237 p_{02} p_{10}^2 p_{11} + 234 p_{01} p_{10}^3 p_{11} - 18 p_{02} p_{11}^2 - 18 p_{01} p_{10} p_{11}^2 + 3366 p_{01}^3 p_{12}\\
&\quad - 336 p_{02} p_{10} p_{12} - 54 p_{01} p_{10}^2 p_{12} - 18 p_{01} p_{11} p_{12} + 85017 p_{01}^4 p_{20} - 2274 p_{02}^2 p_{20} - 9088 p_{01} p_{03} p_{20}\\
&\quad - 4128 p_{01} p_{02} p_{10} p_{20} + 11250 p_{01}^2 p_{10}^2 p_{20} + 477 p_{10}^4 p_{20} + 6515 p_{01}^2 p_{11} p_{20} - 255 p_{10}^2 p_{11} p_{20}\\
&\quad - 18 p_{11}^2 p_{20} - 192 p_{10} p_{12} p_{20} - 1218 p_{02} p_{20}^2 + 2560 p_{01} p_{10} p_{20}^2 - 174 p_{20}^3 - 3872 p_{01} p_{02} p_{21}\\
&\quad - 3078 p_{01}^2 p_{10} p_{21} - 234 p_{10}^3 p_{21} + 18 p_{10} p_{11} p_{21} - 4016 p_{01} p_{20} p_{21} + 9810 p_{01}^3 p_{30}\\
&\quad - 1248 p_{02} p_{10} p_{30} + 702 p_{01} p_{10}^2 p_{30} + 90 p_{01} p_{11} p_{30} - 1104 p_{10} p_{20} p_{30} = 0;
\end{aligned}$$

The size of the polynomial g_i $(i = 4, 5, 6, 7, 8)$ grows exponentially, so we do not present them here, but the interested reader can compute the quantities using any available computer algebra system.

The system of algebraic equations

$$g_1 = g_2 = \dots = g_8 = 0, \tag{2.7}$$

gives us the necessary conditions for the origin of (1.6) to be a center. To find the conditions we have to find the irreducible decomposition of the variety of the ideal I generated by the focus quantities,

$$I = \langle g_1, g_2, \dots, g_8 \rangle. \tag{2.8}$$

Since the decomposition of the variety $\mathbf{V}(I)$ of the ideal I is not possible over the field of rational numbers due to the complexity of calculations, we employ the computational approach based on modular calculations described in [22].

We first compute the minimal associate primes of the ideal I over the field $\mathbb{Z}_{32003}$ using the routine `minAssGTZ` [9] of the computer algebra system Singular which is based on the algorithm of [13]. Computations yield that the minimal associate primes of I are the ideals

$I_1,\ I_2,\ I_3$ in the ring $\mathbb{Z}_{32003}[p_{10}, p_{01}, p_{20}, p_{11}, p_{02}, p_{30}, p_{21}, p_{12}, p_{03}]$ given in the appendix. Then, the rational reconstruction algorithm of [25] applied to $I_1,\ I_2,\ I_3$ gives the polynomials defined by the following conditions, respectively.

Condition 1:

$$p_{01}^2 - \frac{1}{4}p_{11} = 0; \tag{2.9}$$

$$p_{10}^2 + \frac{1}{4}p_{11} = 0; \tag{2.10}$$

$$p_{02} + p_{20} = 0; \qquad p_{12} + 3p_{30} = 0; \qquad p_{03} + \frac{1}{3}p_{21} = 0;$$

$$2p_{01}p_{02} + p_{10}p_{11} = 0; \qquad -2p_{02}p_{10} + p_{01}p_{11} = 0; \qquad \frac{1}{2}p_{02} + p_{01}p_{10} = 0;$$

$$p_{02}^2 + \frac{1}{15}p_{01}p_{21} - \frac{1}{5}p_{10}p_{30} = 0; \qquad p_{02}p_{11} + \frac{2}{15}p_{10}p_{21} + \frac{2}{5}p_{01}p_{30} = 0;$$

$$p_{11}^2 - \frac{4}{15}p_{01}p_{21} + \frac{4}{5}p_{10}p_{30} = 0; \qquad p_{02}p_{10}p_{21} - \frac{1}{30}p_{21}^2 - 3p_{01}p_{02}p_{30} - \frac{3}{10}p_{30}^2 = 0.$$

Condition 2:

$$p_{02} = p_{20} = p_{11} = p_{10} = p_{01} = 0.$$

Condition 3:

$$p_{02} + p_{20} = 0; \tag{2.11}$$

$$p_{01}^2 p_{02} - p_{02}p_{10}^2 + p_{01}p_{10}p_{11} = 0; \tag{2.12}$$

$$p_{03}p_{10} - \frac{1}{3}p_{01}p_{12} + \frac{1}{3}p_{10}p_{21} - p_{01}p_{30} = 0; \tag{2.13}$$

$$p_{01}p_{10}^2 p_{12} - \frac{3}{2}p_{01}^2 p_{10}p_{21} + \frac{1}{2}p_{10}^3 p_{21} + \frac{3}{2}p_{01}^3 p_{30} - \frac{3}{2}p_{01}p_{10}^2 p_{30} = 0; \tag{2.14}$$

$$p_{02}p_{12}^2 - 3p_{02}p_{03}p_{21} + \frac{1}{2}p_{11}p_{12}p_{21} - p_{02}p_{21}^2 - \frac{9}{2}p_{03}p_{11}p_{30} + 3p_{02}p_{12}p_{30} = 0;$$

$$p_{02}p_{10}p_{12} - p_{01}p_{02}p_{21} + \frac{1}{2}p_{10}p_{11}p_{21} - \frac{3}{2}p_{01}p_{11}p_{30} = 0;$$

$$\begin{aligned} &3p_{02}p_{03}^2 + p_{03}p_{11}p_{12} + p_{02}p_{03}p_{21} + \frac{1}{2}p_{11}p_{12}p_{21} \\ &\quad + \frac{3}{2}p_{03}p_{11}p_{30} - p_{02}p_{12}p_{30} + p_{11}p_{21}p_{30} - 3p_{02}p_{30}^2 = 0; \end{aligned}$$

$$3p_{01}p_{02}p_{03} + p_{01}p_{11}p_{12} + \frac{1}{2}p_{10}p_{11}p_{21} - 3p_{02}p_{10}p_{30} + \frac{3}{2}p_{01}p_{11}p_{30} = 0;$$

$$\begin{aligned} &p_{02}^2 p_{03}p_{12} + \frac{3}{2}p_{02}p_{03}p_{11}p_{21} + \frac{1}{2}p_{02}p_{11}p_{21}^2 + \frac{3}{2}p_{03}p_{11}^2 p_{30} \\ &\quad - \frac{1}{2}p_{02}p_{11}p_{12}p_{30} - p_{02}^2 p_{21}p_{30} + \frac{1}{2}p_{11}^2 p_{21}p_{30} - \frac{3}{2}p_{02}p_{11}p_{30}^2 = 0; \end{aligned}$$

$$p_{01}p_{02}^2p_{12} - p_{02}^2p_{10}p_{21} + \frac{3}{2}p_{01}p_{02}p_{11}p_{21} - \frac{3}{2}p_{02}p_{10}p_{11}p_{30} + \frac{3}{2}p_{01}p_{11}^2p_{30} = 0,$$

$$p_{03}p_{12}^3 - \frac{9}{2}p_{03}^2p_{12}p_{21} + \frac{1}{2}p_{12}^3p_{21} - 3p_{03}p_{12}p_{21}^2 - \frac{1}{2}p_{12}p_{21}^3 + \frac{27}{2}p_{03}^3p_{30}$$
$$+ \frac{9}{2}p_{03}p_{12}^2p_{30} + 3p_{12}^2p_{21}p_{30} - \frac{9}{2}p_{03}p_{21}^2p_{30} - p_{21}^3p_{30} + \frac{9}{2}p_{12}p_{21}p_{30}^2 - \frac{27}{2}p_{03}p_{30}^3 = 0;$$

$$p_{01}p_{12}^3 - \frac{9}{2}p_{01}p_{03}p_{12}p_{21} + \frac{1}{2}p_{10}p_{12}^2p_{21} - \frac{3}{2}p_{01}p_{12}p_{21}^2 + \frac{27}{2}p_{01}p_{03}^2p_{30}$$
$$+ \frac{9}{2}p_{01}p_{12}^2p_{30} - \frac{9}{2}p_{01}p_{03}p_{21}p_{30} + 3p_{10}p_{12}p_{21}p_{30} - 3p_{01}p_{21}^2p_{30} + \frac{9}{2}p_{10}p_{21}p_{30}^2 - \frac{27}{2}p_{01}p_{30}^3 = 0;$$

$$p_{01}p_{10}p_{12}^2 - \frac{3}{2}p_{01}^2p_{12}p_{21} + \frac{1}{2}p_{10}^2p_{12}p_{21} + \frac{9}{2}p_{01}^2p_{03}p_{30} + \frac{3}{2}p_{01}p_{10}p_{12}p_{30}$$
$$-3p_{01}^2p_{21}p_{30} + \frac{3}{2}p_{10}^2p_{21}p_{30} - \frac{9}{2}p_{01}p_{10}p_{30}^2 = 0;$$

$$2p_{02}^3p_{03}p_{21} + p_{02}^2p_{11}p_{12}p_{21} + \frac{1}{2}p_{02}p_{11}^2p_{21}^2 + 3p_{02}^2p_{03}p_{11}p_{30}$$
$$- 2p_{02}^3p_{12}p_{30} + p_{02}p_{11}^2p_{12}p_{30} - 2p_{02}^2p_{11}p_{21}p_{30} + \frac{1}{2}p_{11}^3p_{21}p_{30} - \frac{3}{2}p_{02}p_{11}^2p_{30}^2 = 0;$$

$$p_{02}^3p_{03}^2 - \frac{1}{2}p_{02}p_{03}p_{11}^2p_{21} - \frac{1}{4}p_{02}p_{11}^2p_{21}^2 - \frac{1}{2}p_{03}p_{11}^3p_{30}$$
$$+ p_{02}^2p_{11}p_{21}p_{30} - \frac{1}{4}p_{11}^3p_{21}p_{30} - p_{02}^3p_{30}^2 + \frac{3}{4}p_{02}p_{11}^2p_{30}^2 = 0;$$

$$p_{01}p_{02}^3p_{03} + \frac{1}{2}p_{02}^2p_{10}p_{11}p_{21} - \frac{1}{2}p_{01}p_{02}p_{11}^2p_{21}$$
$$- p_{02}^3p_{10}p_{30} + \frac{1}{2}p_{01}p_{02}^2p_{11}p_{30} + \frac{1}{2}p_{02}p_{10}p_{11}^2p_{30} - \frac{1}{2}p_{01}p_{11}^3p_{30} = 0.$$

Let J_1, J_2 and J_3 be the ideals defined by polynomials on the left hand sides of Conditions 1, 2, 3, respectively. Following [22] in order to check the correctness of obtained Conditions 1, 2 and 3 we compute the ideal

$$J = J_1 \cap J_2 \cap J_3, \tag{2.15}$$

which defines the union of the varieties $\mathbf{V}(J_1), \mathbf{V}(J_2)$ and $\mathbf{V}(J_3)$. Then we should check that $\mathbf{V}(J) = \mathbf{V}(I)$. According to the Radical Membership Test (see e.g. [23]), to verify the inclusion

$$\mathbf{V}(J) \supset \mathbf{V}(I) \tag{2.16}$$

it is sufficient to check that the Gröbner bases of all ideals $\langle J, 1 - wg_k \rangle$ (where $k = 1, \dots, 8$ and w is a new variable) computed over $\mathbb{Q}$ are $\{1\}$. The computations show that this is the case.

To check the opposite inclusion,

$$\mathbf{V}(J) \subset \mathbf{V}(I), \tag{2.17}$$

it is sufficient to check that all Gröbner bases of the ideals $\langle I, 1 - wt_i \rangle$ (where the polynomials t_i form a basis of J) computed over $\mathbb{Q}$ are equal to $\{1\}$. Unfortunately, we were not able to perform these computations over $\mathbb{Q}$ using our computational facilities, however we have checked that all the bases are $\{1\}$ over few fields of finite characteristic. It yields that the list of Conditions 1, 2 and 3 give the complete decomposition of the variety $\mathbf{V}(I)$ of the ideal I in the affine space $\mathbb{Q}^9$ with high probability [6] (since (2.16) holds, the obtained Conditions 1, 2 and 3 define correct components of $\mathbf{V}(I)$, but if (2.17) does not holds then the variety can have additional components – in other words, Conditions 1, 2 and 3 are some necessary conditions for a center, but we cannot prove that they represent the *complete list* of necessary center conditions).

We now examine Conditions 1, 2 and 3 more carefully. From equations (2.9) and (2.10) it follows that

$$p_{11} = \frac{1}{4} p_{10}^2 = -\frac{1}{4} p_{01}^2,$$

which implies that $p_{10} = p_{01} = 0$, i.e., $P_1 = 0$, which contradicts the hypothesis of the present theorem. Clearly, Condition 2 also implies $P_1 = 0$ contradicting the hypothesis of the theorem as well. Thus, we have to consider only Condition 3.

Let f_1, f_2, f_3, f_4 be polynomials on the left hand sides of (2.11), (2.12), (2.13) and (2.14), respectively. It is easy to check that the condition

$$f_1 = f_2 = f_3 = f_4 = 0, \qquad p_{10}^2 + p_{02}^2 \neq 0 \tag{2.18}$$

yields condition (2.4). To verify this we consider the ideal

$$Q = \langle 1 - w(p_{10}^2 + p_{01}^2), f_1, f_2, f_3, f_4 \rangle,$$

where w is a new variable. Computing with the routine `eliminate` (which is based on the Elimination Theorem, see e.g. [23]) of Singular we find that the first elimination ideal of Q is the same as the ideal generated by the polynomials on the right hand side of (2.4). This means, that (2.18) yields condition (2.4).

In the following, using Lemma 2.2 we will prove that if condition (2.4) is fulfilled then the origin is a center for (1.6).

Case 1. If $p_{10}p_{01} \neq 0$ by condition (2.4) we get

$$P_2 = -\frac{p_{20}}{p_{01}p_{10}} P_1 \bar{P}_1, \qquad \bar{P}_1 = p_{10}\sin\theta - p_{01}\cos\theta, \qquad P_1 = \frac{d\bar{P}_1}{d\theta},$$

and

$$P_3 = (k_0 + k_1 \bar{P}_1^2) P_1,$$

where

$$k_0 = \frac{1}{4p_{10}p_{01}}(p_{30}p_{01} + p_{03}p_{10} + p_{21}p_{10} + p_{12}p_{01}), \qquad k_1 = \frac{1}{2p_{01}p_{10}^3}(p_{01}p_{30} - p_{10}p_{21}).$$

By Lemma 2.2, $r = 0$ is a center for equation (1.7), i.e., the origin is a center for system (1.6).

Case 2. If $p_{10} = 0$, $p_{01} \neq 0$, then $P_1 = p_{01}\sin\theta$, $\bar{P}_1 = -p_{01}\cos\theta$. By condition (2.4) we obtain $p_{20} = 0$, $p_{12} = p_{30} = 0$, so

$$P_2 = -\frac{p_{11}}{p_{01}^2} P_1 \bar{P}_1,$$

and

$$P_3 = P_1(k_0 + k_1\bar{P}_1^2), \qquad k_0 = \frac{p_{03}}{p_{01}}, \quad k_1 = \frac{p_{21} - p_{03}}{p_{01}^3}.$$

By Lemma 2.2, the origin of system (1.6) is a center.

Case 3. If $p_{01} = 0$, $p_{10} \neq 0$, then $P_1 = p_{10}\cos\theta$, $\bar{P}_1 = p_{10}\sin\theta$. Using condition (2.4) we get $p_{20} = 0$, $p_{21} = p_{03} = 0$, so

$$P_2 = \frac{p_{11}}{p_{10}^2} P_1\bar{P}_1,$$
$$P_3 = P_1(k_0 + k_1\bar{P}_1^2), \qquad k_0 = \frac{p_{30}}{p_{10}}, \quad k_1 = \frac{p_{12} - p_{30}}{p_{10}^3}.$$

By Lemma 2.2, the origin of system (1.6) is a center.

In summary, condition (2.4) is sufficient for the origin to be a center of (1.6). The proof of the present theorem is finished. □

Conjecture 2.4. *If $p_{10}^2 + p_{01}^2 \neq 0$, the origin is a center for (1.6) if and only if condition (2.4) is satisfied.*

As it follows from the proof of Theorem 2.3, Conjecture 2.4 is true if inclusion (2.17) holds. As it is mentioned in the proof of the theorem to check the inclusion it is sufficient to check that Gröbner bases of the ideals $\langle I, 1 - wt_i\rangle$ (where t_i are the polynomials defining a basis of J, and J and I are ideals defined by (2.15) and (2.8), respectively) computed over $\mathbb{Q}$ are equal to $\{1\}$. We were not able to complete our computations of the Gröbner bases over the field $\mathbb{Q}$, however we have checked that all bases are $\{1\}$ computing over a few finite fields. This indicates that Conjecture 2.4 should be true [6].

Corollary 2.5. *If $p_{10}^2 + p_{01}^2 \neq 0$, then $r = 0$ is a center of equation (1.7), if and only if*

$$P_2 = kP_1\bar{P}_1, \qquad P_3 = P_1(k_0 + k_1\bar{P}_1^2), \tag{2.19}$$

where k, k_0, k_1 are constants, $P_1 = p_{10}\cos\theta + p_{01}\sin\theta$, $\bar{P}_1 = p_{10}\sin\theta - p_{01}\cos\theta$.

Proof. Under the condition of the corollary if the origin is a center for system (1.6), then condition (2.4) is satisfied. According to the proof of Theorem 2.3, we know that then the relation (2.19) holds, this means that condition (2.19) is necessary for the origin to be a center of (1.6). On the other hand, by Lemma 2.2, condition (2.19) is sufficient for $r = 0$ of (1.7) (i.e., the origin of (1.6)) to be a center. □

If $P_3 = 0$ then simple computations show that (2.19) holds, so, from Theorem 2.3 and its proof we have the following result.

Corollary 2.6. *The origin is a center of system*

$$x' = -y + x(P_1(x,y) + P_2(x,y)),$$
$$y' = x + y(P_1(x,y) + P_2(x,y)),$$

(where $P_n(x,y) = \sum_{i+j=n} p_{ij}x^iy^j$ $(n = 1,2)$, $p_{i,j}$ $(i,j = 0,1,2)$ are real constants), if and only if

$$p_{20} + p_{02} = 0;$$
$$p_{20}(p_{01}^2 - p_{10}^2) - p_{01}p_{10}p_{11} = 0.$$

This result is the same as Alwash's theorem in [1].

Theorem 2.7. *If* $p_{10}^2 + p_{01}^2 = 0$, $p_{20}^2 + p_{11}^2 + p_{02}^2 \neq 0$, *the origin is a center for system* (1.6), *if and only if the following condition is satisfied*

$$\begin{aligned} p_{20} + p_{02} &= 0; \\ 2p_{20}(p_{12}^2 - p_{21}^2 + 3p_{12}p_{30} - 3p_{21}p_{03}) + p_{11}(9p_{30}p_{03} - p_{12}p_{21}) &= 0; \\ 2p_{20}(3p_{03}^2 - 3p_{30}^2 + p_{03}p_{21} - p_{12}p_{30}) - p_{11}(2p_{03}p_{12} + 2p_{30}p_{21} + p_{12}p_{21} + 3p_{03}p_{30}) &= 0; \\ 2p_{20}^2(p_{12}p_{03} - p_{21}p_{30}) - p_{20}p_{11}(3p_{03}p_{21} - p_{12}p_{30} + p_{21}^2 - 3p_{30}^2) & \\ + p_{11}^2(3p_{30}p_{03} + p_{21}p_{30}) &= 0; \\ 4p_{20}^3(p_{30}^2 - p_{03}^2) + 4p_{20}^2p_{11}p_{21}p_{30} + p_{20}p_{11}^2(p_{21}^2 + 2p_{03}p_{21} - 3p_{30}^2) & \\ - p_{11}^3(2p_{30}p_{03} + p_{21}p_{30}) &= 0. \end{aligned} \tag{2.20}$$

Proof. Since $p_{10}^2 + p_{01}^2 = 0$, that is, $p_{10} = p_{01} = 0$, system (1.6) becomes

$$\begin{aligned} x' &= -y + x(P_2(x,y) + P_3(x,y)), \\ y' &= x + y(P_2(x,y) + P_3(x,y)), \end{aligned} \tag{2.21}$$

where $P_2(x,y) = p_{20}x^2 + p_{11}xy + p_{02}y^2$, $P_3(x,y) = p_{30}x^3 + p_{21}x^2y + p_{12}xy^2 + p_{03}y^3$, p_{ij} $(i,j = 0,1,2,3)$ are real parameters.

In this case we can easily verify that inclusion (2.17) holds, it means, that Conditions 1, 2 and 3 in the proof of Theorem 2.3, give the complete irreducible decomposition of the variety $\mathbf{V}(I)$ of ideal (2.8). Thus, since Condition 3 is the necessary condition for the origin of (2.21) to be a center, the following conditions hold

$$p_{20} + p_{02} = 0; \tag{2.22}$$

$$2p_{20}(p_{12}^2 - p_{21}^2 + 3p_{12}p_{30} - 3p_{21}p_{03}) + p_{11}(9p_{30}p_{03} - p_{12}p_{21}) = 0; \tag{2.23}$$

$$2p_{20}(3p_{03}^2 - 3p_{30}^2 + p_{03}p_{21} - p_{12}p_{30}) - p_{11}(2p_{03}p_{12} + 2p_{30}p_{21} + p_{12}p_{21} + 3p_{03}p_{30}) = 0; \tag{2.24}$$

$$\begin{aligned} &2p_{20}^2(p_{12}p_{03} - p_{21}p_{30}) - p_{20}p_{11}(3p_{03}p_{21} - p_{12}p_{30} + p_{21}^2 - 3p_{30}^2) \\ &\qquad + p_{11}^2(3p_{30}p_{03} + p_{21}p_{30}) = 0; \end{aligned} \tag{2.25}$$

$$\begin{aligned} &4p_{20}^3(p_{30}^2 - p_{03}^2) + 4p_{20}^2p_{11}p_{21}p_{30} + p_{20}p_{11}^2(p_{21}^2 + 2p_{03}p_{21} - 3p_{30}^2) \\ &\qquad - p_{11}^3(2p_{30}p_{03} + p_{21}p_{30}) = 0; \end{aligned} \tag{2.26}$$

$$\begin{aligned} &2(p_{03}p_{12}^3 - p_{30}p_{21}^3) + (p_{12}p_{21} + 9p_{30}p_{03})(p_{12}^2 - p_{21}^2) \\ &\qquad + 6p_{12}p_{21}(p_{12}p_{30} - p_{21}p_{03}) + 9(p_{30}^2 - p_{03}^2)(p_{12}p_{21} - 3p_{30}p_{03}) = 0; \end{aligned} \tag{2.27}$$

$$\begin{aligned} &4p_{20}^3(p_{12}p_{30} - p_{21}p_{03}) + 2p_{20}^2p_{11}(p_{12}p_{21} + 3p_{30}p_{03} - 2p_{21}p_{30}) \\ &\qquad + p_{20}p_{11}^2(3p_{30}^2 - p_{21}^2 - 2p_{12}p_{30}) + p_{11}^3p_{21}p_{30} = 0. \end{aligned} \tag{2.28}$$

From (2.23) and (2.24) it follows that relation (2.27) is the identity. Using (2.24), (2.25) and (2.26) we derive that relation (2.28) holds. It is not difficult to see that relations (2.22)–(2.26) are the same as condition (2.20). Thus, condition (2.20) is the necessary condition for origin to be a center of system (2.21).

Another way to verify that condition (2.20) of the theorem is the necessary condition for existence of a center is as follows.

Let Q be the ideal generated by the polynomials given in condition (2.20). With `eliminate` of Singular we have computed the first elimination ideals $\widetilde{Q}_1$ and $\widetilde{J}_1$ of the ideals

$$\widetilde{Q} = \langle 1 - w(p_{20}^2 + p_{11}^2 + p_{02}^2), Q\rangle \quad \text{and} \quad \widetilde{J} = \langle 1 - w(p_{20}^2 + p_{11}^2 + p_{02}^2), J\rangle$$

in the ring $\mathbb{R}\,[w, p_{20}, p_{11}, p_{02}, p_{30}, p_{21}, p_{12}, p_{03}]$ (where J is defined by (2.15)) and with `reduce` of Singular verified that $\widetilde{Q}_1 = \widetilde{J}_1$. That means that the condition of the present theorem is the necessary condition for existence of a center at the origin of system (2.21).

We now prove that this condition is also the sufficient center condition. The proof is split into five cases.

Case 1. We prove that if $p_{11} = 0$, $p_{20} \neq 0$, then under the condition of the present theorem the origin of (2.21) is a center.

Since $p_{11} = 0$, $p_{20} \neq 0$, $P_2 = p_{20}(\cos^2\theta - \sin^2\theta)$ and relations of (2.20) are equivalent to the following equations

$$p_{20} + p_{02} = 0; \tag{2.29}$$

$$p_{12}^2 - p_{21}^2 + 3(p_{12}p_{30} - p_{21}p_{03}) = 0; \tag{2.30}$$

$$3(p_{03}^2 - p_{30}^2) + p_{03}p_{21} - p_{30}p_{12} = 0; \tag{2.31}$$

$$p_{12}p_{03} - p_{21}p_{30} = 0; \tag{2.32}$$

$$p_{30}^2 - p_{03}^2 = 0. \tag{2.33}$$

From (2.30)–(2.33) we get $p_{30} = \pm p_{03}$, $p_{21} = \pm p_{12}$.

1^0. If $p_{30} = p_{03}$, $p_{12} = p_{21}$, then $P_3 = \frac{1}{2}u'(p_{30} + p_{21} + (p_{30} - p_{21})u^2))$, $P_2 = p_{20}u'u$, $u = \sin\theta - \cos\theta$.

2^0. If $p_{30} = p_{03}$, $p_{12} = -p_{21}$, by (2.32) we have $p_{30}p_{21} = 0$.
If $p_{30} = 0$, then $P_3 = \frac{p_{21}}{2}u'(u^2 - 1)$, $P_2 = p_{20}u'u$, $u = \sin\theta + \cos\theta$.
If $p_{21} = 0$, then $P_3 = \frac{p_{30}}{2}u'(1 + u^2)$, $P_2 = p_{20}u'u$, $u = \sin\theta - \cos\theta$.

3^0. If $p_{30} = -p_{03}$, $p_{12} = -p_{21}$, then $P_3 = \frac{1}{2}u'(p_{30} - p_{21} + (p_{21} + p_{30})u^2)$, $P_2 = p_{20}u'u$, $u = \sin\theta + \cos\theta$.

4^0. If $p_{30} = -p_{03}$, $p_{12} = p_{21}$, by (2.32) we get $p_{30}p_{21} = 0$.
If $p_{30} = 0$, then $P_3 = \frac{p_{21}}{2}u'(1 - u^2)$, $P_2 = p_{20}u'u$, $u = \sin\theta - \cos\theta$.
If $p_{21} = 0$, then $P_3 = \frac{p_{30}}{2}u'(1 + u^2)$, $P_2 = p_{20}u'u$, $u = \sin\theta + \cos\theta$.

By Lemma 2.2 the origin of (2.21) is a center.

Case 2. We now show that if $p_{20} = 0$, $p_{11} \neq 0$, under the condition of the present theorem the origin of (2.21) is a center.

Since $p_{20} = 0$, $p_{11} \neq 0$, we have that $P_2 = p_{11}\cos\theta\sin\theta$ and condition (2.20) is equivalent to the following relations

$$p_{30}p_{03} = p_{12}p_{21} = p_{21}p_{30} = p_{03}p_{12} = 0.$$

1^0. If $p_{21} \neq 0$, then $p_{12} = p_{30} = 0$, $P_3 = -u'(p_{03} + (p_{21} - p_{03})u^2)$, $P_2 = -p_{11}u'u$, $u = \cos\theta$.

2^0. If $p_{12} \neq 0$, then $p_{21} = p_{03} = 0$, $P_3 = u'(p_{30} + (p_{12} - p_{30})u^2)$, $P_2 = p_{11}u'u$, $u = \sin\theta$.

By Lemma 2.2 the origin of (2.21) is a center.

Case 3. We check that if $p_{11}p_{20} \neq 0$, $p_{30}p_{21} - p_{03}p_{12} = 0$ then under condition (2.20) the origin of (2.21) is a center.

Subcase a. If $p_{30}p_{03} \neq 0$, by $p_{30}p_{21} - p_{03}p_{12} = 0$, we get $p_{21} = kp_{03}$, $p_{12} = kp_{30}$. Substituting it into (2.23)–(2.26) we obtain

$$(k+3)(2p_{20}k(p_{30}^2 - p_{03}^2) + (3-k)p_{11}p_{30}p_{03}) = 0; \tag{2.34}$$

$$(k+3)(2p_{20}(p_{03}^2 - p_{30}^2) - (k+1)p_{11}p_{30}p_{03}) = 0; \tag{2.35}$$

$$(k+3)(p_{11}p_{30}p_{03} - p_{20}(kp_{03}^2 - p_{30}^2)) = 0; \tag{2.36}$$

$$4p_{20}^3(p_{30}^2 - p_{03}^2) + 4kp_{20}^2p_{11}p_{30}p_{03} + p_{20}p_{11}^2(k^2p_{03}^2 + 2kp_{03}^2 - 3p_{30}^2) - (k+2)p_{11}^3p_{30}p_{03} = 0. \tag{2.37}$$

1^*. If $k = -3$, then relations (2.34)–(2.36) are identities. Substituting $k = -3$ into (2.37) we obtain

$$\mu^3 p_{30}p_{03} - 3\mu^2(p_{30}^2 - p_{03}^2) - 12\mu p_{30}p_{03} + 4(p_{30}^2 - p_{03}^2) = 0, \qquad \mu := \frac{p_{11}}{p_{20}}. \tag{2.38}$$

Since $p_{21} = -3p_{03}$, $p_{12} = -3p_{30}$,

$$P_3 = p_{30}\cos^3\theta - 3p_{03}\cos^2\theta\sin\theta - 3p_{30}\cos\theta\sin^2\theta + p_{03}\sin^3\theta,$$

$$P_2 = p_{20}(\cos\theta - \delta\sin\theta)\left(\cos\theta + \frac{1}{\delta}\sin\theta\right), \qquad \delta := \frac{-\mu + \sqrt{\mu^2+4}}{2}, \qquad \mu = \delta^{-1} - \delta.$$

By (2.38), we have

$$((3\delta^2 - 1)p_{03} - \delta(\delta^2 - 3)p_{30})(\delta(3 - \delta^2)p_{03} - (3\delta^2 - 1)p_{30}) = 0.$$

If $(3\delta^2 - 1)p_{03} - \delta(\delta^2 - 3)p_{30} = 0$, then $\cos\theta - \delta\sin\theta / P_3$ and

$$P_2 = \delta^{-1}p_{20}u'u, \qquad P_3 = \frac{p_{30}}{1 - 3\delta^2}u'(\delta^2 + 1 - 4u^2), \qquad u := \sin\theta + \delta\cos\theta,$$

If $\delta(3 - \delta^2)p_{03} - (3\delta^2 - 1)p_{30} = 0$, then $\cos\theta + \delta^{-1}\sin\theta / P_3$ and

$$P_2 = -\delta p_{20}u'u, \qquad P_3 = \frac{p_{30}}{\delta^2 - 3}u'(\delta^2 + 1 - 4\delta^2u^2), \qquad u = \sin\theta - \delta^{-1}\cos\theta.$$

On the other hand, we can prove that $\delta^2 \neq 3$ and $\delta^2 \neq \frac{1}{3}$. Otherwise, if $\delta^2 = 3$ or $\delta^2 = \frac{1}{3}$, we get $\mu^2 = \frac{4}{3}$, substituting it into (2.41) we have $p_{03}p_{30} = 0$, this is inconsistent with the previous hypothesis, so $\delta^2 \neq 3$ and $\delta^2 \neq \frac{1}{3}$.

By Lemma 2.2, the origin of (2.21) is a center.

2^*. If $k \neq -3$, (2.34)–(2.36) imply

$$2p_{20}k(p_{30}^2 - p_{03}^2) + (3-k)p_{11}p_{30}p_{03} = 0; \tag{2.39}$$

$$2p_{20}(p_{03}^2 - p_{30}^2) - (k+1)p_{11}p_{30}p_{03} = 0; \tag{2.40}$$

$$p_{11}p_{30}p_{03} - p_{20}(kp_{03}^2 - p_{30}^2) = 0. \tag{2.41}$$

Using (2.39) and (2.40) we get $(k+3)(k-1)p_{11}p_{30}p_{03} = 0$, which implies $k = 1$. Thus, $p_{21} = p_{03}$, $p_{12} = p_{30}$, and (2.39)–(2.41) are equivalent to

$$p_{11}p_{30}p_{03} + p_{20}(p_{30}^2 - p_{03}^2) = 0. \tag{2.42}$$

Equation (2.37) can be reduced to

$$(p_{11}p_{30}p_{03} + p_{20}(p_{30}^2 - p_{03}^2))(3p_{11}^2 - 4p_{20}^2) = 0. \tag{2.43}$$

By (2.42) we see that relation (2.43) holds.

Thus,

$$P_3 = p_{30}\cos\theta + p_{03}\sin\theta, \qquad P_2 = -\frac{p_{20}}{p_{30}p_{03}}P_3\bar{P}_3, \qquad \bar{P}_3 = p_{30}\sin\theta - p_{03}\cos\theta.$$

By Lemma 2.2, the origin point of (2.21) is a center.

Subcase b. If $p_{30} = 0$, $p_{03} \neq 0$, then $p_{12} = 0$ and condition (2.20) yields

$$p_{21} = -3p_{03},\ 3p_{11}^2 = 4p_{20}^2,$$

which implies that $\mu^2 = (\frac{p_{11}}{p_{20}})^2 = \frac{4}{3}$, so $\delta = \frac{1}{\sqrt{3}}$ or $\delta = \sqrt{3}$.

If $\delta = \sqrt{3}$, then

$$P_2 = -\sqrt{3}p_{20}uu', \qquad P_3 = -\frac{\sqrt{3}p_{03}}{2}u'(1-3u^2), \qquad u = \sin\theta - \frac{\sqrt{3}}{3}\cos\theta.$$

If $\delta = \frac{1}{\sqrt{3}}$, then

$$P_2 = \sqrt{3}p_{20}u'u, \qquad P_3 = \frac{\sqrt{3}p_{03}}{2}u'(1-3u^2), \qquad u = \sin\theta + \frac{\sqrt{3}}{3}\cos\theta.$$

By Lemma 2.2 the origin of (2.21) is a center.

Subcase c. If $p_{03} = 0$, $p_{30} \neq 0$, then $p_{21} = 0$ and condition (2.20) implies

$$p_{12} = -3p_{30}, \qquad 3p_{11}^2 = 4p_{20}^2.$$

It follows that $\mu^2 = (\frac{p_{11}}{p_{20}})^2 = \frac{4}{3}$, so $\delta = \sqrt{3}$ or $\delta = \frac{1}{\sqrt{3}}$.

If $\delta = \sqrt{3}$, then

$$P_2 = \frac{\sqrt{3}p_{20}}{3}uu', \qquad P_3 = -\frac{p_{30}}{2}u'(1-u^2), \qquad u = \sin\theta + \sqrt{3}\cos\theta.$$

If $\delta = \frac{1}{\sqrt{3}}$, then

$$P_2 = -\sqrt{3}p_{20}u'u, \qquad P_3 = -\frac{1}{2}p_{30}u'(1-u^2), \qquad u = \sin\theta - \sqrt{3}\cos\theta.$$

By Lemma 2.2 the origin of (2.21) is a center.

Subcase d. If $p_{30} = p_{03} = 0$, then from condition (2.20) it follows that $p_{12}^2 + p_{21}^2 = 0$. This implies $P_3 \equiv 0$, and the equation (1.7) becomes $r' = P_2r^3$, since $p_{20} + p_{02} = 0$. Then, $r(\theta + 2\pi) = r(\theta)$, that is, the origin of (2.21) is a center.

Case 4. We show that if $p_{11}p_{20} \neq 0$, $p_{20} + p_{02} = 0$ and $p_{30}p_{21} - p_{03}p_{12} \neq 0$, then

$$P_3 = u'(\lambda_0 + \lambda_1 u^2), \qquad u = \sin\theta + \delta\cos\theta$$

or

$$P_3 = v'(\tilde{\lambda}_0 + \tilde{\lambda}_1 v^2), \qquad v = \sin\theta - \delta^{-1}\cos\theta$$

if and only if,

$$p_{30}k_2(3k_1^2 - k_2^2) + p_{03}k_1(k_1^2 - 3k_2^2) + k_1k_2(k_2^2 - k_1^2) = 0 \tag{2.44}$$

and

$$k_1k_2\mu - k_2^2 + k_1^2 = 0, \tag{2.45}$$

where

$$\mu = \frac{p_{11}}{p_{20}}, \qquad \delta = \frac{1}{2}\left(-\mu + \sqrt{\mu^2+4}\right), \qquad k_1 = p_{12} + 3p_{30}, \qquad k_2 = p_{21} + 3p_{03}$$

and λ_1, λ_2, $\tilde{\lambda}_1$, $\tilde{\lambda}_2$ are constants.

To verify the claim we first note that by $p_{20} + p_{02} = 0$ we have

$$P_2 = (\cos\theta - \delta\sin\theta)(\cos\theta + \delta^{-1}\sin\theta),$$

where $\delta = \frac{1}{2}(-\mu + \sqrt{\mu^2+4})$, $\mu = \frac{p_{11}}{p_{20}}$.

Since $p_{30}p_{21} - p_{03}p_{12} \neq 0$, we have that $(p_{12} + 3p_{30})^2 + (p_{21} + 3p_{03})^2 \neq 0$.

Subcase a. If $p_{12} + 3p_{30} \neq 0$, then $P_3 = u'(\lambda_0 + \lambda_1 u^2)$, $u = \sin\theta + \delta\cos\theta$ if and only if

$$p_{30}\delta^3 + p_{21}\delta^2 + p_{12}\delta + p_{03} = 0, \qquad \delta = -\frac{k_2}{k_1}.$$

That is,

$$p_{30}k_2^3 - p_{21}k_2^2k_1 + p_{12}k_2k_1^2 - p_{03}k_1^3 = 0;$$
$$k_1k_2\mu + k_1^2 - k_2^2 = 0.$$

These equalities are equivalent to (2.34) and (2.45).

Subcase b. If $p_{21} + 3p_{03} \neq 0$, then $P_3 = v'(\lambda_0 + \lambda_1 v^2)$, $v = \sin\theta - \delta^{-1}\cos\theta$, if and only if

$$p_{30} - p_{21}\delta + p_{12}\delta^2 - p_{03}\delta^3 = 0, \ \delta = \frac{k_1}{k_2}.$$

Thus,

$$p_{30}k_2^3 - p_{21}k_2^2k_1 + p_{12}k_2k_1^2 - p_{03}k_1^3 = 0;$$
$$k_1k_2\mu + k_1^2 - k_2^2 = 0.$$

Therefore, $P_3 = u'(\lambda_0 + \lambda_1 u^2)$ or $P_3 = v'(\tilde{\lambda}_0 + \tilde{\lambda}_1 v^2)$, if and only if relations (2.44) and (2.45) are held.

Case 5. Finally, we check that if $p_{11}p_{20} \neq 0$ and $p_{30}p_{21} - p_{03}p_{12} \neq 0$, then under the condition (2.20) the origin of (2.21) is a center.

Denote $k_1 = p_{12} + 3p_{30}$, $k_2 = p_{21} + 3p_{03}$. Then relations (2.23) and (2.24) can be rewritten as

$$2(k_1^2 - k_2^2) - 6(k_1p_{30} - k_2p_{03}) = \mu(k_1k_2 - 3p_{03}k_1 - 3p_{30}k_2); \tag{2.46}$$

$$2(p_{03}k_2 - p_{30}k_1) = \mu(k_1k_2 - p_{03}k_1 - k_2p_{30}). \tag{2.47}$$

Substituting (2.50) into (2.49) we get

$$k_2^2 - k_1^2 = \mu k_1 k_2,$$

that is, relation (2.45) holds. Form this relation, since $p_{11}p_{20} \neq 0$ and $p_{30}p_{21} - p_{03}p_{12} \neq 0$, it follows that $k_1k_2 \neq 0$ and $\mu = \frac{k_2^2 - k_1^2}{k_1k_2}$. Substituting the latter relation into (2.47) we obtain

$$p_{30}(3k_2k_1^3 - k_2^3) + p_{03}(k_1^3 - 3k_1k_2^2) + k_1k_2(k_2^2 - k_1^2) = 0.$$

This means that relation (2.43) holds. Thus, as it was shown in the study of Case 4 the conditions of Lemma 2.2 are satisfied. Hence, the origin of (2.21) is a center.

Summarizing, condition (2.20) is the necessary and sufficient condition for the origin of system (2.21) to be a center. Therefore, the proof of the present theorem is finished. □

Remark 2.8. If $p_{10} = p_{01} = p_{20} = p_{11} = p_{02} = 0$, then equation (1.7) becomes $r' = P_3(\cos\theta, \sin\theta)r^4$. Since P_3 is a cubic polynomial in $\cos\theta$ and $\sin\theta$, we have that

$$\int_0^{2\pi} P_3(\cos\theta, \sin\theta)d\theta = 0,$$

so $r(\theta + 2\pi) = r(\theta)$. Therefore, $r = 0$ is a center of (1.7).

To summarize, we have shown that in all center cases considered in this paper the Composition Condition holds. Thus, we have the following result.

Corollary 2.9. *If the origin is a center for system* (1.7) *satisfying one of Theorems 2.3, 2.7, or Corollary 2.6, then P_1, P_2 and P_3 satisfy Composition Condition* (2.1) *of Lemma 2.2.*

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