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Positive solutions of second-order problem with dependence on derivative in nonlinearity under Stieltjes integral boundary condition

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Abstract. In this paper, we investigate the second-order problem with dependence on derivative in nonlinearity and Stieltjes integral boundary condition

$$\begin{cases} -u''(t) = f(t, u(t), u'(t)), & t \in [0,1], \\ u(0) = \alpha[u], \quad u'(1) = 0, \end{cases}$$

where $f\colon [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and $\alpha[u]$ is a linear functional. Some inequality conditions on nonlinearity f and the spectral radius conditions of linear operators are presented that guarantee the existence of positive solutions to the problem by the theory of fixed point index on a special cone in $C^1[0,1]$. The conditions allow that $f(t, x_1, x_2)$ has superlinear or sublinear growth in x_1, x_2. Some examples are given to illustrate the theorems respectively under multi-point and integral boundary conditions with sign-changing coefficients.

Keywords: positive solution, fixed point index, cone, spectral radius.

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1 Introduction

In this paper, we discuss the existence of positive solutions for second-order boundary value problem (BVP) with dependence on derivative in nonlinearity and Stieltjes integral boundary conditions

$$\begin{cases} -u''(t) = f(t, u(t), u'(t)), & t \in [0,1], \\ u(0) = \alpha[u], \quad u'(1) = 0, \end{cases} \tag{1.1}$$

where α denotes linear functional given by $\alpha[u] = \int_0^1 u(t)dA(t)$ involving Stieltjes integral with suitable function A of bounded variation.

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Recently, Li [7] considered the existence of solutions for second order boundary value problem

$$\begin{cases} -u''(t) = f(t, u(t), u'(t)), & t \in [0,1], \\ u(0) = 0, \quad u(1) = 0, \end{cases} \tag{1.2}$$

where $f\colon [0,1] \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ is continuous. Under the conditions that the nonlinearity $f(t, x_1, x_2)$ may be superlinear or sublinear growth on x_1 and x_2, the existence of positive solutions are obtained. It should be remarked that the constant π^2 in the discussion plays an important role and is the first eigenvalue of the linear eigenvalue problem corresponding to BVP (1.2) which is related to the spectral radius of linear operator. The results of [7] extend those of [14] in which only sublinear problem was treated. The fourth-order problem with fully nonlinear terms was also investigated in [8].

Webb and Infante [11] gave a unified method of establishing the existence of positive solutions for a large number of local and nonlocal boundary problems by applying the theory of fixed point index on cones when f does not depend on u'. They dealt with the boundary problems involving Stieltjes integrals with signed measures and their results included the multipoint and integral boundary conditions as special cases. We also refer some other relevant articles, for example, [3, 9–13] and references therein. In these works the nonlinearity is independent of derivative term.

Inspired and motived by those previous works, we investigate BVP (1.1) in which not only the nonlinearity depends on derivative term but also the boundary conditions involves Stieltjes integral. Some inequality conditions on nonlinearity f and the spectral radius conditions of linear operators are presented that guarantee the existence of positive solutions to BVP (1.1) by the theory of fixed point index on a special cone in $C^1[0,1]$. The conditions allow that $f(t, x_1, x_2)$ has superlinear or sublinear growth in x_1, x_2. The readers can also refer to [4, 5, 15] for some pertinent questions.

The organization of this paper is as follows. In Section 2, we give some lemmas useful for our main results and present a reproducing cone P and a cone K which play important roles in calculating fixed point indexes of nonlinear operator. In Section 3, we discuss the existence of positive solutions to problem mentioned above and give its complete proof. At last in Section 4, some examples are given to illustrate the theorems respectively under multi-point and integral boundary conditions with sign-changing coefficients.

2 Preliminaries

Let $C^1[0,1]$ denote the Banach space of all continuously differentiable functions on $[0,1]$ with the norm $\|u\|_{C^1} = \max\{\|u\|_C, \|u'\|_C\}$. We first make the assumption:

(C_1) $f\colon [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, here $\mathbb{R}^+ = [0, \infty)$.

As shown by Webb and Infante [11] BVP (1.1) has a solution if and only if there exists a solution in $C^1[0,1]$ for the following integral equation

$$u(t) = \alpha[u] + \int_0^1 k(t,s) f(s, u(s), u'(s))\, ds =: (Tu)(t), \tag{2.1}$$

where

$$k(t,s) = \begin{cases} s, & 0 \leqslant s \leqslant t \leqslant 1, \\ t, & 0 \leqslant t \leqslant s \leqslant 1, \end{cases} \tag{2.2}$$

and $\alpha[u] = \int_0^1 u(t)dA(t)$. We set

$$(Fu)(t) := \int_0^1 k(t,s)f(s,u(s),u'(s))ds,$$

so that $(Tu)(t) = \alpha[u] + (Fu)(t)$.

We also impose the following hypotheses:

(C_2) A is of bounded variation and

$$\mathcal{K}_A(s) := \int_0^1 k(t,s)dA(t) \geq 0, \qquad \forall s \in [0,1];$$

(C_3) $0 \leq \alpha[1] < 1$.

Adopting the notations and ideas in [11], define the operator S as

$$(Su)(t) = \frac{\alpha[Fu]}{1-\alpha[1]} + (Fu)(t),$$

and thus S can be written in the form as follows,

$$\begin{aligned}(Su)(t) &= \frac{1}{1-\alpha[1]}\int_0^1 \mathcal{K}_A(s)f(s,u(s),u'(s))ds + \int_0^1 k(t,s)f(s,u(s),u'(s))ds \\ &=: \int_0^1 k_S(t,s)f(s,u(s),u'(s))ds,\end{aligned}$$

i.e.,

$$(Su)(t) = \int_0^1 k_S(t,s)f(s,u(s),u'(s))ds, \tag{2.3}$$

where

$$k_S(t,s) = \frac{1}{1-\alpha[1]}\mathcal{K}_A(s) + k(t,s). \tag{2.4}$$

Lemma 2.1. *If (C_2) and (C_3) hold, then there exists a nonnegative function $\Phi(s)$ satisfying*

$$t\Phi(s) \leq k_S(t,s) \leq \Phi(s) \quad \text{for } t,s \in [0,1],$$

where

$$\Phi(s) = \frac{1}{1-\alpha[1]}\mathcal{K}_A(s) + s.$$

Define two cones in $C^1[0,1]$ and several linear operators as follow.

$$P = \{u \in C^1[0,1] : u(t) \geq 0,\ u'(t) \geq 0,\ \forall t \in [0,1]\}, \tag{2.5}$$

$$K = \Big\{u \in P : u(t) \geq t\|u\|_C,\ \forall t \in [0,1],\ \alpha[u] \geq 0,\ u'(1) = 0\Big\}, \tag{2.6}$$

$$(L_i u)(t) = \int_0^1 k_S(t,s)(a_i u(s) + b_i u'(s))ds \qquad (i = 1,2), \tag{2.7}$$

$$(L_3 u)(t) = a_1 \int_0^1 k_S(t,s)u(s)ds, \tag{2.8}$$

where a_i, b_i $(i = 1,2)$ are nonnegative constants and $\tau \in (0,1)$. We write $u \preceq v$ equivalently $v \succeq u$ if and only if $v - u \in P$, to denote the cone ordering induced by P.

Lemma 2.2. *If* (C_1)–(C_3) *hold, then* $S\colon P \to K$ *and* $L_i\colon C^1[0,1] \to C^1[0,1]$ *are completely continuous operators with* $L_i(P) \subset K$ $(i = 1,2,3)$.

Proof. From (2.3), (2.4) and (C_1)–(C_3) we have for $u \in P$ that $(Su)(t) \geq 0$ and

$$(Su)'(t) = \int_t^1 f(s,u(s),u'(s))ds \geq 0, \qquad \forall t \in [0,1].$$

It is easy to see from (C_1) that $S\colon P \to C^1[0,1]$ and $L_i\colon C^1[0,1] \to C^1[0,1]$ $(i = 1,2,3)$ are continuous. Let F is a bounded set in P and there exists $M > 0$ such that $\|u\|_{C^1} \leq M$ for all $u \in F$. By (C_1) and Lemma 2.1 we have that $\forall u \in F$ and $t \in [0,1]$,

$$(Su)(t) \leq \Big(\max_{(s,x,y)\in[0,1]\times[0,M]^2} f(s,x,y) \Big) \int_0^1 \Phi(s)ds,$$

$$(Su)'(t) \leq \Big(\max_{(s,x,y)\in[0,1]\times[0,M]^2} f(s,x,y) \Big) \int_t^1 ds \leq \max_{(s,x,y)\in[0,1]\times[0,M]^2} f(s,x,y),$$

then $S(F)$ is uniformly bounded in $C^1[0,1]$. Moreover $\forall u \in F$ and $t_1,t_2 \in [0,1]$ with $t_1 < t_2$,

$$\begin{aligned}|(Su)(t_1) - (Su)(t_2)| &\leq \int_0^1 |k_S(t_1,s) - k_S(t_2,s)| f(s,u(s),u'(s))ds \\ &\leq \Big(\max_{(s,x,y)\in[0,1]\times[0,M]^2} f(s,x,y) \Big) \int_0^1 |k_S(t_1,s) - k_S(t_2,s)|ds,\end{aligned}$$

$$|(Su)'(t_1) - (Su)'(t_2)| = \int_{t_1}^{t_2} f(s,u(s),u'(s))ds \leq \Big(\max_{(s,x,y)\in[0,1]\times[0,M]^2} f(s,x,y) \Big) |t_2 - t_1|,$$

thus $S(F)$ and $S'(F) =: \{v' : v'(t) = (Su)'(t), u \in F\}$ are equicontinuous.

Therefore $S\colon P \to C^1[0,1]$ is completely continuous by the Arzelà–Ascoli theorem and so are $L_i\colon C^1[0,1] \to C^1[0,1]$ $(i = 1,2,3)$ similarly.

For $u \in P$ it follows from Lemma 2.1 that

$$\|Su\|_C = \int_0^1 k_S(1,s) f(s,u(s),u'(s))ds \leq \int_0^1 \Phi(s) f(s,u(s),u'(s))ds,$$

and hence for $t \in [0,1]$,

$$(Su)(t) = \int_0^1 k_S(t,s) f(s,u(s),u'(s))ds \geq t \int_0^1 \Phi(s) f(s,u(s),u'(s))ds \geq t\|Su\|_C.$$

From (C_1)–(C_3) it can easily be checked that $\alpha[Su] \geq 0$ and $(Su)'(1) = 0$. Thus $S\colon P \to K$.

By the same way, we have $L_i\colon P \to K$ $(i = 1,2,3)$. □

Lemma 2.3 ([11]). *If* (C_1)–(C_3) *hold, then* S *and* T *have the same fixed points in* K. *As a result, BVP* (1.1) *has a solution if and only if* S *has a fixed point.*

3 Main results

In order to prove the main theorems, we need the following properties of fixed point index, see for example [1,2].

Lemma 3.1. *Let Ω be a bounded open subset of X with $0 \in \Omega$ and K be a cone in X. If $A\colon K \cap \overline{\Omega} \to K$ is a completely continuous operator and $\mu A u \neq u$ for $u \in K \cap \partial\Omega$ and $\mu \in [0,1]$, then the fixed point index $i(A, K \cap \Omega, K) = 1$.*

Lemma 3.2. *Let Ω be a bounded open subset of X and K be a cone in X. If $A\colon K \cap \overline{\Omega} \to K$ is a completely continuous operator and there exists $v_0 \in K \setminus \{0\}$ such that $u - Au \neq \nu v_0$ for $u \in K \cap \partial\Omega$ and $\nu \geq 0$, then the fixed point index $i(A, K \cap \Omega, K) = 0$.*

Recall that a cone P in Banach space X is said to be reproducing if $X = P - P$.

Lemma 3.3 (Krein–Rutman). *Let P be a reproducing cone in Banach space X and $L\colon X \to X$ be a completely continuous linear operator with $L(P) \subset P$. If the spectral radius $r(L) > 0$, then there exists $\varphi \in P \setminus \{0\}$ such that $L\varphi = r(L)\varphi$, where 0 denotes the zero element in X.*

In the sequel, let $X = C^1[0,1]$ and denote $\Omega_r = \{u \in C^1[0,1] : \|u\|_{C^1} < r\}$ for $r > 0$.

Theorem 3.4. *Under the hypotheses (C_1)–(C_3) suppose that*

(F_1) *there exist constants $a_1 > 0,\ b_1 \geq 0,\ c_1 \geq 0$ such that*

$$f(t, x_1, x_2) \leq a_1 x_1 + b_1 x_2 + c_1, \tag{3.1}$$

for all $(t, x_1, x_2) \in [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+$, moreover the spectral radius $r(L_1) < 1$;

(F_2) *there exist constants $a_2 > 0,\ b_2 \geq 0$ and $r > 0$ such that*

$$f(t, x_1, x_2) \geq a_2 x_1 + b_2 x_2, \tag{3.2}$$

for all $(t, x_1, x_2) \in [0,1] \times [0,r]^2$, moreover the spectral radius $r(L_2) \geq 1$, where $L_i\colon C^1[0,1] \to C^1[0,1]$ $(i = 1,2)$ are defined by (2.7).

Then BVP (1.1) has at least one nondecreasing positive solution.

Proof. Let $W = \{u \in K : u = \mu S u,\ \mu \in [0,1]\}$ where S and K are respectively defined in (2.3) and (2.6).

We first assert that W is a bounded set. In fact, if $u \in W$, then $u = \mu S u$ for some $\mu \in [0,1]$. From (2.7) and (3.1) we have that

$$\begin{aligned} u(t) = \mu (Su)(t) &= \mu \int_0^1 k_S(t,s) f(s, u(s), u'(s)) ds \\ &\leq \int_0^1 k_S(t,s)[a_1 u(s) + b_1 u'(s) + c_1] ds = (L_1 u)(t) + c_1 \int_0^1 k_S(t,s) ds \end{aligned}$$

and

$$((I - L_1)u)(t) \leq c_1 \int_0^1 k_S(t,s) ds =: v(t).$$

Obviously $v \in P$ and it is easy to see from (2.4) that

$$u'(t) = \mu \int_t^1 f(s, u(s), u'(s)) ds \leq \int_t^1 [a_1 u(s) + b_1 u'(s) + c_1] ds = (L_1 u)'(t) + v'(t),$$

that is, $((I - L_1)u)'(t) \leq v'(t)$ for $t \in [0,1]$, and thus $(I - L_1)u \preceq v$. Because of the spectral radius $r(L_1) < 1$, we know that $I - L_1$ has a bounded inverse operator $(I - L_1)^{-1}$ which can be written as

$$(I - L_1)^{-1} = I + L_1 + L_1^2 + \cdots + L_1^n + \cdots.$$

Since $L_1(P) \subset K \subset P$ by Lemma 2.2, we have $(I - L_1)^{-1}(P) \subset P$ which implies the inequality $u \preceq (I - L_1)^{-1}v$. Therefore,

$$u(t) \leq ((I - L_1)^{-1}v)(t), \quad u'(t) \leq ((I - L_1)^{-1}v)'(t), \qquad \forall t \in [0,1]$$

and hence $\|u\|_{C^1} \leq \|(I - L_1)^{-1}v\|_{C^1}$, i.e. W is bounded.

Now select $R > \max\{r, \sup W\}$, then $\mu Au \neq u$ for $u \in K \cap \partial\Omega_R$ and $\mu \in [0,1]$, and $i(S, K \cap \Omega_R, K) = 1$ follows from Lemma 3.1.

It is easy to verify that P is a solid cone, i.e. the interior point set $\mathring{P} \neq \emptyset$, then P is reproducing (cf. [1,2,6]). Since $L_2 \colon P \to K \subset P$ and $r(L_2) \geq 1$, it follows from Lemma 3.3 that there exists $\varphi_0 \in P \setminus \{0\}$ such that $L_2\varphi_0 = r(L_2)\varphi_0$. Furthermore, $\varphi_0 = (r(L_2))^{-1}L_2\varphi_0 \in K$.

We may suppose that S has no fixed points in $K \cap \partial\Omega_r$ and will show that $u - Su \neq \nu\varphi_0$ for $u \in K \cap \partial\Omega_r$ and $\nu \geq 0$.

Otherwise, there exist $u_0 \in K \cap \partial\Omega_r$ and $\nu_0 \geq 0$ such that $u_0 - Su_0 = \nu_0\varphi_0$, and it is clear that $\nu_0 > 0$. Since $u_0 \in K \cap \partial\Omega_r$, we have $0 \leq u_0(t), u_0'(t) \leq r, \forall t \in [0,1]$. It follows from (2.4), (2.7) and (3.2) that $(Su_0)(t) \geq (L_2u_0)(t)$ and

$$(Su_0)'(t) = \int_t^1 f(s, u_0(s), u_0'(s))ds \geq \int_t^1 [a_2u_0(s) + b_2u_0'(s)]ds = (L_2u_0)'(t), \qquad \forall t \in [0,1]$$

which imply that

$$u_0 = \nu_0\varphi_0 + Su_0 \succeq \nu_0\varphi_0 + L_2u_0 \succeq \nu_0\varphi_0. \tag{3.3}$$

Set $\nu^* = \sup\{\nu > 0 : u_0 \succeq \nu\varphi_0\}$, then $\nu_0 \leq \nu^* < +\infty$ and $u_0 \succeq \nu^*\varphi_0$. Thus it follows from (3.3) that

$$u_0 \succeq \nu_0\varphi_0 + L_2u_0 \succeq \nu_0\varphi_0 + \nu^*L_2\varphi_0 \succeq \nu_0\varphi_0 + \nu^*r(L_2)\varphi_0.$$

But $r(L_2) \geq 1$, so $u_0 \succeq (\nu_0 + \nu^*)\varphi_0$, which is a contradiction to the definition of ν^*. Therefore $u - Au \neq \nu\varphi_0$ for $u \in K \cap \partial\Omega_r$ and $\nu \geq 0$.

From Lemma 3.2 it follows that $i(S, K \cap \Omega_r, K) = 0$.

Making use of the properties of fixed point index, we have that

$$i(S, K \cap (\Omega_R \setminus \overline{\Omega}_r), K) = i(S, K \cap \Omega_R, K) - i(S, K \cap \Omega_r, K) = 1$$

and hence S has at least one fixed point in K. Therefore, BVP (1.1) has at least one nondecreasing positive solution by Lemma 2.3. □

Theorem 3.5. *Under the hypotheses* (C_1)*–*(C_3) *suppose that*

(F_3) *there exist positive constants* a_1*,* c_1 *satisfying*

$$a_1 \int_0^1 s\Phi(s)ds > 1 \tag{3.4}$$

such that

$$f(t, x_1, x_2) \geq a_1x_1 - c_1, \tag{3.5}$$

for all $(t, x_1, x_2) \in [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+$*;*

(F_4) *there exist constants* $a_2 > 0$*,* $b_2 \geq 0$ *and* $r > 0$ *such that*

$$f(t, x_1, x_2) \leq a_2x_1 + b_2x_2, \tag{3.6}$$

for all $(t, x_1, x_2) \in [0,1] \times [0,r]^2$*, moreover the spectral radius* $r(L_2) < 1$*, where* L_2 *is defined by* (2.7)*.*

If the following Nagumo condition is fulfilled, i.e.

(F_5) *for any* $M > 0$ *there is a positive continuous function* $H_M(\rho)$ *on* $\mathbb{R}^+$ *satisfying*

$$\int_0^{+\infty} \frac{\rho d\rho}{H_M(\rho)+1} = +\infty \tag{3.7}$$

such that

$$f(t,x,y) \le H_M(y), \qquad \forall (t,x,y) \in [0,1]\times[0,M]\times\mathbb{R}^+, \tag{3.8}$$

then BVP (1.1) has at least one nondecreasing positive solution.

Proof. (i) First we prove that $\mu S u \neq u$ for $u \in K \cap \partial\Omega_r$ and $\mu \in [0,1]$. In fact, if there exist $u_1 \in K\cap\partial\Omega_r$ and $\mu_0 \in [0,1]$ such that $u_1 = \mu_0 S u_1$, then we deduce from

$$0 \le u_1(t),\ u_1'(t) \le r, \qquad \forall t \in [0,1]$$

and (3.6) that

$$\begin{aligned} u_1(t) = \mu_0 (Su_1)(t) &= \mu_0 \int_0^1 k_S(t,s) f(s,u_1(s),u_1'(s))ds \\ &\le \int_0^1 k_S(t,s)[a_2u_1(s) + b_2u_1'(s)]ds = (L_2u_1)(t) \end{aligned}$$

and from (2.4) that

$$u_1'(t) = \mu_0 \int_t^1 f(s,u_1(s),u_1'(s))ds \le \int_t^1 [a_2u_1(s)+b_2u_1'(s)]ds = (L_2u_1)'(t), \qquad \forall t\in[0,1],$$

thus $(I-L_2)u_1 \preceq 0$. Because of the spectral radius $r(L_2) < 1$, we know that $I - L_2$ has a bounded inverse operator $(I-L_2)^{-1}\colon P \to P$ and $u_1 \preceq (I-L_2)^{-1}0 = 0$ which contradicts $u_1 \in K\cap\partial\Omega_r$.

Therefore, $i(S, K\cap\Omega_r, K) = 1$ follows from Lemma 3.1.

(ii) Let

$$M = \frac{c_1\int_0^1 \Phi(s)ds}{a_1\int_0^1 s\Phi(s)ds - 1}. \tag{3.9}$$

By (3.7) it is easy to see that

$$\int_0^{+\infty} \frac{\rho d\rho}{H_M(\rho)+c_1} = +\infty,$$

hence there exists $M_1 > M$ such that

$$\int_0^{M_1} \frac{\rho d\rho}{H_M(\rho)+c_1} > M. \tag{3.10}$$

(iii) For $u \in P$ define

$$(S_1u)(t) = \int_0^1 k_S(t,s)[f(s,u(s),u'(s)) + c_1]ds. \tag{3.11}$$

Similar to the proof in Lemma 2.2, we know that $S_1\colon P\to K$ is completely continuous.

Let $R > \max\{r, M_1\}$ and we will show that

$$(1-\lambda)Su + \lambda S_1 u \neq u, \qquad \forall u \in K\cap\partial\Omega_R,\ \lambda\in[0,1]. \tag{3.12}$$

If it does not hold, there exist $u_2 \in K \cap \partial\Omega_R$ and $\lambda_0 \in [0,1]$ such that

$$(1-\lambda_0)Su_2 + \lambda_0 S_1 u_2 = u_2, \tag{3.13}$$

thus by (3.5) and Lemma 2.1 we obtain that

$$\begin{aligned}
\|u_2\|_C &= (1-\lambda_0)\int_0^1 k_S(1,s) f(s,u_2(s),u_2'(s))ds + \lambda_0 \int_0^1 k_S(1,s)[f(s,u_2(s),u_2'(s)) + c_1]ds \\
&\geq \int_0^1 k_S(1,s)[f(s,u_2(s),u_2'(s)) + \lambda_0 c_1]ds \geq \int_0^1 k_S(1,s)[a_1 u_2(s) - c_1 + \lambda_0 c_1]ds \\
&\geq \int_0^1 k_S(1,s)[a_1 u_2(s) - c_1]ds \geq a_1 \int_0^1 \Phi(s) u_2(s) ds - c_1 \int_0^1 \Phi(s)ds \\
&\geq a_1 \|u_2\|_C \int_0^1 s\Phi(s)ds - c_1 \int_0^1 \Phi(s)ds
\end{aligned}$$

which implies that

$$\|u_2\|_C \leq \frac{c_1 \int_0^1 \Phi(s)ds}{a_1 \int_0^1 s\Phi(s)ds - 1} = M. \tag{3.14}$$

We can derive from (3.8), (3.13) and (3.14) that

$$\begin{aligned}
-u_2''(t) &= (1-\lambda_0) f(t,u_2(t),u_2'(t)) + \lambda_0 (f(t,u_2(t),u_2'(t)) + c_1) \\
&= f(t,u_2(t),u_2'(t)) + \lambda_0 c_1 \leq f(t,u_2(t),u_2'(t)) + c_1 \\
&\leq H_M(u_2'(t)) + c_1.
\end{aligned}$$

Multiplying both sides of the above inequality by $u_2'(t) \geq 0$, we have that

$$\frac{-u_2'(t)u_2''(t)}{H_M(u_2'(t)) + c_1} \leq u_2'(t), \qquad t \in [0,1]. \tag{3.15}$$

Then integrating the inequality (3.15) over $[0,1]$ and making the variable transformation $\rho = u_2'(t)$, we also obtain from (3.14) that

$$\int_0^{\|u_2'\|_C} \frac{\rho d\rho}{H_M(\rho) + c_1} = \int_{u_2'(1)}^{u_2'(0)} \frac{\rho d\rho}{H_M(\rho) + c_1} \leq u_2(1) - u_2(0) \leq \|u_2\|_C \leq M$$

since $u_2''(t) \leq 0$ and $u_2'(1) = 0$. Hence by (3.10) and (3.14) we have that $\|u_2'\|_C \leq M_1$ and $\|u_2\|_{C^1} \leq M_1$, which is a contradiction to $\|u_2\|_{C^1} = R > M_1$.

From (3.12) it follows that

$$i(S, K \cap \Omega_R, K) = i(S_1, K \cap \Omega_R, K) \tag{3.16}$$

by the homotopy invariance property of fixed point index.

(iv) For the function $h(t) = t$, we have from Lemma 2.1 that

$$(L_3 h)(t) = a_1 \int_0^1 s k_S(t,s)ds \geq a_1 t \int_0^1 s\Phi(s)ds = \Big(a_1 \int_0^1 s\Phi(s)ds\Big) h(t),$$

so by a result of Krasnosel'skii [6, p. 76, Theorem 2.5], there exist $\lambda_1 \geq a_1 \int_0^1 s\Phi(s)ds > 1$ and $\varphi_0 \in C[0,1]$ with $\varphi_0(t) \geq 0$ such that $\varphi_0 = \lambda_1^{-1} L_3 \varphi_0$ as L_3 is a completely continuous linear operator in $C[0,1]$. Since

$$\varphi_0'(t) = a_1 \lambda_1^{-1} \int_t^1 \varphi_0(s)ds \geq 0, \qquad t \in [0,1],$$

we have $\varphi_0 \in P$ and thus $\varphi_0 \in K$ by Lemma 2.2.

(v) In this step we prove that $u - S_1 u \neq \nu\varphi_0$ for $u \in K \cap \partial\Omega_R$ and $\nu \geq 0$, where φ_0 is as in step (iv), and hence

$$i(S_1, K \cap \Omega_R, K) = 0 \tag{3.17}$$

holds by Lemma 3.2.

If there exist $u_0 \in K \cap \partial\Omega_R$ and $\nu_0 \geq 0$ such that $u_0 - S_1 u_0 = \nu_0 \varphi_0$. Obviously $\nu_0 > 0$ by (3.12) and

$$u_0(t) = (S_1 u_0)(t) + \nu_0 \varphi_0(t) \geq \nu_0 \varphi_0(t) \tag{3.18}$$

for $t \in [0,1]$. Set

$$\nu^* = \sup\{\nu > 0 : u_0(t) \geq \nu\varphi_0(t),\ \forall t \in [0,1]\},$$

then $\nu_0 \leq \nu^* < +\infty$ and $u_0(t) \geq \nu^* \varphi_0(t)$ for $t \in [0,1]$. From (3.5) and (3.18) we have that for $t \in [0,1]$,

$$\begin{aligned} u_0(t) &= (S_1 u_0)(t) + \nu_0\varphi_0(t) \geq (L_3 u_0)(t) + \nu_0 \varphi_0(t) \\ &\geq \nu^*(L_3\varphi_0)(t) + \nu_0\varphi_0(t) = \lambda_1 \nu^* \varphi_0(t) + \nu_0 \varphi_0(t). \end{aligned}$$

Since $\lambda_1 > 1$, we have $r(L_3)\nu^* + \nu_0 > \nu^*$ which contradicts the definition of ν^*.

(vi) From (3.16) and (3.17) it follows that $i(S, K \cap \Omega_R, K) = 0$ and

$$i(S, K \cap (\Omega_R \setminus \overline{\Omega}_r), K) = i(S, K \cap \Omega_R, K) - i(S, K \cap \Omega_r, K) = -1.$$

Hence S has at least one fixed solution and BVP (1.1) has at least one nondecreasing positive solution by Lemma 2.3. □

4 Examples

For the sake of providing some examples to illustrate the theorems, we roughly estimate some coefficients by inequalities in advance.

Consider 4-point boundary problem with sign-changing coefficients as follows:

$$\begin{cases} -u''(t) = f(t, u(t), u'(t)), & t \in [0,1], \\ u(0) = \frac{1}{4}u(\frac{1}{4}) - \frac{1}{12}u(\frac{3}{4}), & u'(1) = 0. \end{cases} \tag{4.1}$$

Then $\alpha[u] = \frac{1}{4}u(\frac{1}{4}) - \frac{1}{12}u(\frac{3}{4})$ from which it follows that

$$\mathcal{K}_A(s) = \int_0^1 k(t,s)dA(t) = \frac{1}{4}k\Big(\frac{1}{4}, s\Big) - \frac{1}{12}k\Big(\frac{3}{4}, s\Big) = \begin{cases} \frac{s}{6}, & 0 \leq s \leq \frac{1}{4}, \\ \frac{3-4s}{48}, & \frac{1}{4} < s \leq \frac{3}{4}, \\ 0, & \frac{3}{4} < s \leq 1. \end{cases}$$

It is easy to see that $0 \leq \mathcal{K}_A(s) \leq \frac{1}{24}$, $\alpha[1] = \frac{1}{6}$, thus (C_2) and (C_3) are satisfied. From (2.4) follows that

$$k_S(t,s) = \frac{6}{5}\mathcal{K}_A(s) + k(t,s) \leq \frac{21}{20},$$

$$\Phi(s) = \frac{6}{5}\mathcal{K}_A(s) + s.$$

Since for $u \in C^1[0,1]$ and $t \in [0,1]$,

$$|(L_i u)(t)| \leq \frac{21}{20}\int_0^1 (a_i|u(s)| + b_i|u'(s)|)ds \leq \frac{21}{20}(a_i + b_i)\|u\|_{C^1},$$

$$|(L_i u)'(t)| \leq \int_0^1 (a_i|u(s)| + b_i|u'(s)|)ds \leq (a_i + b_i)\|u\|_{C^1} \qquad (i = 1, 2),$$

we have that $r(L_i) \leq \|L_i\| \leq (21/20)(a_i + b_i) < 1$ when $a_i + b_i < 20/21$ $(i = 1, 2)$.

From Lemma 2.1 and Lemma 2.2 we have that for $u \in K \setminus \{0\}$ and $t \in [0,1]$,

$$(L_i u)(t) \geq a_i t \int_0^1 \Phi(s)u(s)ds \geq a_i t \int_0^1 s\Phi(s)\|u\|_C ds = a_i t\|u\|_C \int_0^1 s\Phi(s)ds$$

and

$$\|(L_i u)\|_C = (L_i u)(1) \geq a_i\|u\|_C \int_0^1 s\Phi(s)ds \qquad (i = 1, 2),$$

hence

$$\begin{aligned}(L_i^2 u)(t) &\geq a_i \int_0^1 k_S(t,s)(L_i u)(s)ds \geq a_i t \int_0^1 \Phi(s)(L_i u)(s)ds \\ &\geq a_i t \int_0^1 s\Phi(s)\|(L_i u)\|_C ds \geq a_i^2 t\|u\|_C \Big(\int_0^1 s\Phi(s)ds\Big)^2\end{aligned}$$

and

$$\|(L_i^2 u)\|_C = (L_i^2 u)(1) \geq a_i^2\|u\|_C \Big(\int_0^1 s\Phi(s)ds\Big)^2.$$

By induction,

$$\|(L_i^n u)\|_C = (L_i^n u)(1) \geq a_i^n\|u\|_C \Big(\int_0^1 s\Phi(s)ds\Big)^n.$$

Consequently for $u \in K \setminus \{0\}$,

$$\|L_i^n\|\|u\|_{C^1} \geq \|L_i^n u\|_{C^1} \geq \|L_i^n u\|_C \geq a_i^n\|u\|_C \Big(\int_0^1 s\Phi(s)ds\Big)^n,$$

and by virtue of Gelfand's formula, the spectral radius

$$r(L_i) = \lim_{n\to\infty} \|L_i^n\|^{1/n} \geq a_i \Big(\int_0^1 s\Phi(s)ds\Big) \lim_{n\to\infty} \Big(\frac{\|u\|_C}{\|u\|_{C^1}}\Big)^{1/n} = a_i \Big(\int_0^1 s\Phi(s)ds\Big). \tag{4.2}$$

Hence we can obtain that $r(L_i) \geq \frac{163}{480} a_i$ $(i = 1, 2)$.

Example 4.1. If $f(t, x_1, x_2) = \sqrt[3]{x_1} + \sqrt[3]{x_2}$, while $a_1 = \frac{1}{4}$, $b_1 = \frac{1}{6}$ and c_1 is large enough for (F_1), and $a_2 = 3$, $b_2 = 1$, $r = \frac{\sqrt{3}}{9}$ for (F_2). By Theorem 3.4 we know that BVP (4.1) has at least one positive solution.

Example 4.2. If

$$f(t, x_1, x_2) = \frac{\frac{1}{4}{x_1}^4 + \frac{1}{6}x_2^4}{1 + x_1^2 + x_2^2},$$

while $a_1 = 3$ and c_1 is large enough for (F_3), and $a_2 = \frac{1}{4}$, $b_2 = \frac{1}{6}$, $r < 1$ for (F_4), $H_M(\rho) = M^2 + \rho^2$. By Theorem 3.5 we know that BVP (4.1) has at least one positive solution.

Our last example is the following problem with integral boundary condition in which one should notice that $\cos(\pi t)$ is sign-changing over $[0,1]$:

$$\begin{cases} -u''(t) = f(t, u(t), u'(t)), \quad t \in [0,1], \\ u(0) = -\int_0^1 u(t)\cos(\pi t)dt, \quad u'(1) = 0, \end{cases} \tag{4.3}$$

Then $\alpha[u] = -\int_0^1 u(t)\cos(\pi t)dt$ from which it follows that

$$0 \le \mathcal{K}_A(s) = -\int_0^1 k(t,s)\cos(\pi t)dt = \frac{1-\cos(\pi s)}{\pi^2} \le \frac{2}{\pi^2}$$

and $\alpha[1] = -\int_0^1 \cos(\pi t)dt = 0$. Thus (C_2) and (C_3) are satisfied and from (2.4) follows that

$$k_S(t,s) = \frac{1-\cos(\pi s)}{\pi^2} + k(t,s) \le \frac{2+\pi^2}{\pi^2}.$$

$$\Phi(s) = K_A(s) + s.$$

Since for $u \in C^1[0,1]$ and $t \in [0,1]$,

$$|(L_i u)(t) \le \frac{2+\pi^2}{\pi^2}\int_0^1 (a_i|u(s)| + b_i|u'(s)|)ds \le \frac{2+\pi^2}{\pi^2}(a_i+b_i)\|u\|_{C^1},$$

$$|(L_i u)'(t)| \le \int_0^1 (a_i|u(s)| + b_i|u'(s)|)ds \le (a_i+b_i)\|u\|_{C^1} \qquad (i=1,2),$$

we have that

$$r(L_i) \le \|L_i\|_{C^1} \le \frac{2+\pi^2}{\pi^2}(a_i+b_i) < 1$$

when $a_i + b_i < \frac{\pi^2}{2+\pi^2}$ $(i = 1,2)$. Moreover, (4.2) holds, from which we can obtain $r(L_i) \ge \frac{2\pi^4+3\pi^2+12}{6\pi^4}a_i$ $(i=1,2)$.

Example 4.3. If $f(t,x_1,x_2) = \sqrt{x_1} + \sqrt{x_2}$, while $a_1 = \frac{1}{4}$, $b_1 = \frac{1}{5}$ and c_1 is large enough for (F_1), and $a_2 = 3$, $b_2 = 1$, $r = \frac{1}{9}$ for (F_2). By Theorem 3.4 we know that BVP (4.3) has at least one positive solution.

Example 4.4. If

$$f(t,x_1,x_2) = \frac{\frac{1}{4}{x_1}^4 + \frac{1}{5}x_2^4}{1+x_1^2+x_2^2},$$

while $a_1 = 3$ and c_1 is large enough for (F_3), and $a_2 = \frac{1}{4}$, $b_2 = \frac{1}{5}$, $r = \frac{3}{4}$ for (F_4), $H_M(\rho) = M^2 + \rho^2$. By Theorem 3.5 we know that BVP (4.3) has at least one positive solution.

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