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Bautin bifurcations of a financial system

Bo Sang ✉ [1,2] and **Bo Huang** [3,4]

[1] School of Mathematical Sciences, Liaocheng University, Liaocheng 252059, China
[2] Guangxi Colleges and Universities Key Laboratory of Symbolic Computation and Engineering Data Processing, Hezhou University, Hezhou 542899, China
[3] School of Mathematics and Systems Science, Beihang University, Beijing 100191, China
[4] LIMB of the Ministry of Education, Beijing 100191, China

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Abstract. This paper is concerned with the qualitative analysis of a financial system. We focus our interest on the stability and cyclicity of the equilibria. Based on some previous results, some notes are given for a class of systems concerning focus quantities, center manifolds and Hopf bifurcations. The analysis of Hopf bifurcations on the center manifolds is carried out based on the computation of focus quantities and other analytical techniques. For each equilibrium, the structure of the bifurcation set is explored in depth. It is proved through the study of Bautin bifurcations that the system can have at most four small limit cycles (on the center manifolds) in two nests and this bound is sharp.

Keywords: focus quantity, limit cycle, Bautin bifurcation.

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1 Introduction

Hopf bifurcation is the simplest way in which limit cycles can emerge from an equilibrium point. This phenomenon is an attractive subject of analysis for mathematicians as well as for economists, see [2, 5, 8, 10, 13, 18, 19, 21, 23, 26, 31] and the references therein. It occurs when a pair of complex conjugate eigenvalues of an equilibrium point cross the imaginary axis as the bifurcation parameter is varied. We recall that a limit cycle is a periodic orbit isolated in the set of all the periodic oribits of the system.

Hopf bifurcations have been studied in many business models, see, for instance, [14, 18, 28]. For three-dimensional autonomous systems, Asada and Semmler [1] provided rigorous treatments on the analysis of Hopf bifurcations; Makovínyiová [20] proved the existence and stability of business cycles; Guirao, García-Rubio and Vera [7] studied the stability and the Hopf bifurcations of a generalized IS-LM macroeconomic model; Přibylová [23] investigated the Hopf bifurcations in an idealized macroeconomic model with foreign capital investment.

✉ Corresponding author. Email: sangbo_76@163.com

The Hopf bifurcations of a 3-dimensional financial system were firstly discussed in a series of two papers by Ma and Chen [16, 17], which were far from complete because the focus quantities that characterize the criticality of the bifurcations were not obtained, and the Bautin bifurcations (also known as the generalized Hopf bifurcations) were not taken into account. The same model was later considered in [15] based on computing Lyapunov coefficients (which are equilivalent to focus quantities, see [24, Theorem 6.2.3 (page 261)]) by the method of Kuznetsov [9]. However the results in [15] were still far from complete because some parameters were kept fixed. Thus, a more complete mathematical treatment of Hopf bifurcations and Bautin bifurcations in this model is necessary and important.

Consider the model proposed in [16, 17], i.e.,

$$\begin{cases} \dfrac{dx}{dt} = z + (y-a)x, \\ \dfrac{dy}{dt} = 1 - by - x^2, \\ \dfrac{dz}{dt} = -x - cz, \end{cases} \tag{1.1}$$

describing the development of interest rate x, investment demand y and price index z. The parameters $a > 0, b > 0$ and $c > 0$ denote the saving amount, the per-investment cost, and the demand elasticity of commercials, respectively. This system is invariant under the transformation $(x, y, z) \to (-x, y, -z)$. Despite its simplicity the system exhibits mathematically rich dynamics: from stable equilibria to periodic and even chaotic oscillations depending on the parameter values, see [15–17, 27].

The rest of the paper is organized as follows. In order to acquaint the reader with the focus quantities, center manifolds and Hopf bifurcations in three dimensional systems, in section 2, we gives some notes on these topics based on some works [3, 4, 6, 9, 11, 12, 24, 26, 29–31]. In section 3 the linear stability analysis is performed for the equilibria. In sections 4 and 5, the Hopf and Bautin bifurcations are studied for the equilibrium on the axis and two interior equilibria, respectively. Finally the concluding remarks are presented.

2 Focus quantities, center manifolds and Hopf bifurcations in $\mathbb{R}^3$

Consider the following 3-dimensional differential system

$$\begin{cases} \dfrac{dx}{dt} = \epsilon\, x - \omega\, y + P_1(x, y, z) = X_\epsilon(x, y, z), \\ \dfrac{dy}{dt} = \omega\, x + \epsilon\, y + P_2(x, y, z) = Y_\epsilon(x, y, z), \\ \dfrac{dz}{dt} = -\delta\, z + Q(x, y, z) = Z_\epsilon(x, y, z), \end{cases} \tag{2.1}$$

where ω, δ are positive constants, P_j, Q are real analytical functions without constant and linear terms, defined in a neighborhood of the origin, $j = 1, 2$, and ϵ is considered as a real parameter. When $\epsilon = 0$, the Jacobian matrix at the origin has a pair of purely imaginary eigenvalues $\lambda_{1,2} = \pm \mathrm{i}\omega$ and a negative eigenvalue $\lambda_3 = -\delta$, so the origin is a Hopf point (see [3]) associated to the simple Hopf bifurcation. The simple Hopf bifurcation is a special type of Hopf bifurcations, where a pair of complex conjugate eigenvalues of the Jacobian matrix passes through the imaginary axis while all other eigenvalues have negative real parts, see [11].

For later use, let us write

$$\begin{cases} P_1(x,y,z) = \sum\limits_{|p|+q=2}^{\infty} c^{(1)}_{p_1,p_2,q} x^{p_1} y^{p_2} z^q, \\ P_2(x,y,z) = \sum\limits_{|p|+q=2}^{\infty} c^{(2)}_{p_1,p_2,q} x^{p_1} y^{p_2} z^q, \\ Q(x,y,z) = \sum\limits_{|p|+q=2}^{\infty} d_{p_1,p_2,q} x^{p_1} y^{p_2} z^q, \end{cases}$$

where $p = (p_1, p_2)$ and $|p| = p_1 + p_2$.

By introducing the transformation

$$x = \frac{1}{2}(u+v), \qquad y = \frac{\mathtt{i}}{2}(v-u), \qquad z = w, \tag{2.2}$$

system (2.1)$\big|_{\epsilon=0}$ can be transformed into the following form:

$$\begin{cases} \dfrac{du}{dt} = \mathtt{i}\omega\, u + R_1(u,v,w) = U(u,v,w), \\ \dfrac{dv}{dt} = -\mathtt{i}\omega\, v + R_2(u,v,w) = V(u,v,w), \\ \dfrac{dw}{dt} = -\delta\, w + S(u,v,w) = W(u,v,w), \end{cases} \tag{2.3}$$

where

$$R_2(u,\overline{u},w) = \overline{R_1(u,\overline{u},w)};$$

$S(u,\overline{u},w)$ is real-valued for all $u \in \mathbb{C}$ and $w \in \mathbb{R}$; and

$$\begin{cases} R_1(u,v,w) = \sum\limits_{|p|+q=2}^{\infty} a^{(1)}_{p_1,p_2,q} u^{p_1} v^{p_2} w^q, \\ R_2(u,v,w) = \sum\limits_{|p|+q=2}^{\infty} a^{(2)}_{p_1,p_2,q} u^{p_1} v^{p_2} w^q, \\ S(u,v,w) = \sum\limits_{|p|+q=2}^{\infty} b_{p_1,p_2,q} u^{p_1} v^{p_2} w^q, \end{cases}$$

with

$$\overline{a^{(1)}_{p_1,p_2,q}} = a^{(2)}_{p_2,p_1,q}, \qquad \overline{b_{p_1,p_2,q}} = b_{p_2,p_1,q}. \tag{2.4}$$

2.1 Focus quantities

Before we introduce the concept of focus quantities, we need a theorem, which is a generalization of [26, Theorem 3.1].

Theorem 2.1. *For system (2.3), we can derive successively the terms of the following formal series:*

$$\begin{aligned} F(u,v,w) &= uv + \sum_{|p|+q=3}^{\infty} C_{p_1,p_2,q} u^{p_1} v^{p_2} w^q \\ &\triangleq \sum_{|p|+q=2}^{\infty} C_{p_1,p_2,q} u^{p_1} v^{p_2} w^q, \end{aligned} \tag{2.5}$$

such that

$$\left.\frac{dF}{dt}\right|_{(2.3)}=\frac{\partial F}{\partial u}U+\frac{\partial F}{\partial v}V+\frac{\partial F}{\partial w}W=\sum_{n=1}^{\infty}V_n(uv)^{n+1}. \tag{2.6}$$

For $(p_1,p_2,q)\neq(p_1,p_1,0)$, where $|p|+q\geq 3$, the coefficients $C_{p_1,p_2,q}$ in (2.5) are determined by the recursive formula

$$\begin{aligned}C_{p_1,p_2,q}=\frac{1}{-\mathtt{i}\omega(p_1-p_2)+\delta q}\sum_{|j|+s=3}^{|p|+q}\Big[&(p_1-j_1+1)a^{(1)}_{j_1,j_2-1,s}+(p_2-j_2+1)a^{(2)}_{j_1-1,j_2,s}\\&+(q-s)b_{j_1-1,j_2-1,s+1}\Big]C_{p_1-j_1+1,p_2-j_2+1,q-s},\end{aligned} \tag{2.7}$$

where $|j|=j_1+j_2$.

For $(p_1,p_2,q)=(p_1,p_1,0)$, where $p_1\geq 2$, we set

$$C_{p_1,p_1,0}=0. \tag{2.8}$$

The V_n in (2.6) are determined by

$$V_n=\sum_{j_1+j_2=3}^{2(n+1)}\Big[(n-j_1+2)a^{(1)}_{j_1,j_2-1,0}+(n-j_2+2)a^{(2)}_{j_1-1,j_2,0}\Big]C_{n-j_1+2,n-j_2+2,0}. \tag{2.9}$$

Proof. By direct computation, we find that

$$\begin{aligned}\left.\frac{dF}{dt}\right|_{(2.3)}&=\frac{\partial F}{\partial u}U+\frac{\partial F}{\partial v}V+\frac{\partial F}{\partial w}W\\&=\sum_{|p|+q=3}^{\infty}u^{p_1}v^{p_2}w^{q}\Big\{[\mathtt{i}\omega(p_1-p_2)-\delta q]C_{p_1,p_2,q}\\&\qquad+\sum_{|j|+s=3}^{|p|+q}\Big[(p_1-j_1+1)a^{(1)}_{j_1,j_2-1,s}+(p_2-j_2+1)a^{(2)}_{j_1-1,j_2,s}\\&\qquad+(q-s)b_{j_1-1,j_2-1,s+1}\Big]C_{p_1-j_1+1,p_2-j_2+1,q-s}\Big\}.\end{aligned}$$

Comparing the above power series with the right side of (2.6), we can obtain the recursive formulas (2.7) and (2.9). This completes the proof. □

Remark 2.2. From (2.6), we can see that in order to compute V_n, we only need to find a polynomial in the following form

$$F_{2n+2}(u,v,w)=uv+\sum_{|p|+q=3}^{2n+2}C_{p_1,p_2,q}u^{p_1}v^{p_2}w^{q},$$

which is an approximation of (2.5) up to $(2n+2)$-th order.

The following result can be proved using an argument similar to the proof of Theorem 2.1.

Corollary 2.3. *For $(p_2, p_1, q) \neq (p_1, p_1, 0)$, where $|p| + q \geq 3$, the coefficients $C_{p_2,p_1,q}$ in (2.5) are determined by the recursive formula*

$$C_{p_2,p_1,q} = \frac{1}{-i\omega(p_2 - p_1) + \delta q} \sum_{|j|+s=3}^{|p|+q} \Big[(p_2 - j_2 + 1)a^{(1)}_{j_2,j_1-1,s} + (p_1 - j_1 + 1)a^{(2)}_{j_2-1,j_1,s} + (q - s)b_{j_2-1,j_1-1,s+1}\Big] C_{p_2-j_2+1,p_1-j_1+1,q-s}, \tag{2.10}$$

where $|j| = j_1 + j_2$.

Using the structure of F in Theorem 2.1, we obtain the following result.

Corollary 2.4. *$F(u, \overline{u}, w)$ is real-valued for $u \in \mathbb{C}$ and $w \in \mathbb{R}$.*

Proof. In order to prove the conclusion, we only need to show that

$$\overline{C_{p_1,p_2,q}} = C_{p_2,p_1,q}.$$

We use induction on $|p| + q = p_1 + p_2 + q$. The statement is obviously true for $|p| + q = 2$, because we have already set $C_{1,1,0} = 1$ and $C_{2,0,0} = C_{1,0,1} = C_{0,1,1} = C_{0,2,0} = C_{0,0,2} = 0$.

Assume that the statement holds true for $(p_1, p_2, q) :\ 2 \leq |p| + q < N$.

By the induction hypothesis and in view of (2.4), (2.7), (2.8) and (2.10), the statement holds true for $|p| + q = N$. This completes the proof of Corollary 2.4. □

Let

$$u = x + \mathrm{i}y, \qquad v = x - \mathrm{i}y, \qquad w = z,$$

be the inverse of the transformation (2.2) and F be the formal series in Theorem 2.1, then $G := F(u, v, w)$ is in the following form:

$$G(x, y, z) = (x^2 + y^2) + \sum_{|p|+q=3}^{\infty} g_{p_1,p_2,q} x^{p_1} y^{p_2} z^q, \tag{2.11}$$

and satisfies

$$\begin{aligned}
\left.\frac{dG}{dt}\right|_{(2.1)|_{\epsilon=0}} &= \frac{\partial G}{\partial x} X_0 + \frac{\partial G}{\partial y} Y_0 + \frac{\partial G}{\partial z} Z_0 \\
&= \left(\frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x}\right) X_0 + \left(\frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y}\right) Y_0 + \frac{\partial F}{\partial w}\frac{dw}{dz} Z_0 \\
&= \frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x} X_0 + \frac{\partial u}{\partial y} Y_0\right) + \frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x} X_0 + \frac{\partial v}{\partial y} Y_0\right) + \frac{\partial F}{\partial w} W \\
&= \frac{\partial F}{\partial u} U + \frac{\partial F}{\partial v} V + \frac{\partial F}{\partial w} W \\
&= \sum_{n=1}^{\infty} V_n (uv)^{n+1} \\
&= \sum_{n=1}^{\infty} V_n (x^2 + y^2)^{n+1}.
\end{aligned} \tag{2.12}$$

Definition 2.5. The functions V_n in (2.12), which can be expressed as polynomials in the coefficients of $(2.1)\big|_{\epsilon=0}$, i.e.,

$$c^{(1)}_{p_1,p_2,q},\, c^{(2)}_{p_1,p_2,q},\, d_{p_1,p_2,q},$$

are called the n-th order focus quantities of system $(2.1)\big|_{\epsilon=0}$.

Remark 2.6. The definition is a natural extension of the focus quantities for two-dimensional systems. For the latter case, see [12, Definition 2.2.3] and [24, Definition 3.3.3].

In Theorem 2.1, if we try any other choice of $C_{p_1,p_1,0}$ for $p_1 \geq 2$, we may get different focus quantities V'_n. However using the same idea (based on normal form theory) as in [6, 24], we can prove that: for any $s \geq 1$, we have

$$\langle V_1, V_2, \cdots, V_s \rangle = \langle V'_1, V'_2, \cdots, V'_s \rangle,$$

i.e., these two ideals are the same. Thus our definition for focus quantities is well-defined.

2.2 Focus quantities, center manifolds and Hopf bifurcations

Returning to system $(2.1)\big|_{\epsilon=0}$, for every $r \in \mathbb{N}$, according to the center manifold theorem [4, Theorem 1, Theorem 2, Theorem 3], there exists, in a sufficiently small neighborhood of the origin, a C^{r-1} center manifold $z = h(x, y)$ (which need not to be unique) such that

$$h(0, 0) = 0, \qquad Dh(0, 0) = 0$$

and

$$\frac{\partial h}{\partial x} X_0(x, y, h) + \frac{\partial h}{\partial y} Y_0(x, y, h) = Z_0(x, y, h). \tag{2.13}$$

Moreover, system $(2.1)\big|_{\epsilon=0}$ is locally topologically equivalent near the origin to the system

$$\begin{cases} \dfrac{dx}{dt} = X_0(x, y, h), \\ \dfrac{dy}{dt} = Y_0(x, y, h), \\ \dfrac{dz}{dt} = -\delta z. \end{cases}$$

In general the closed-form solution $h(x, y)$ of (2.13) is very difficult to be found. However using formal Taylor series method, we can compute an approximate cener manifold to any desired degree of accuracy.

Let

$$w = \tilde{h}(u, v) = h\left(\frac{u+v}{2}, \frac{\mathtt{i}\,(v-u)}{2}\right),$$

where the function h is the center manifold of system $(2.1)\big|_{\epsilon=0}$. In view of (2.13), we obtain by direct computation that

$$\begin{aligned}
\frac{\partial \tilde{h}}{\partial u}U+\frac{\partial \tilde{h}}{\partial v}V &= \left(\frac{\partial h}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial h}{\partial y}\frac{\partial y}{\partial u}\right)U+\left(\frac{\partial h}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial h}{\partial y}\frac{\partial y}{\partial v}\right)V \\
&= \frac{\partial h}{\partial x}\left(\frac{\partial x}{\partial u}U+\frac{\partial x}{\partial v}V\right)+\frac{\partial h}{\partial y}\left(\frac{\partial y}{\partial u}U+\frac{\partial y}{\partial v}V\right) \\
&= \frac{\partial h}{\partial x}X_0(x,y,h)+\frac{\partial h}{\partial y}Y_0(x,y,h) \\
&= Z_0(x,y,h) \\
&= W(u,v,\tilde{h}),
\end{aligned}$$

which implies that $w=\tilde{h}(u,v)$ is the center manifold of system (2.3). Using similar arguments, we can prove that: if $w=\tilde{h}(u,v)$ is a center manifold of system (2.3), then $z=h(x,y):=\tilde{h}(x+\mathrm{i}y,x-\mathrm{i}y)$ is a center manifold of system $(2.1)\big|_{\epsilon=0}$.

Now we consider the restriction of system $(2.1)\big|_{\epsilon=0}$ to the center manifold, i.e.,

$$\begin{cases} \dfrac{dx}{dt}=X_0(x,y,h), \\ \dfrac{dy}{dt}=Y_0(x,y,h). \end{cases} \tag{2.14}$$

From (2.11), we can construct

$$\widetilde{G}(x,y)=G(x,y,h)=(x^2+y^2)+\sum_{|p|+q=3}^{\infty} g_{p_1,p_2,q}x^{p_1}y^{p_2}h^q.$$

In view of (2.12), (2.13), we obtain by direct computation that

$$\begin{aligned}
\frac{d\widetilde{G}}{dt}\bigg|_{(2.14)} &= \frac{\partial \widetilde{G}}{\partial x}X_0+\frac{\partial \widetilde{G}}{\partial y}Y_0 \\
&= \left(\frac{\partial G}{\partial x}+\frac{\partial G}{\partial z}\frac{\partial h}{\partial x}\right)X_0+\left(\frac{\partial G}{\partial y}+\frac{\partial G}{\partial z}\frac{\partial h}{\partial y}\right)Y_0 \\
&= \frac{\partial G}{\partial x}X_0+\frac{\partial G}{\partial y}Y_0+\frac{\partial G}{\partial z}Z_0 \\
&= \sum_{n=1}^{\infty} V_n(x^2+y^2)^{n+1}.
\end{aligned}$$

From the identity above and [12, Definition 2.2.3] we know that V_n are also the n-th focus quantities of the restriction system (2.14).

Remark 2.7. From the above discussion, we know that the focus quantities of system $(2.1)\big|_{\epsilon=0}$ and system (2.14) are the same. This conclusion is of great importance: on the one hand, we can compute the focus quantities without recourse to center manifold reduction; on the other hand, just as in the 2-dimensional case, we can use focus quantities to analysis the Hopf bifurcations occurring on the center manifolds.

Focus quantities indicate the level of degeneration of the system $(2.1)\big|_{\epsilon=0}$. When $V_1\neq 0$, on a two-dimensional center manifold of the origin, the Hopf bifurcation occurring at $\epsilon=0$ is non-degenerate. If $V_1<0$ then there is a stable limit cycle on the center manifold for $\epsilon>0$;

the Hopf bifurcation is then called supercritical. If $V_1 > 0$ then there is an unstable limit cycle on the center manifold for $\epsilon < 0$; the Hopf bifurcation is then called subcritical.

In order to describe the occurrence of Bautin bifurcation (the co-dimensional two Hopf bifurcation), we need to consider a special type of system (2.1), i.e.,

$$\begin{cases} \dfrac{dx}{dt} = \epsilon\, x - \omega\, y + P_{1,a}(x,y,z) = X_{\epsilon,a}(x,y,z), \\ \dfrac{dy}{dt} = \omega\, x + \epsilon\, y + P_{2,a}(x,y,z) = Y_{\epsilon,a}(x,y,z), \\ \dfrac{dz}{dt} = -\delta\, z + Q_a(x,y,z) = Z_a(x,y,z), \end{cases} \tag{2.15}$$

where ω, δ are positive constants, $P_{j,a}, Q_a$ (a is a parameter) are real analytical functions without constant and linear terms, defined in a neighborhood of the origin, $j = 1,2$, and $\epsilon, a \in \mathbb{R}$ are considered as two parameters. Let $V_1(a), V_2(a)$ be the first two focus quantities of system $(2.15)\big|_{\epsilon=0}$. Suppose that $V_1(a_0) = 0, V_2(a_0) \neq 0$ and the map $(\epsilon, a) \mapsto (\epsilon, V_1(a))$ is regular (see [9,31]), then a Bautin bifurcation occurs at $\epsilon = 0, a = a_0$ on a two-dimensional center manifold of the origin. Moreover, if $V_2(a_0) < 0$ then the Bautin bifurcation is supercritical; if $V_2(a_0) > 0$ then the Bautin bifurcation is subcritical. In both cases, at most two limit cycles (on the local center manifold of the origin) can be found for the system by varying the parameters.

3 Linear stability of the equilibria

For convenience we define some test functions of the three positive parameters by

$$\begin{aligned} &k_1 = abc + b - c, \qquad k_2 = ab + bc - 1, \\ &k_3 = bc + c^2 - 1, \qquad k_4 = bc^4 + b^2c^3 - 2\,ab^2c^2 + \left(2\,ab - 3\,b^2 - 2\right)c + 3\,b, \end{aligned} \tag{3.1}$$

which are needed hereafter.

If $k_1 \geq 0$, then (1.1) has a unique equilibrium at $E_1 = (0, 1/b, 0)$; if $k_1 < 0$, then besides E_1 it has two other equilibria $E_2 = (x_0, (ac+1)/c, -x_0/c)$ and $E_3 = (-x_0, (ac+1)/c, x_0/c)$ in the fifth octant and second octant respectively, where $x_0 = \sqrt{-k_1/c}$.

Proposition 3.1. *If $k_1 > 0, k_2 > 0$, then E_1 is asymptotically stable and no other equilibrium exists for the system.*

Proof. The Jacobian matrix evaluated at E_1 is

$$J_1 := \begin{bmatrix} -a + 1/b & 0 & 1 \\ 0 & -b & 0 \\ -1 & 0 & -c \end{bmatrix}.$$

Let us denote the corresponding characteristic polynomial by

$$\begin{aligned} g_1(\lambda) &= \lambda^3 + p_{1,1}\lambda^2 + p_{1,2}\lambda + p_{1,3} \\ &= (\lambda + b)\left(\lambda^2 + \frac{(ab + bc - 1)}{b}\lambda + \frac{abc + b - c}{b}\right) \\ &= (\lambda + b)\left(\lambda^2 + \frac{k_2}{b}\lambda + \frac{k_1}{b}\right). \end{aligned} \tag{3.2}$$

If $k_1 > 0, k_2 > 0$, then this polynomial has three roots with negative real parts, which implies that the equilibrium is asymptotically stable. Because $k_1 > 0$, the system has a unique equilibrium at $E_1 = (0, 1/b, 0)$, and thus the presence of E_2 and E_3 is impossible. This completes the proof. □

Proposition 3.2. *If $k_3 > 0, k_4 > 0$, then the equilibria E_2 and E_3 are asymptotically stable.*

Proof. Due to the symmetry, we only consider the stability of E_2.

The Jacobian matrix evaluated at this equilibrium is

$$J_2 := \begin{bmatrix} 1/c & x_0 & 1 \\ -2\,x_0 & -b & 0 \\ -1 & 0 & -c \end{bmatrix}.$$

Let us denote the corresponding characteristic polynomial by

$$g_2(\lambda) = \lambda^3 + p_{2,1}\lambda^2 + p_{2,2}\lambda + p_{2,3}, \tag{3.3}$$

where the coefficients are defined by

$$p_{2,1} = \frac{bc + c^2 - 1}{c}, \qquad p_{2,2} = \frac{bc^2 + 2c{x_0}^2 - b}{c}, \qquad p_{2,3} = 2c{x_0}^2. \tag{3.4}$$

By the Routh–Hurwitz criteria, this polynomial has three roots with negative real parts if and only if

$$p_{2,1} > 0, \qquad p_{2,3} > 0, \qquad p_{2,1}p_{2,2} - p_{2,3} > 0.$$

It can be easily checked that these inequalities are equivalent to $k_3 > 0, k_4 > 0$. Thus if $k_3 > 0, k_4 > 0$, then E_2 is asymptotically stable. This completes the proof. □

4 Hopf and Bautin bifurcations of the system at E_1

In this section we study the Hopf and Bautin bifurcations at E_1. Taking a as the bifurcation parameter, that is, the coefficients in (3.2) can be rewritten as follows:

$$p_{1,1} = p_{1,1}(a), p_{1,2} = p_{1,2}(a), p_{1,3} = p_{1,3}(a).$$

According to the criterion [1, Proposition], a Hopf bifurcation occurs at a certain value of a, say $a = a_0 > 0$, if

$$p_{1,1}(a_0) \neq 0, \quad p_{1,2}(a_0) > 0, \quad p_{1,1}(a_0)p_{1,2}(a_0) - p_{1,3}(a_0) = 0, \quad \left.\frac{d[p_{1,1}p_{1,2} - p_{1,3}]}{da}\right|_{a=a_0} \neq 0.$$

More specifically, by solving this semi-algebraic system, we can concluded that a Hopf bifurcation occurs at E_1 for $a = a_0$, where $a_0 = -c + 1/b > 0$ with $b > 0, 0 < c < 1$.

For a near a_0, the Jacobian matrix at E_1 has a pair of complex conjugate eigenvalues $\lambda_{1,2}(a) = \delta(a) \pm \omega(a)\mathrm{i}$ and a negative eigenvalue $\lambda_3(a) = -b$, where $\delta(a_0) = 0, \omega(a_0) > 0$ and $\frac{d\delta}{da}(a_0) \neq 0$. This claim implies that the bifurcation at $a = a_0$ is the simple Hopf bifurcation, see [11] or Section 2 of this paper for the definition.

In order to determine the sign of $\frac{d\delta}{da}(a_0)$ and the value of $\omega(a_0)$, we consider the quadratic factor of (3.2), i.e.,

$$g_{1,1}(\lambda) := \lambda^2 + \frac{(ab+bc-1)}{b}\lambda + \frac{abc+b-c}{b}.$$

Since $g_{1,1}(\lambda_{1,2}) = 0$, we have $\delta(a) = -(ab+bc-1)/(2b)$, so

$$\frac{d\delta}{da}(a_0) = \frac{d\delta}{da}(a) = -\frac{1}{2}; \tag{4.1}$$

and

$$\lambda_1(a_0)\lambda_2(a_0) = \omega^2(a_0) = \left(\frac{abc+b-c}{b}\right)\bigg|_{a=a_0},$$

which implies that $\omega(a_0) = \sqrt{1-c^2}$.

Proposition 4.1. *For a near a_0, where $a_0 = -c+1/b > 0$ with $b > 0, 0 < c < 1$, system (1.1) has a unique equilibrium E_1, implying that Hopf bifurcation occurs at E_1 in the absence of any other equilibrium.*

Proof. From the discussion above, we know that a Hopf bifurcation occurs at E_1 for $a_0 = -c+1/b > 0$ with $b > 0, 0 < c < 1$.

Recall from (3.1) that $k_1 = abc + b - c$, thus $k_1|_{a=a_0} = b(1-c^2) > 0$. Let us think of k_1 as a function of a, which is continuous for all $a > 0$. From the continuity of this function at $a = a_0$, we have $k_1 > 0$ for a near a_0. In this case, system (1.1) has a unique equilibrium E_1, and thus the presence of the other equilibria E_2 and E_3 is impossible for a near a_0.

This completes the proof. □

By introducing the transformation

$$\begin{cases} x = \left(-c - \mathrm{i}\sqrt{1-c^2}\right)u + \left(-c + \mathrm{i}\sqrt{1-c^2}\right)v, \\ y = w + 1/b, \\ z = u + v, \end{cases} \tag{4.2}$$

the system (1.1) with $a = a_0$ becomes

$$\begin{cases} \dfrac{du}{dt} = \mathrm{i}\omega_0 u + \dfrac{\mathrm{i}\,(\mathrm{i}\omega_0 - c)}{2\omega_0}vw - \dfrac{\mathrm{i}\,(c+\mathrm{i}\omega_0)}{2\omega_0}uw, \\ \dfrac{dv}{dt} = -\mathrm{i}\omega_0 v - \dfrac{\mathrm{i}\,(\mathrm{i}\omega_0 - c)}{2\omega_0}vw + \dfrac{\mathrm{i}\,(c+\mathrm{i}\omega_0)}{2\omega_0}uw, \\ \dfrac{dw}{dt} = -bw - \mathrm{i}\left(2\,\mathrm{i}{\omega_0}^2 - 2\,\omega_0 c - \mathrm{i}\right)v^2 - 2\,uv - \mathrm{i}\left(2\,\mathrm{i}{\omega_0}^2 + 2\,\omega_0 c - \mathrm{i}\right)u^2, \end{cases} \tag{4.3}$$

where $\omega_0 := \omega(a_0) = \sqrt{1-c^2}$.

By performing computation on the first focus quantity $V_1(b,c)$ at $(u,v,w) = (0,0,0)$ of system (4.3), we get

$$V_1(b,c) = \frac{8\,c^2 + 2\,bc - 3\,b^2 - 8}{b\,(-4\,c^2 + b^2 + 4)}. \tag{4.4}$$

Let $S = \bigcup_{j=1}^{3} S_j$ be a subset of $\{(b,c) : b > 0, 0 < c < 1\}$, where

$$S_1 = \left\{ (b,c) : 0 < b < 2/3, 0 < c < \frac{-b + \sqrt{25b^2 + 64}}{8} \right\},$$
$$S_2 = \{(b,c) : 2/3 \le b \le 1, 0 < c < 1\},$$
$$S_3 = \{(b,c) : b > 1, 0 < c < 1/b\},$$

and let $U = \left\{(b,c) : 0 < b < 2/3, \frac{-b+\sqrt{25b^2+64}}{8} < c < 1\right\}$.

Before we discuss the Hopf and Bautin bifurcations of system (1.1) at E_1, we should know that these bifurcations occur on a center manifold of E_1. Due to the complexity of the expression, we only give the approximate center manifold of system (4.3) up to third order, i.e.,

$$\begin{aligned} w = &- \frac{\left(2\,\mathrm{i}c^2 - 2\sqrt{-(c-1)(c+1)}c - \mathrm{i}\right)}{\mathrm{i}b - 2\sqrt{-(c-1)(c+1)}} u^2 - \frac{2}{b} uv \\ &- \frac{\left(2\,\mathrm{i}c^2 + 2\sqrt{-(c-1)(c+1)}c - \mathrm{i}\right)}{\mathrm{i}b + 2\sqrt{-(c-1)(c+1)}} v^2 + O(|u,v|^4), \end{aligned}$$

where the cubic terms are all zero.

Theorem 4.2. *On a center manifold of system (1.1) at E_1, a supercritical Hopf bifurcation occurs at $a = a_0 = -c + 1/b$ with $(b,c) \in S$, leading to a stable limit cycle on the center manifold for $a < a_0$ and near a_0; and a subcritical Hopf bifurcation occurs at $a = a_0$ with $(b,c) \in U$, leading to an unstable limit cycle on the center manifold for $a > a_0$ and near a_0.*

Proof. Since we have assumed that $b > 0$ and $0 < c < 1$, the denominator of $V_1(b,c)$ is positive, thus the sign of $V_1(b,c)$ is only determined by its numerator. Under the constraints $a_0 = -c + 1/b > 0, b > 0$ and $0 < c < 1$, the solving of $V_1(b,c) < 0$ and $V_1(b,c) > 0$ yield the two sets of parameters: S and U, respectively. Thus the conclusion of this theorem follows from the Hopf bifurcation theorem [22, Theorem 3.15] along with the transversality condition (4.1). This completes the proof. □

By performing the computation on the second focus quantity $V_2(b,c)$, we get

$$V_2(b,c) = \frac{27\,b^3 - 54\,b^2c + 120\,b - 64\,c}{16\,(b^2 - 4c^2 + 4)^2\,(1 - c^2)}, \tag{4.5}$$

where the numerator of $V_2(b,c)$ is reduced w.r.t. that of $V_1(b,c)$. For further simplification of this quantity, we solve $V_1(b,c) = 0$ for c and obtain a unique solution $c = c_0 := -b/8 + 1/8\sqrt{25\,b^2 + 64}$, with $0 < b < 2/3$(this constraint is to make $0 < c_0 < 1$). By substituting it into $V_2(b,c)$, we get

$$\tilde{V}_2(b,c_0) = -\frac{7\,b\sqrt{25\,b^2 + 64} + 35\,b^2 + 64}{4\,b^3} < 0. \tag{4.6}$$

Hence a supercritical Bautin bifurcation may occur at E_1 for $(a,c) = (a_0^{(1)}, c_0)$, where $a_0^{(1)} = -c_0 + 1/b$ with $0 < b < 2/3$. It can easily be checked that $a_0^{(1)} > 0$.

Theorem 4.3. *On a center manifold of system* (1.1) *at* E_1 *, a supercritical Bautin bifurcation occurs at* $(a,c) = (a_0^{(1)}, c_0)$ *with* $0 < b < 2/3$*. This bifurcation generates two small amplitude limit cycles on the center manifold for the fixed parameters in the set* $\{(a,b,c) : 0 < a - a_0^{(1)} \ll c - c_0 \ll 1, 0 < b < 2/3\}$*, with the outermost cycle stable and the inner cycle unstable. Moreover in this case the equilibria* E_2, E_3 *don't exist.*

Proof. Since $0 < b < 2/3$, we have $0 < c_0 < 1$, thus from $0 < c - c_0 \ll 1$, we also have $0 < c < 1$. Moreover since $0 < b < 2/3$, we have $a_0^{(1)} = -c_0 + 1/b > 0$, thus from $0 < a - a_0^{(1)} \ll 1$, we find that $a > 0$.

To facilitate the proof, the real part of the eigenvalues $\lambda_{1,2} = \delta \pm \omega \mathtt{i}$, i.e., δ , will be treated as a function of a, b and c.

A simple computation gives

$$\delta(a_0^{(1)}, b, c_0) = V_1(b, c_0) = 0, \tag{4.7}$$

$$\frac{\partial \delta}{\partial a}(a_0^{(1)}, b, c_0) = -1/2 < 0, \qquad \frac{\partial V_1}{\partial c}(b, c_0) = \frac{5\,b\sqrt{25\,b^2 + 64} + 25\,b^2 + 64}{4\,b^2} > 0. \tag{4.8}$$

Recall from (4.6) that $\tilde{V}_2(b, c_0) < 0$.

It follows from (4.8) that the Jacobian determinant

$$\begin{vmatrix} \frac{\partial \delta}{\partial a}(a_0^{(1)}, b, c_0) & \frac{\partial \delta}{\partial c}(a_0^{(1)}, b, c_0) \\ \frac{\partial V_1}{\partial a}(b, c_0) & \frac{\partial V_1}{\partial c}(b, c_0) \end{vmatrix} = \begin{vmatrix} \frac{\partial \delta}{\partial a}(a_0^{(1)}, b, c_0) & \frac{\partial \delta}{\partial c}(a_0^{(1)}, b, c_0) \\ 0 & \frac{\partial V_1}{\partial c}(b, c_0) \end{vmatrix} < 0,$$

i.e., the map $(a,c) \mapsto (\delta(a,b,c), V_1(b,c))$ is regular at $a = a_0^{(1)}, c = c_0$.

Thus all the conditions of Bautin bifurcation are satisfied (see [9, Theorem 8.2] or [31, Theorem 2.3]), so that the conclusion on the limit cycles is proved.

For any $b \in (0, 2/3)$, we have $k_1(a_0^{(1)}, b, c_0) = \frac{b^2(\sqrt{25b^2+64} - 13b)}{32} > 0$. Thus by the continuity of $k_1(a,b,c)$ in a and c, we also have $k_1(a,b,c) > 0$ for (a,b,c) in the set $\{(a,b,c) : 0 < a - a_0^{(1)} \ll c - c_0 \ll 1, 0 < b < 2/3\}$, which implies that the equilibria E_2 and E_3 don't exist.

In summary, we complete the proof. □

5 The Hopf and Bautin bifurcations at E_2

Let us choose a as the bifurcation parameter, that is, the coefficients of (3.3) can be rewritten as follows:

$$p_{2,1} = p_{2,1}(a), \qquad p_{2,2} = p_{2,2}(a), \qquad p_{2,3} = p_{2,3}(a).$$

According to the criterion [1, Proposition], a Hopf bifurcation occurs at E_2 for a certain value of a, say $a = a_1 > 0$, if

$$p_{2,1}(a_1)p_{2,2}(a_1) - p_{2,3}(a_1) = 0, \quad p_{2,1}(a_1) \neq 0, \quad p_{2,2}(a_1) > 0, \quad \left.\frac{d\left[p_{2,1}p_{2,2} - p_{2,3}\right]}{da}\right|_{a=a_1} \neq 0.$$

More specifically, by solving this semi-algebraic system for the critical value a_1, we can concluded that a Hopf bifurcation occurs at E_2 for $a = a_1$ with $h_1 > 0, h_2 > 0, h_3 > 0$, where

$$a_1 = \frac{b^2c^3 + bc^4 - 3\,b^2c + 3\,b - 2\,c}{2\,(bc - 1)\,bc} \tag{5.1}$$

and

$$h_1 = (bc-1)\left(1-c^2\right), \quad h_2 = bc + c^2 - 1, \quad h_3 = (b^2c^3 + bc^4 - 3\,b^2c + 3\,b - 2\,c)(bc-1). \tag{5.2}$$

The condition $h_1 > 0$ follows from $p_{2,2}(a_1) > 0$. The condition $h_2 > 0$ follows from the presence of E_2. The condition $h_3 > 0$ follows from the fact that the critical value a_1 must be positive.

Proposition 5.1. *The Hopf bifurcation occurs at E_2 for $a = a_1$ is a simple Hopf bifurcation.*

Proof. For a near a_1, let $\lambda_{1,2}(a) = \delta_1(a) \pm \omega_1(a)\mathrm{i}$ and $\lambda_3(a)$ denote the three roots of (3.3). From (3.4) we know that $p_{2,3}(a) > 0$. According to Vieta's formulas, this implies $\lambda_1(a)\lambda_2(a)\lambda_3(a) = -p_{2,3}(a) < 0$. Furthermore we note that $\lambda_1(a)\lambda_2(a) = \delta_1^2(a) + \omega_1^2(a) > 0$ for a near a_1, so that $\lambda_3(a) < 0$ for a near a_1. This claim implies that the bifurcation at $a = a_1$ is a simple Hopf bifurcation. Thus we end the proof. □

With the same notations for the roots as in the proof of Proposition 5.1. If we set $a = a_1$ in (3.3), then the characteristic polynomial becomes

$$\begin{aligned} g_2(\lambda)\big|_{a=a_1} &= \lambda^3 + \frac{\left(bc + c^2 - 1\right)}{c}\lambda^2 - \frac{bc\left(c^2-1\right)}{bc-1}\lambda - \frac{b\left(bc^3 + c^4 - bc - 2\,c^2 + 1\right)}{bc-1} \\ &= \frac{\left(-1 + c^2 + (b+\lambda)\,c\right)\left(bc^3 + \left(-\lambda^2 - 1\right)bc + \lambda^2\right)}{c\,(1-bc)}. \end{aligned}$$

Thus

$$\delta_1(a_1) = 0, \qquad \omega_1(a_1) = \sqrt{\frac{bc(c^2-1)}{1-bc}}, \qquad \lambda_3(a_1) = \frac{1 - c^2 - bc}{c}.$$

Let

$$\begin{aligned} h_{3,0} &= b^2c^3 + bc^4 - 3\,b^2c + 3\,b - 2\,c \\ &= c\left(c^2-3\right)b^2 + \left(c^4+3\right)b - 2\,c, \end{aligned} \tag{5.3}$$

which is a factor of h_3, seen in (5.2). Before checking the sign of $\delta_1'(a_1)$, we need the following lemma.

Lemma 5.2. *If $h_1 > 0, h_3 > 0$, then $c > 1, bc < 1$ and $h_{3,0} < 0$.*

Proof. Since $h_1 > 0$, $c \neq 1$. Suppose that $0 < c < 1$. Since $h_1 > 0$, $bc - 1 > 0$.

According to the Taylor expansion formula, we rewrite $h_{3,0}$ in the following way:

$$h_{3,0} = c^3 - c + \left(c^4 + 2\,c^2 - 3\right)\left(b - c^{-1}\right) + \left(c^3 - 3\,c\right)\left(b - c^{-1}\right)^2.$$

It follows from $h_3 > 0$ and $bc - 1 > 0$ that $h_{3,0} > 0$. However, $b - c^{-1} > 0$ and $c^3 - c < 0$, $c^4 + 2\,c^2 - 3 < 0$, $c^3 - 3\,c < 0$ for $0 < c < 1$, which makes $h_{3,0} < 0$. Thus we have reached a contradiction and so that $c > 1$.

Since $c > 1$ and $h_1 > 0$, $bc < 1$. Thus it follows from $h_3 > 0$ that $h_{3,0} < 0$.

In summary, we end the proof of this lemma. □

Corollary 5.3. *The conditions $h_1 > 0, h_3 > 0$ imply $h_2 > 0$.*

Proof. Assume $h_1 > 0, h_3 > 0$, it follows from Lemma 5.2 that $c > 1$. Recalling from (5.2) the expression of h_2, we have $h_2 > 0$. This completes the proof. □

As a direct consequence of Corollary 5.3, we have the following result.

Corollary 5.4. *The Hopf bifurcation set is*

$$S := \{(a, b, c) : a = a_1, h_1 > 0, h_3 > 0\}. \tag{5.4}$$

With the same notations for the roots as in the proof of Proposition 5.1, we have the following result.

Corollary 5.5. *The conditions $h_1 > 0, h_3 > 0$ imply $\delta_1'(a_1) < 0$.*

Proof. Suppose that $h_1 > 0, h_3 > 0$. Then, by Lemma 5.2 and Corallary 5.4, we know that $c > 1$ and a Hopf bifurcation occurs at $a = a_1$.

Recalling (3.3), the occurrence of Hopf bifurcation implies that

$$[p_{2,1}p_{2,2} - p_{2,3}]'(a_1) = \frac{2\,b\,(1 - bc)}{c} \neq 0. \tag{5.5}$$

Since $c > 1$ and $h_1 > 0$, we have $1 - bc > 0$ and (5.5) is positive. From the proof of [1, Proposition], we know that the sign of $\delta_1'(a_1)$ is different from that of (5.5), so we complete the proof. □

Remark 5.6. If we treat δ_1 as a function of a, b and c, then by Corollary 5.5, we have $\frac{\partial \delta_1}{\partial a}(a_1, b, c) < 0$. This fact will be used in the future.

The following result is the converse of Lemma 5.2.

Lemma 5.7. *If $c > 1, bc < 1$ and $h_{3,0} < 0$, then $h_1 > 0, h_3 > 0$.*

Proof. Since $c > 1$ and $bc < 1$, $h_1 > 0$. Since $bc < 1$ and $h_{3,0} < 0$, $h_3 > 0$. In summary, we complete the proof of this lemma. □

As a direct consequence of Lemma 5.2, Lemma 5.7 and Corallary 5.4, we have the following result.

Corollary 5.8. *The Hopf bifurcation set S defined by* (5.4) *can be implicitly rewritten as follows:*

$$S = \{(a, b, c) : a = a_1,\ c > 1,\ bc < 1,\ h_{3,0} < 0\}. \tag{5.6}$$

For later use, let

$$S^* := \{(b, c) : c > 1,\ bc < 1,\ h_{3,0} < 0\}. \tag{5.7}$$

We now seek to find the explicit representation of the bifurcation set S, which is described by (5.6). To achieve this goal, we need the following lemmas, which are related to the roots of polynomial $h_{3,0}$ in b, seen in (5.3).

For $c > 1$ and $c \neq \sqrt{3}$, let Δ be the discriminant of $h_{3,0}$ with respect to b, i.e.,

$$\Delta = (c^4 + 3)^2 + 8c^2(c^2 - 3). \tag{5.8}$$

Lemma 5.9. *For $c > 1$, we have $\Delta > 0$.*

Proof. This inequality can be proved by noting that $c^2 > 1$ and (5.8) can be rewritten as follows:

$$\Delta = (c^2 - 1)[(c^2 - 1)(c^4 + 2c^2 + 17) + 8],$$

which is positive when $c > 1$. So that $\Delta > 0$. Thus we complete the proof of this lemma. $\square$

For $c > 1$ and $c \neq \sqrt{3}$, according to Lemma 5.9, the quadratic polynomial $h_{3,0}$ has two distinct roots for b. By the direct computations, these roots can be represented by

$$\tau_1 := \frac{-c^4 - 3 + \sqrt{\Delta}}{2c(c^2 - 3)}, \qquad \tau_2 := \frac{-c^4 - 3 - \sqrt{\Delta}}{2c(c^2 - 3)}.$$

It can be easily checked that $\tau_1 > 0$ for $c > 1$ and $c \neq \sqrt{3}$. The sign of τ_2 is positive for $1 < c < \sqrt{3}$ and negative for $c > \sqrt{3}$.

Lemma 5.10. *For $1 < c < \sqrt{3}$, we have*

$$\tau_1 < \frac{1}{c} < \tau_2.$$

Proof. Since $1 < c < \sqrt{3}$, the left inequality is equivalent to

$$c^4 + 3 - 2(3 - c^2) < \sqrt{(c^4 + 3)^2 + 8c^2(c^2 - 3)}. \tag{5.9}$$

Note that $c^4 + 3 - 2(3 - c^2) = (c^2 - 1)(c^2 + 3) > 0$. By squaring and rearranging, the desired inequality (5.9) can be reduced to $c^2 > 1$, which is obviously true.

Since $1 < c < \sqrt{3}$, the right inequality is equivalent to

$$\sqrt{(c^4 + 3)^2 + 8c^2(c^2 - 3)} > 2(3 - c^2) - (c^4 + 3).$$

This is obviously true because the right hand side equals to $(-c^2 + 1)(c^2 + 3)$, which is negative for $1 < c < \sqrt{3}$.

In summary, we complete the proof of this lemma. $\square$

Lemma 5.11. *For $c > \sqrt{3}$, we have*

$$\tau_1 < \frac{1}{c}.$$

Proof. Since $c > \sqrt{3}$, the inequality is equivalent to

$$c^4 + 3 - 2(3 - c^2) > \sqrt{(c^4 + 3)^2 + 8c^2(c^2 - 3)}, \tag{5.10}$$

Both sides are positive. By squaring and rearranging, the desired inequality (5.10) can be reduced to $c^2 > 1$ which is obviously true. This completes the proof. $\square$

Theorem 5.12. *The Hopf bifurcation set S defined by (5.4) can be rewritten as follows:*

$$S = \{(a, b, c) : a = a_1, (b, c) \in S_1 \cup S_2 \cup S_3\}, \tag{5.11}$$

where

$$S_1 = \left\{(b, c) : 0 < b < \tau_1, 1 < c < \sqrt{3}\right\},$$
$$S_2 = \left\{(b, c) : 0 < b < \frac{\sqrt{3}}{6}, c = \sqrt{3}\right\},$$
$$S_3 = \left\{(b, c) : 0 < b < \tau_1, c > \sqrt{3}\right\}.$$

Proof. According to Corollary 5.8, it suffices to get the solution set of the following inequalities:

$$c > 1, \qquad bc < 1, \qquad h_{3,0} < 0. \tag{5.12}$$

To prove (5.11), we consider three cases.

(1) Assume that $1 < c < \sqrt{3}$. Then according to Lemma 5.10, the solving of $bc < 1$, $h_{3,0} < 0$ for b yields $0 < b < \tau_1$.

(2) Assume that $c = \sqrt{3}$. Then $h_{3,0} = 12b - 2\sqrt{3}$, and the solving of $bc < 1$, $h_{3,0} < 0$ for b yields $0 < b < \frac{\sqrt{3}}{6}$.

(3) Assume that $c > \sqrt{3}$. Then according to Lemma 5.11, the solving of $bc < 1$, $h_{3,0} < 0$ for b yields $0 < b < \tau_1$.

Summing up these conclusions, we complete the proof. □

If $a = a_1$, then

$$E_2 = \left(\frac{\sqrt{2}}{2} m, \frac{b^2c^3 + bc^4 - b^2c + b - 2\,c}{2\,c\,(bc-1)\,b}, -\frac{\sqrt{2}m}{2\,c} \right),$$

where

$$m = \sqrt{\frac{b\,(c^2-1)\,(bc+c^2-1)}{c\,(1-bc)}}.$$

By introducing the transformation

$$\begin{cases} x = (s_1 + s_2\mathrm{i})u + (s_1 - s_2\mathrm{i})v + s_3 w + \dfrac{\sqrt{2}}{2} m, \\ y = (s_4 + s_5\mathrm{i})u + (s_4 - s_5\mathrm{i})v + s_6 w + \dfrac{b^2c^3 + bc^4 - b^2c + b - 2\,c}{2\,c\,(bc-1)\,b} \\ z = u + v + w - \dfrac{\sqrt{2}m}{2\,c}, \end{cases}$$

where

$$s_1 = -c, \qquad s_2 = -\omega_1\,(a_1, b, c)\,, \qquad s_3 = \frac{bc-1}{c},$$

$$s_4 = \frac{\sqrt{2}c^2 m}{bc + c^2 - 1}, \qquad s_5 = \frac{\sqrt{2}\,(bc-1)\,m\omega_1\,(a_1, b, c)}{b\,(bc+c^2-1)}, \qquad s_6 = \frac{\sqrt{2}\,(bc-1)\,m}{c^2-1}$$

and the notation ω_1, which appeared in the proof of Proposition 5.1, is now considered as a function of a, b and c, system $(1.1)|_{a=a_1}$ becomes

$$\begin{cases} \dfrac{du}{dt} = \mathrm{i}\omega_1(a_1, b, c)u + P_1(u, v, w), \\ \dfrac{dv}{dt} = -\mathrm{i}\omega_1(a_1, b, c)v + P_2(u, v, w), \\ \dfrac{dw}{dt} = \lambda_3(a_1, b, c)w + P_3(u, v, w), \end{cases} \tag{5.13}$$

where $P_j(u, v, w)$, $j = 1, 2, 3$ are homogeneous quadratic polynomials, which are too complicated to be presented here.

By performing computation on the first two focus quantities for system (5.13), we get

$$V_1(b,c) = \frac{2\,bc^3 n_{11}}{n_{12} n_{13}}, \tag{5.14}$$

$$V_2(b,c) = -\frac{c^4 n_{21}}{3\,n_{22}}, \tag{5.15}$$

where

$$\begin{aligned}
n_{11} = {} & 3\,b^5c^4 + 13\,b^4c^5 - 15\,b^3c^6 - 5\,b^2c^7 + 4\,bc^8 - 21\,b^4c^3 - 23\,b^3c^4 + 37\,b^2c^5 - 7\,bc^6 - 2\,c^7 \\
& + 45\,b^3c^2 + 5\,b^2c^3 - 10\,bc^4 + 8\,c^5 - 39\,b^2c + bc^2 - 10\,c^3 + 12\,b + 4\,c, \\
n_{12} = {} & b^4c^4 + 3\,b^3c^5 - b^2c^6 - 3\,bc^7 - 4\,b^3c^3 - 5\,b^2c^4 + 2\,bc^5 - c^6 + 6\,b^2c^2 \\
& + 5\,bc^3 + 3\,c^4 - 4\,bc - 3\,c^2 + 1, \\
n_{13} = {} & b^3c^3 + 2\,b^2c^4 - 3\,b^2c^2 - 3\,bc^3 - c^4 + 3\,bc + 2\,c^2 - 1, \\
n_{22} = {} & (c-1)\,(c+1)\left(b^3c^3 + 2\,b^2c^4 - 8\,bc^5 - 3\,b^2c^2 + 5\,bc^3 - c^4 + 3\,cb + 2\,c^2 - 1\right) \\
& \times \left(b^3c^3 + 2\,b^2c^4 - 3\,bc^5 - 3\,b^2c^2 - c^4 + 3\,cb + 2\,c^2 - 1\right)^2 \left(cb + c^2 - 1\right)^3 \\
& \times \left(b^3c^3 + 2\,b^2c^4 - 3\,b^2c^2 - 3\,bc^3 - c^4 + 3\,cb + 2\,c^2 - 1\right)^3,
\end{aligned}$$

and the expression of n_{21} is somewhat complicated and can be found in the Appendix.

Before we discuss the Hopf and Bautin bifurcations of system (1.1) at E_2, we should know that these bifurcations occur on a center manifold of E_2. Due to the complexity of the quadratic approximation of center manifold of system (5.13), we will not present here.

According to the Hopf bifurcation theorem [22, Theorem 3.15] and Remark 5.6, we have the following theorem.

Theorem 5.13. *Assume that $(b,c) \in S^*$, where S^* is described by (5.7) and $V_1(b,c) \neq 0$. On a center manifold of system (1.1) at E_2, a Hopf bifurcation occurs at $a = a_1$. More precisely, the bifurcation is supercritical for $V_1(b,c) < 0$, giving rise to a stable limit cycle on the center manifold for $a < a_1$; and subcritical for $V_1(b,c) > 0$ giving rise to an unstable limit cycle on the center manifold for $a > a_1$.*

In Maple 2016 (a computer algebra system), the command *RootFinding[Isolate]* isolates the real roots of univariate polynomials and polynomial systems with a finite number of solutions. By default it computes isolating intervals for each of the roots and numerically evaluates the midpoints of those intervals at the current setting of digits. All significant digits returned by the program are correct, and unlike purely numerical methods no roots are ever lost.

Now we consider the semi-algebraic system

$$n_{11} = n_{21} = 0, \qquad (b,c) \in S^*, \tag{5.16}$$

where S^* is described by (5.7).

Using the command *RootFinding[Isolate]*, we find there is no solution to (5.16) with $(b,c) \in S^*$. So there is no need to calculate $V_3(b,c)$, and thus the system can have at most two small limit cycles in some neighborhood of E_2. Due to the symmetry, at most four small limit cycles can be found on the center manifolds that spiral around the equilibria E_2 and E_3.

Theorem 5.14. *On a center manifold of system* $(1.1)|_{c=3/2}$ *at* E_2*, a subcritical Bautin bifurcation occurs at* $(a,b) = (9/2, 1/6)$*, which leads to two small amplitude limit cycles on the center manifold, with the outermost cycle unstable and the inner cycle stable, for*

$$0 < 9/2 - a \ll 1/6 - b \ll 1. \tag{5.17}$$

Moreover in this case both E_1 *and* E_2 *are unstable.*

Proof. Let

$$b = b_1 := 1/6, \qquad c = c_1 := 3/2. \tag{5.18}$$

Then from (5.1) we obtain $a_1 = 9/2$. It can be easily checked that $(a,b,c) = (9/2, 1/6, 3/2) \in S$, where S is the bifurcation set described by (5.11).

Let $\delta_1(a,b,c)$ be the real part of the complex conjugate roots of (3.3). A simple computation gives

$$\begin{aligned} \delta_1(9/2, 1/6, 3/2) &= 0, & V_1(1/6, 3/2) &= 0, \\ V_2(1/6, 3/2) &= 44000/969 > 0, & \frac{\partial V_1}{\partial b}(1/6, 3/2) &= 27/17 > 0. \end{aligned} \tag{5.19}$$

Recall from Remark 5.6 that

$$\frac{\partial \delta_1}{\partial a}(9/2, 1/6, 3/2) < 0. \tag{5.20}$$

From (5.19)–(5.20), we can concluded that the map $(a,b) \mapsto (\delta_1(a,b,3/2), V_1(b,3/2))$ is regular at $a = 9/2, b = 1/6$.

Thus the conditions of Bautin bifurcation are fullfilled, so that the conclusion on the limit cycles is proved.

Since the inner cycle is asymptotically stable, E_2 is unstable. We recall from (3.1) the definition of k_1 and k_2. For system $(1.1)|_{c=3/2}$, we have $k_1 = -5/24$, $k_2 = 0$ if $a = a_1 = 9/2$, $b = b_1 = 1/6$. By imposing a small perturbation satisfying (5.17) on (a,b), we have $k_1 < 0$ by the continuity, and $k_2 < 0$ because

$$\frac{\partial k_2}{\partial a}\Big|_{a=a_1, b=b_1, c=3/2} = 1/6, \qquad \frac{\partial k_2}{\partial b}\Big|_{a=a_1, b=b_1, c=3/2} = 6,$$

which implies (3.2) has one positive root, and thus E_1 is unstable. □

Remark 5.15. We have tried the cases with $c = 2$ and $c = \sqrt{3}$. For each case, if a Bautin bifurcation occurs at E_2, we can checked that it is also subcritical.

6 Concluding remarks

We investigated a financial system that describes the development of interest rate, investment demand and price index. By performing computations on focus quantities using the recursive formula, we derived the conditions at which limit cycles can bifurcate from the equilibria E_1 and $E_{2,3}$, respectively. The stabilities of the bifurcated limit cycles were also investigated in detail. Based on the analysis of Bautin bifurcations, it was proved that the system have at most four small limit cycles on the center manifolds and this bound is sharp.

Appendix: The expression of n_{21}

$$
\begin{aligned}
n_{21} = {} & 1188\, b^{19}c^{18} + 13272\, b^{18}c^{19} + 37844\, b^{17}c^{20} - 40132\, b^{16}c^{21} - 291708\, b^{15}c^{22} - 128756\, b^{14}c^{23} \\
& + 605812\, b^{13}c^{24} + 345204\, b^{12}c^{25} - 473048\, b^{11}c^{26} - 250196\, b^{10}c^{27} + 139752\, b^{9}c^{28} \\
& + 56768\, b^{8}c^{29} - 12928\, b^{7}c^{30} - 3072\, b^{6}c^{31} - 1431\, b^{19}c^{16} - 38598\, b^{18}c^{17} - 247295\, b^{17}c^{18} \\
& - 334960\, b^{16}c^{19} + 1246686\, b^{15}c^{20} + 3106212\, b^{14}c^{21} - 1028810\, b^{13}c^{22} - 5559912\, b^{12}c^{23} \\
& - 296783\, b^{11}c^{24} + 3582082\, b^{10}c^{25} + 522393\, b^{9}c^{26} - 873144\, b^{8}c^{27} - 142792\, b^{7}c^{28} \\
& + 55920\, b^{6}c^{29} + 11968\, b^{5}c^{30} - 1536\, b^{4}c^{31} + 27189\, b^{18}c^{15} + 439329\, b^{17}c^{16} + 1861676\, b^{16}c^{17} \\
& + 527976\, b^{15}c^{18} - 10176796\, b^{14}c^{19} - 12244022\, b^{13}c^{20} + 12466048\, b^{12}c^{21} + 18577936\, b^{11}c^{22} \\
& - 5772331\, b^{10}c^{23} - 9479363\, b^{9}c^{24} + 1155748\, b^{8}c^{25} + 2004560\, b^{7}c^{26} + 21386\, b^{6}c^{27} \\
& - 135760\, b^{5}c^{28} + 6968\, b^{4}c^{29} + 1376\, b^{3}c^{30} - 236115\, b^{17}c^{14} - 2726802\, b^{16}c^{15} \\
& - 7857945\, b^{15}c^{16} + 4608714\, b^{14}c^{17} + 41182386\, b^{13}c^{18} + 23063252\, b^{12}c^{19} - 45484929\, b^{11}c^{20} \\
& - 29581420\, b^{10}c^{21} + 20582413\, b^{9}c^{22} + 10664314\, b^{8}c^{23} - 5218255\, b^{7}c^{24} - 1976930\, b^{6}c^{25} \\
& + 389644\, b^{5}c^{26} + 41348\, b^{4}c^{27} - 8383\, b^{3}c^{28} - 524\, b^{2}c^{29} + 1252125\, b^{16}c^{13} + 10918551\, b^{15}c^{14} \\
& + 21245184\, b^{14}c^{15} - 28757748\, b^{13}c^{16} - 100970458\, b^{12}c^{17} - 18645262\, b^{11}c^{18} \\
& + 85763195\, b^{10}c^{19} + 20207125\, b^{9}c^{20} - 30605620\, b^{8}c^{21} - 1905598\, b^{7}c^{22} + 6857765\, b^{6}c^{23} \\
& + 399881\, b^{5}c^{24} - 216087\, b^{4}c^{25} + 6765\, b^{3}c^{26} - 3256\, b^{2}c^{27} - 770\, bc^{28} - 4557735\, b^{15}c^{12} \\
& - 30754338\, b^{14}c^{13} - 39581691\, b^{13}c^{14} + 81588724\, b^{12}c^{15} + 165215238\, b^{11}c^{16} \\
& - 4326132\, b^{10}c^{17} - 92274188\, b^{9}c^{18} + 5569390\, b^{8}c^{19} + 22435760\, b^{7}c^{20} - 6611584\, b^{6}c^{21} \\
& - 3811154\, b^{5}c^{22} - 52316\, b^{4}c^{23} - 15539\, b^{3}c^{24} + 34034\, b^{2}c^{25} + 6757\, bc^{26} - 10\, c^{27} \\
& + 12110553\, b^{14}c^{11} + 64170297\, b^{13}c^{12} + 53791584\, b^{12}c^{13} - 143868664\, b^{11}c^{14} \\
& - 190762524\, b^{10}c^{15} + 19355706\, b^{9}c^{16} + 50687426\, b^{8}c^{17} - 23208704\, b^{7}c^{18} - 7524470\, b^{6}c^{19} \\
& + 6037900\, b^{5}c^{20} + 1821722\, b^{4}c^{21} + 367794\, b^{3}c^{22} - 78085\, b^{2}c^{23} - 25401\, bc^{24} + 130\, c^{25} \\
& - 24351327\, b^{13}c^{10} - 102580170\, b^{12}c^{11} - 57504784\, b^{11}c^{12} + 170813154\, b^{10}c^{13} \\
& + 160635804\, b^{9}c^{14} - 6959884\, b^{8}c^{15} + 1759762\, b^{7}c^{16} + 23307350\, b^{6}c^{17} + 975470\, b^{5}c^{18} \\
& - 3605064\, b^{4}c^{19} - 1503672\, b^{3}c^{20} - 32152\, b^{2}c^{21} + 51339\, bc^{22} - 770\, c^{23} + 37857105\, b^{12}c^{9} \\
& + 128522559\, b^{11}c^{10} + 54630019\, b^{10}c^{11} - 137438047\, b^{9}c^{12} - 99311849\, b^{8}c^{13} \\
& - 14665085\, b^{7}c^{14} - 25553443\, b^{6}c^{15} - 14620145\, b^{5}c^{16} + 1193479\, b^{4}c^{17} + 2733231\, b^{3}c^{18} \\
& + 477303\, b^{2}c^{19} - 52325\, bc^{20} + 2750\, c^{21} - 46042425\, b^{11}c^{8} - 128091942\, b^{10}c^{9} \\
& - 51612909\, b^{9}c^{10} + 68816794\, b^{8}c^{11} + 42425383\, b^{7}c^{12} + 20105230\, b^{6}c^{13} + 20030083\, b^{5}c^{14} \\
& + 4744022\, b^{4}c^{15} - 2441874\, b^{3}c^{16} - 1047648\, b^{2}c^{17} - 1890\, bc^{18} - 6600\, c^{19} + 43996095\, b^{10}c^{7} \\
& + 102307227\, b^{9}c^{8} + 47060600\, b^{8}c^{9} - 11723762\, b^{7}c^{10} - 8446819\, b^{6}c^{11} - 10728251\, b^{5}c^{12} \\
& - 6717822\, b^{4}c^{13} + 821340\, b^{3}c^{14} + 1255302\, b^{2}c^{15} + 90246\, bc^{16} + 11220\, c^{17} - 32945913\, b^{9}c^{6} \\
& - 65403030\, b^{8}c^{7} - 36340229\, b^{7}c^{8} - 11528220\, b^{6}c^{9} - 3566953\, b^{5}c^{10} + 1770458\, b^{4}c^{11} \\
& - 81455\, b^{3}c^{12} - 1005780\, b^{2}c^{13} - 151578\, bc^{14} - 13860\, c^{15} + 19142487\, b^{8}c^{5} + 33056253\, b^{7}c^{6} \\
& + 21400794\, b^{6}c^{7} + 10951260\, b^{5}c^{8} + 3792381\, b^{4}c^{9} + 853915\, b^{3}c^{10} + 663614\, b^{2}c^{11} \\
& + 150372\, bc^{12} + 12540\, c^{13} - 8464365\, b^{7}c^{4} - 12858582\, b^{6}c^{5} - 8983149\, b^{5}c^{6} - 4666242\, b^{4}c^{7}
\end{aligned}
$$

$$- 1483503\, b^3c^8 - 446054\, b^2c^9 - 104575\, bc^{10} - 8250\, c^{11} + 2754675\, b^6c^3 + 3671691\, b^5c^4$$
$$+ 2511300\, b^4c^5 + 1094354\, b^3c^6 + 279679\, b^2c^7 + 52835\, bc^8 + 3850\, c^9 - 622485\, b^5c^2$$
$$- 709902\, b^4c^3 - 418506\, b^3c^4 - 125336\, b^2c^5 - 18753\, bc^6 - 1210\, c^7 + 87291\, b^4c + 79881\, b^3c^2$$
$$+ 32515\, b^2c^3 + 4179\, bc^4 + 230\, c^5 - 5724\, b^3 - 3612\, b^2c - 436\, bc^2 - 20\, c^3.$$

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