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Combined effects of concave and convex nonlinearities in nonperiodic fourth-order equations

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Abstract. In this paper, we consider the multiplicity of nontrivial solutions for a class of nonperiodic fourth-order equation with concave and convex nonlinearities. Based on the Nehari manifold and Ekeland variational principle, we prove that the equation has at least two solutions under some proper assumptions. Moreover, one solution is a ground state solution.

Keywords: nonperiodic fourth-order equation, Nehari manifold, Ekeland variational principle, ground state solution.

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1 Introduction

The purpose of this paper is to consider the multiplicity of nontrivial solutions for the following fourth-order differential equation:

$$u^{(4)} + wu'' + a(x)u = f(x)|u|^{q-2}u + g(x)|u|^{p-2}u, \qquad x \in \mathbb{R}, \tag{1.1}$$

where $1 < q < 2 < p < +\infty$, $a(x)$, $f(x)$ and $g(x)$ are continuous functions and satisfy suitable conditions. This equation has been used to solve some problems associated to mathematical model for the study of pattern formation in physic and mechanics. There are many papers considered fourth-order differential equations, see [1,2,6–8,10–12,14–16,21] for example. Some authors researched the well-known extended Fisher–Kolmogorov equations (see [4,5]) and the Swift–Hohenberg equations (see [9,17]). With suitable changes of variables, the stationary solutions to the above equations lead to consider the following fourth-order equation

$$u^{(4)} + wu'' - u + u^3 = 0,$$

where $w > 0$ corresponds to the extended Fisher–Kolmogorov equations and $w < 0$ to the Swift–Hohenberg equations. In the past years, by critical point theory and variational methods, many researchers are interested in the existence of homoclinic solutions for the following equation

$$u^{(4)} + wu'' + a(x)u = c(x)u^2 + d(x)u^3,$$

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where $a(x)$, $c(x)$, $d(x)$ are independent of x or T-periodic in x, see [7, 13, 14, 18] and the reference therein. In [18], applying the mountain pass theorem, the authors showed that the equation possesses one nontrivial homoclinic solution $u \in H^2(\mathbb{R})$, when $a(x)$, $c(x)$ and $d(x)$ are continuous periodic functions and satisfy some other assumptions. If there is no periodicity assumption of $a(x)$, $c(x)$ and $d(x)$, then the study will be more difficult. Very recently, Sun and Wu [15] considered a class of fourth order differential equations with a perturbation:

$$u^{(4)} + wu'' + a(x)u = f(x,u) + \lambda h(x)|u|^{p-2}u, \qquad x \in \mathbb{R}$$

where $\lambda > 0$ is a parameter, $1 \leq p < 2$ and $h \in L^{\frac{2}{2-p}}(\mathbb{R})$. By using variational methods, the existence result of two homoclinic solutions for the above equation is obtained if the parameter λ is small enough. In [11, 16], the authors considered the equation

$$u^{(4)} + wu'' + \lambda a(x)u = f(x,u), \qquad x \in \mathbb{R},$$

by using variational methods, they get the existence of homoclinic solutions. Motivated by these papers mentioned above, we consider the fourth-order differential equation (1.1) with concave-convex nonlinearities on the whole space $\mathbb{R}$. To our best knowledge, there are few papers which deal with this type of fourth-order differential equation by using Nehari manifold. The main difficulties lie in the boundedness of the domain $\mathbb{R}$ and the presence of the concave-convex nonlinearities.

In order to get our main results, we assume that $a(x)$, $f(x)$ and $g(x)$ satisfy the following conditions:

(H_1) $a \in C(\mathbb{R}, \mathbb{R})$, there exists a positive constant a_1 such that $0 < a_1 < a(x) \to +\infty$ as $|x| \to +\infty$ and $w \leq 2\sqrt{a_1}$;

(H_2) $f \in C(\mathbb{R}) \bigcap L^{q^*}(\mathbb{R})$, $q^* = \frac{p}{p-q}$;

(H_3) $g \in C(\mathbb{R}) \bigcap L^{\infty}(\mathbb{R})$ and $g(x) > 0$, for almost every $x \in \mathbb{R}$.

In the problem (1.1), the presence of the concave-convex nonlinearities prevents us from using the Nehari manifold method in a standard way. Motivated by [3, 19], we split the Nehari manifold into three parts which are then considered separately. Here are our main results:

Theorem 1.1. *Under the assumptions (H_1)–(H_3), if $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0, \sigma)$, then the problem (1.1) has at least two nontrivial solutions, one of which corresponds to negative energy and the other corresponds to positive energy, where $\sigma = (p-2)(2-q)^{(2-q)/(p-2)}(S_p/(p-q))^{(p-q)/(p-2)}$ and S_p is the best Sobolev constant described in Section 2.*

Remark 1.2. In problem (1.1), because of the unboundedness of the domain $\mathbb{R}$, we need the hypothesis (H_1), which is used to establish the corresponding compact embedding lemmas on suitable functional spaces, see Lemma 2 in [8], Lemma 2.2 in [15] and Lemma 2.2 in [10].

Theorem 1.3. *Under the assumptions (H_1)–(H_3), if $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0, \sigma^*)$, then the problem (1.1) has at least two nontrivial solutions, one of which corresponds to negative energy and the other corresponds to positive energy. Moreover, the solution corresponding to the negative energy is a ground state solution, where $0 < \sigma^* := \frac{q}{2}\sigma < \sigma$.*

Remark 1.4. On the one hand, from the condition (H_2), we can easily conclude that $f(x)$ is allowed to be sign-changing. On the other hand, to the best of our knowledge, there are few papers which obtain the ground state solutions of fourth-order equations, so our results complete the existence of solutions for fourth-order differential equations.

2 Preliminaries

First, we present the definition of ground state solutions, Palais–Smale (denoted by (PS)) sequences and (PS) value for J as follows.

Definition 2.1.

(i) u is called a ground state solution of equation (1.1), if $J(u)$ is the least level for J at the nontrivial solutions of (1.1), where J denotes the energy functional corresponding to (1.1).

(ii) For $c \in \mathbb{R}$, a sequence $\{u_n\}$ is a $(\mathrm{PS})_c$-sequence in $H^2(\mathbb{R})$ for J if $J(u_n) = c + o(1)$ and $J'(u_n) = o(1)$ strongly in $(H^2(\mathbb{R}))'$ as $n \to \infty$, where $(H^2(\mathbb{R}))'$ is the dual space of $H^2(\mathbb{R})$.

(iii) $c \in \mathbb{R}$ is a (PS)-value in $H^2(\mathbb{R})$ for J if there is a $(\mathrm{PS})_c$-sequence in $H^2(\mathbb{R})$ for J.

Lemma 2.2 (See Lemma 8 in [18]). *Assume that $a(x) \geq a_1 > 0$ and $w \leq 2\sqrt{a_1}$. Then there exists a constant $c_0 > 0$, such that*

$$\int_{\mathbb{R}} [u''(x)^2 - wu'(x)^2 + a(x)u(x)^2]dx \geq c_0 \|u\|_{H^2}^2, \tag{2.1}$$

for all $u \in H^2(\mathbb{R})$, where $\|u\|_{H^2} = \left(\int_{\mathbb{R}} [u''(x)^2 + u(x)^2]dx \right)^{1/2}$ is the norm of Sobolev space $H^2(\mathbb{R})$.

By Lemma 2.2, we define

$$X := \left\{ u \in H^2(\mathbb{R}) \;\middle|\; \int_{\mathbb{R}} [u''(x)^2 - wu'(x)^2 + a(x)u(x)^2]dx < +\infty \right\},$$

with the inner product

$$(u,v) = \int_{\mathbb{R}} [u''(x)v''(x) - wu'(x)v'(x) + a(x)u(x)v(x)]dx,$$

and the corresponding norm

$$\|u\| = \left(\int_{\mathbb{R}} [u''(x)^2 - wu'(x)^2 + a(x)u(x)^2]dx \right)^{1/2}.$$

It is easy to verify that X is a Hilbert space.

Now we begin describing the variational formulation of the problem (1.1). Consider the functional $J : X \to \mathbb{R}$, defined by

$$\begin{aligned} J(u) &= \frac{1}{2} \int_{\mathbb{R}} [u''(x)^2 - wu'(x)^2 + a(x)u(x)^2]dx - \frac{1}{q} \int_{\mathbb{R}} f(x)|u|^q dx - \frac{1}{p} \int_{\mathbb{R}} g(x)|u|^p dx \\ &= \frac{1}{2}\|u\|^2 - \frac{1}{q} \int_{\mathbb{R}} f(x)|u|^q dx - \frac{1}{p} \int_{\mathbb{R}} g(x)|u|^p dx, \qquad u \in X. \end{aligned} \tag{2.2}$$

Lemma 2.3. *If (H_1)–(H_3) hold, then the functional $J \in C^1(X, \mathbb{R})$, and for any $u,\ v \in X$,*

$$\begin{aligned} \langle J'(u), v \rangle = &\int_{\mathbb{R}} [u''(x)v''(x) - wu'(x)v'(x) + a(x)u(x)v(x)]dx \\ &- \int_{\mathbb{R}} f(x)|u|^{q-2}uv dx - \int_{\mathbb{R}} g(x)|u|^{p-2}uv dx. \end{aligned} \tag{2.3}$$

The proof of Lemma 2.3 is a direct computation under (H_1)–(H_3). Then we can infer that $u \in X$ is a critical point of J if and only if it is a solution of problem (1.1). Moreover, as pointed out previously, assumption (H_1) is used to recover compactness of embedding theorem, which is given below.

Lemma 2.4 (See [15]). *Assume that condition (H_1) holds, then the embedding $X \hookrightarrow L^p(\mathbb{R})$ is continuous for $p \in [2, \infty]$, and compact for $p \in [2, \infty)$.*

Throughout this paper, we denote by S_p the best Sobolev constant for the embedding $X \hookrightarrow L^p(\mathbb{R})$, which is given by

$$S_p = \inf_{u \in X \setminus \{0\}} \frac{\|u\|^2}{\left(\int_{\mathbb{R}} |u|^p dx\right)^{2/p}} > 0.$$

In particular, for $\forall u \in X \setminus \{0\}$, $|u|_p \leq S_p^{-1/2} \|u\|$, where $|\cdot|_p$ is the L^p-norm, $2 \leq p < \infty$.

As usual, some energy functionals such as J in (2.2) are not bounded from below on X, but are bounded from below on an appropriate subset of X, and a minimizer on this set (if it exists) may give rise to a solution of corresponding differential equation. A good example for an appropriate subset of X is the so-called Nehari manifold

$$\mathcal{N} = \{u \in X : \langle J'(u), u \rangle = 0\},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between X and X'. It is obvious to see that $u \in \mathcal{N}$ if and only if

$$\|u\|^2 = \int_{\mathbb{R}} f(x)|u|^q dx + \int_{\mathbb{R}} g(x)|u|^p dx. \tag{2.4}$$

Obviously, $\mathcal{N}$ contains all solutions of (1.1). In the following, we will use the Nehari manifold methods to find critical points for J. The Nehari manifold $\mathcal{N}$ is closely linked to the behavior of functions of the form $N_u : t \to J(tu)$ for $t > 0$. For $u \in X$, let

$$N_u(t) = J(tu) = \frac{1}{2} t^2 \|u\|^2 - \frac{1}{q} t^q \int_{\mathbb{R}} f(x)|u|^q dx - \frac{1}{p} t^p \int_{\mathbb{R}} g(x)|u|^p dx.$$

Because $N_u'(t) = \langle J'(tu), u \rangle = \frac{1}{t} \langle J'(tu), tu \rangle$ for $u \in X \setminus \{0\}$ and $t > 0$, then $tu \in \mathcal{N}$ if and only if $N_u'(t) = 0$, that is, the critical points of $N_u(t)$ correspond to the points on the Nehari manifold. In particular, $u \in \mathcal{N}$ if and only if $N_u'(1) = 0$. Then we define

$$\begin{aligned}
\mathcal{N}^+ &= \{u \in N : N_u''(1) > 0\}, \\
\mathcal{N}^0 &= \{u \in N : N_u''(1) = 0\}, \\
\mathcal{N}^- &= \{u \in N : N_u''(1) < 0\}.
\end{aligned}$$

Let

$$\begin{aligned}
\psi(u) &= N_u'(1) = \langle J'(u), u \rangle \\
&= \|u\|^2 - \int_{\mathbb{R}} f(x)|u|^q dx - \int_{\mathbb{R}} g(x)|u|^p dx.
\end{aligned} \tag{2.5}$$

Then, for $u \in \mathcal{N}$,

$$\begin{aligned}
\frac{d}{dt} \psi(tu)|_{t=1} &= \langle \psi'(u), u \rangle = \langle \psi'(u), u \rangle - \langle J'(u), u \rangle \\
&= \|u\|^2 - \int_{\mathbb{R}} f(x)|u|^q dx - \int_{\mathbb{R}} g(x)|u|^p dx.
\end{aligned}$$

For each $u \in \mathcal{N}$, $\psi(u) = N_u'(1) = 0$. Thus, we have

$$N_u''(1) = N_u''(1) - (q-1)\psi(u) = (2-q)\|u\|^2 - (p-q)\int_{\mathbb{R}} g(x)|u|^p dx, \tag{2.6}$$

$$N_u''(1) = N_u''(1) - (p-1)\psi(u) = (2-p)\|u\|^2 + (p-q)\int_{\mathbb{R}} f(x)|u|^q dx. \tag{2.7}$$

In order to ensure the Nehari manifold to be a C^1-manifold, we need the following lemmas.

Lemma 2.5. *If $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0,\sigma)$, then the set $\mathcal{N}^0 = \{0\}$, where*

$$\sigma = (p-2)(2-q)^{(2-q)/(p-2)}(S_p/(p-q))^{(p-q)/(p-2)}.$$

Proof. Suppose that there exists $u \in \mathcal{N}\backslash\{0\}$, such that $N_u''(1) = 0$. By Lemma 2.4,

$$\int_{\mathbb{R}} g(x)|u|^p dx \le |g|_\infty S_p^{-p/2}\|u\|^p. \tag{2.8}$$

Noting that $1 < q < 2 < p < +\infty$, from (2.6), we have

$$(2-q)\|u\|^2 \le (p-q)|g|_\infty S_p^{-p/2}\|u\|^p,$$

and then

$$\|u\| \ge \left(\frac{(2-q)S_p^{p/2}}{(p-q)|g|_\infty}\right)^{1/(p-2)}. \tag{2.9}$$

Moreover, by Hölder inequality and Lemma 2.4, one obtains

$$\begin{aligned}\int_{\mathbb{R}} f(x)|u|^q dx &\le \left(\int_{\mathbb{R}} |f(x)|^{q^*} dx\right)^{1/q^*}\left(\int_{\mathbb{R}} |u|^p dx\right)^{q/p} \\ &= |f|_{q^*}|u|_p^q \le |f|_{q^*} S_p^{-q/2}\|u\|^q.\end{aligned} \tag{2.10}$$

From (2.7), we have $(p-2)\|u\|^2 \le (p-q)|f|_{q^*} S_p^{-q/2}\|u\|^q$, which implies that

$$\|u\| \le \left(\frac{(p-q)|f|_{q^*}}{(p-2)S_p^{q/2}}\right)^{1/(2-q)}. \tag{2.11}$$

Combining (2.9) and (2.11), we deduce that

$$\begin{aligned}|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} &\ge \left(\frac{(2-q)S_p^{p/2}}{p-q}\right)^{(2-q)/(p-2)} \frac{p-2}{p-q} S_p^{q/2} \\ &= (p-2)(2-q)^{(2-q)/(p-2)}(S_p/(p-q))^{(p-q)/(p-2)},\end{aligned}$$

which contradicts the assumptions. □

For each $u \in X\backslash\{0\}$, let $h(t) = t^{2-q}\|u\|^2 - t^{p-q}\int_{\mathbb{R}} g(x)|u|^p dx$ for $t \ge 0$, then we have $h(0) = 0$, $h(t) > 0$ for t small enough, and $h(t) \to -\infty$ as $t \to \infty$. By $1 < q < 2 < p < +\infty$ and

$$h'(t) = t^{p-q-1}\left((2-q)t^{2-p}\|u\|^2 - (p-q)\int_{\mathbb{R}} g(x)|u|^p dx\right) = 0,$$

we can obtain that there is a unique

$$t_{\max} = \left[\frac{(2-q)\|u\|^2}{(p-q)\int_{\mathbb{R}} g(x)|u|^p dx} \right]^{1/(p-2)}$$

such that $h(t)$ achieves its maximum at $t_{\max}$, increasing for $t \in [0, t_{\max})$, and decreasing for $t \in [t_{\max}, \infty)$. Then we have the lemma below.

Lemma 2.6. *Suppose that $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0, \sigma)$ and $u \in X \backslash \{0\}$. Then*

(*i*) *if $\int_{\mathbb{R}} f(x)|u|^q dx = 0$, then there is a unique $t^- > t_{\max}$, such that $t^- u \in \mathcal{N}^-$ and*

$$J(t^- u) = \sup_{t \geq 0} J(tu);$$

(*ii*) *if $\int_{\mathbb{R}} f(x)|u|^q dx > 0$, then there are unique t^+ and t^- with $t^- > t_{\max} > t^+ > 0$, such that $t^- u \in \mathcal{N}^-$, $t^+ u \in \mathcal{N}^+$ and*

$$J(t^+ u) = \inf_{0 \leq 0 \leq t_{\max}} J(tu), \quad J(t^- u) = \sup_{t \geq t_{\max}} J(tu).$$

Proof. By the Sobolev embedding theorem, we have that

$$\begin{aligned} h(t_{\max}) = & \left[\frac{(2-q)\|u\|^2}{(p-q)\int_{\mathbb{R}} g(x)|u|^p dx} \right]^{(2-q)/(p-2)} \|u\|^2 \\ & - \left[\frac{(2-q)\|u\|^2}{(p-q)\int_{\mathbb{R}} g(x)|u|^p dx} \right]^{(p-q)/(p-2)} \left(\int_{\mathbb{R}} g(x)|u|^p dx \right) \\ & \geq \|u\|^q \frac{p-2}{p-q} \left(\frac{(2-q)S_p^{p/2}}{(p-q)|g|_\infty} \right)^{(2-q)/(p-2)} . \end{aligned} \tag{2.12}$$

(*i*) If $\int_{\mathbb{R}} f(x)|u|^q dx = 0$, there exists a unique positive number $t^- > t_{\max}$ such that $h(t^-) = \int_{\mathbb{R}} f(x)|u|^q dx = 0$, and $h'(t^-) < 0$. Then

$$\begin{aligned} \frac{d}{dt} J(tu)\Big|_{t-t^-} &= \left[\frac{1}{t} (\|tu\|^2 - \int_{\mathbb{R}} g(x)|tu|^p dx - \int_{\mathbb{R}} f(x)|tu|^q) \right]\Big|_{t=t^-} = 0, \\ \frac{d^2}{dt^2} J(tu)\Big|_{t-t^-} &= \left[\frac{1}{t^2} (\|tu\|^2 - (p-1) \int_{\mathbb{R}} g(x)|tu|^p dx - (q-1) \int_{\mathbb{R}} f(x)|tu|^q) \right]\Big|_{t=t^-} < 0, \end{aligned}$$

and $J(tu) \to -\infty$ as $t \to \infty$. Moreover, for $1 < q < 2 < p$, it is easy to check that $t^- u \in \mathcal{N}^-$, and $J(t^- u) = \sup_{t \geq 0} J(tu)$.

(*ii*) If $\int_{\mathbb{R}} f(x)|u|^q dx > 0$, by (2.10) and (2.12), then

$$h(0) = 0 < \int_{\mathbb{R}} f(x)|u|^q dx \leq \|u\|^q \frac{p-2}{p-q} \left(\frac{(2-q)S_p^{p/2}}{(p-q)|g|_\infty} \right)^{(2-q)/(p-2)} \leq h(t_{\max}).$$

It follows that there exist unique positive numbers t^+ and t^- such that $t^+ < t_{\max} < t^-$, $h(t^+) = \int_{\mathbb{R}} f(x)|u|^q dx = h(t^-)$ and $h'(t^-) < 0 < h'(t^+)$. Similarly, we have that $t^+ u \in \mathcal{N}^+$, $t^- u \in \mathcal{N}^-$, $J(t^+ u) \leq J(tu) \leq J(t^- u)$ for each $t \in [t^+, t^-]$, and $J(t^+ u) \leq J(tu)$ for each $t \in [0, t_{\max}]$. Hence, $J(t^+ u) = \inf_{0 \leq 0 \leq t_{\max}} J(tu)$, $J(t^- u) = \sup_{t \geq t_{\max}} J(tu)$. □

In the following, we will give some lemmas to obtain the minimizing sequence of the energy functional J on Nehari manifold $\mathcal{N}$.

Lemma 2.7. *The energy functional J is coercive and bounded from below on $\mathcal{N}$.*

Proof. For $u \in \mathcal{N}$, by Hölder's inequality and Lemma 2.4,

$$\begin{aligned} J(u) &= J(u) - \frac{1}{p}\langle J'(u), u\rangle \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^2 - \left(\frac{1}{q} - \frac{1}{p}\right)\int_{\mathbb{R}} f(x)|u|^q dx \\ &\geq \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^2 - \left(\frac{1}{q} - \frac{1}{p}\right)|f|_{q^*} S_p^{-q/2}\|u\|^q. \end{aligned} \tag{2.13}$$

For $1 < q < 2 < p$, thus we get the conclusion. □

Lemma 2.8. *If $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0, \sigma)$, the set $\mathcal{N}^-$ is closed in X.*

Proof. Let $\{u_n\} \subset \mathcal{N}^-$ such that $u_n \to u$ in X. In the following, we prove $u \in \mathcal{N}^-$. Indeed, by $\langle J'(u_n), u_n\rangle = 0$, and

$$\langle J'(u_n), u_n\rangle - \langle J'(u), u\rangle = \langle J'(u_n) - J'(u), u\rangle + \langle J'(u_n), u_n - u\rangle \to 0$$

as $n \to \infty$, we have $\langle J'(u), u\rangle = 0$, that is, $u \in \mathcal{N}$. For any $u \in \mathcal{N}^-$, from (2.6), one obtains

$$(2 - q)\|u\|^2 < (p - q)\int_{\mathbb{R}} g(x)|u|^p dx.$$

Similar to the proof of (2.9), we have

$$\|u\| \leq \left(\frac{(2-q)S_p^{p/2}}{(p-q)|g|_\infty}\right)^{1/(p-2)}. \tag{2.14}$$

Thus, $\mathcal{N}^-$ is bounded away from 0. By (2.6), it follows that $N''_{u_n}(1) \to N''_u(1)$. Combining with $N''_{u_n}(1) < 0$, we have $N''_u(1) \leq 0$. By Lemma 2.5, for $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0, \sigma)$, $N''_u(1) < 0$. Thus we deduce $u \in \mathcal{N}^-$. □

Lemma 2.9. *If $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0, \sigma)$, then for each $u \in \mathcal{N}^+$, there exist $\epsilon > 0$ and a differential function $\varphi_1 : (-\epsilon, \epsilon) \to \mathbb{R}_+ = (0, +\infty)$ such that*

$$\varphi_1(0) = 1, \quad \varphi_1(w)(u - w) \in \mathcal{N}^+, \qquad \forall w \in (-\epsilon, \epsilon),$$

$$\langle \varphi_1(0), w\rangle = \frac{L(u, w)}{N''_u(1)}, \tag{2.15}$$

where

$$L(u, w) = 2\langle u, w\rangle - q\int_{\mathbb{R}} f(x)|u|^{q-2}uw dx - p\int_{\mathbb{R}} g(x)|u|^{p-2}uw dx.$$

Moreover, for any $C_1,\ C_2 > 0$, there exists $C > 0$, such that if $C_1 \leq \|u\| \leq C_2$, then $|\langle \varphi_1'(0), w\rangle| \leq C\|w\|$.

Proof. First, we define $F : \mathbb{R} \times X \to \mathbb{R}$ by $F(t,w) = N''_{u-w}(t)$, it is easy to obtain that F is differentiable. Since $F(1,0) = 0$ and $F'_t(1,0) = N''_u(1) > 0$, according to the implicit function theorem at point $(1,0)$, one can get the existence of $\epsilon > 0$, and differentiable function $\varphi_1 : (-\epsilon,\epsilon) \to \mathbb{R}_+ = (0,+\infty)$ such that

$$\varphi_1(0) = 1, \quad F(\varphi_1(w),w) = 0, \qquad \forall w \in (-\epsilon,\epsilon).$$

Thus, $\varphi_1(w)(u-w) \in \mathcal{N}$, $\forall w \in (-\epsilon,\epsilon)$. Next, we prove $\varphi_1(w)(u-w) \in \mathcal{N}^+$, $\forall w \in (-\epsilon,\epsilon)$. Indeed, by $u \in \mathcal{N}^+$ and $\mathcal{N}^- \cup \mathcal{N}^0$ is closed, we know $\operatorname{dist}(u,\mathcal{N}^- \cup \mathcal{N}^0) > 0$. Since $\varphi_1(w)(u-w)$ is continuous with respect to w, when $\epsilon > 0$ small enough, for $w \in (-\epsilon,\epsilon)$, one has

$$\|\varphi_1(w)(u-w) - u\| < \frac{1}{2}\operatorname{dist}(u,\mathcal{N}^- \cup \mathcal{N}^0),$$

and thus

$$\begin{aligned}\operatorname{dist}(\varphi_1(w)(u-w),\mathcal{N}^- \cup \mathcal{N}^0) &\geq \operatorname{dist}(u,\mathcal{N}^- \cup \mathcal{N}^0) - \|\varphi_1(w)(u-w) - u\| \\ &> \frac{1}{2}\operatorname{dist}(u,\mathcal{N}^- \cup \mathcal{N}^0) > 0.\end{aligned}$$

Thus, $\varphi_1(w)(u-w) \in \mathcal{N}^+$, $\forall w \in (-\epsilon,\epsilon)$. Also by the differentiability of the implicit function theorem, we have

$$\langle \varphi_1'(0), w\rangle = -\frac{\langle F'_w(1,0), w\rangle}{F'_t(1,0)}.$$

Note that $L(u,w) = -\langle F'_w(1,0), w\rangle$ and $N''_u(1) = F_t(1,0)$. So we prove (2.15).

Then we prove that for any $C_1,\ C_2 > 0$, if $C_1 \leq \|u\| \leq C_2$, $u \in \mathcal{N}$, there exists $\delta > 0$, such that $N''_u(1) \geq \delta > 0$. On the contrary, if there exists a sequence $\{u_n\} \subset \mathcal{N}^+$, $C_1 \leq \|u_n\| \leq C_2$, such that for any δ_n small enough, $N''_{u_n}(1) \leq \delta_n$, $\delta_n \to 0$ as $n \to \infty$. From (2.8) we have

$$(2-q)\|u_n\|^2 \leq (p-q)|g|_\infty S_p^{-p/2}\|u_n\|^p + O(\delta_n)$$

and so

$$\|u_n\| \geq \left(\frac{(2-q)S_p^{p/2}}{(p-q)|g|_\infty}\right)^{1/(p-2)} + O(\delta_n). \tag{2.16}$$

From (2.7), we also have

$$(p-2)\|u_n\|^2 = (p-q)\int_{\mathbb{R}} f(x)|u_n|^q dx + O(\delta_n).$$

In view of (2.10), we obtain

$$(p-2)\|u_n\|^2 \leq (p-q)|f|_{q^*} S_p^{-q/2}\|u_n\|^q + O(\delta_n),$$

which implies

$$\|u_n\| \leq \left(\frac{(p-q)|f|_{q^*}}{(p-2)S_p^{q/2}}\right)^{1/(2-q)} + O(\delta_n). \tag{2.17}$$

Combining (2.16) and (2.17) as $n \to \infty$, we deduce a contradiction. Thus if $C_1 \leq \|u\| \leq C_2$, then $|\langle \varphi_1'(0), w\rangle| \leq C\|w\|$. This completes the proof. □

Similarly, we establish the lemma below.

Lemma 2.10. *If $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0,\sigma)$, then for each $u \in \mathcal{N}^-$, there exist $\varepsilon > 0$ and a differential function $\varphi_2 : (-\varepsilon,\varepsilon) \to \mathbb{R}_+ = (0,+\infty)$ such that*

$$\varphi_2(0) = 1, \quad \varphi_2(w)(u-w) \in \mathcal{N}^-, \qquad \forall w \in (-\varepsilon,\varepsilon),$$

$$\langle \varphi_2(0), w\rangle = \frac{L(u,w)}{N''_u(1)},$$

where $L(u,w)$ is defined in Lemma 2.9. Moreover, for any $C_1,\ C_2 > 0$, there exists $C > 0$, such that if $C_1 \le \|u\| \le C_2$, then $|\langle \varphi_2'(0), w\rangle| \le C\|w\|$.

The following lemma aims at obtaining the critical point of J on whole space from the local minimizer for J on Nehari manifold.

Lemma 2.11. *Suppose that u is a local minimizer for J on $\mathcal{N}^+$ (or $\mathcal{N}^-$). Then $J'(u) = 0$.*

Proof. If $u \ne 0$, u is a local minimizer for J on $\mathcal{N}^+$ (or $\mathcal{N}^-$), then u is a nontrivial solution of the optimization problem: minimize J subject to $\psi'(u) = 0$, where $\psi(u)$ is defined in (2.5). By $\psi'(u) \ne 0$, $\mathcal{N}^+$ (or $\mathcal{N}^-$) is a local differential manifold. So by the theory of Lagrange multipliers, there exists $\lambda \in \mathbb{R}$ such that $J'(u) = \lambda\psi'(u)$, thus $\langle J'(u), u\rangle = \lambda\langle\psi'(u), u\rangle$. Since $u \in \mathcal{N}^+$ (or $\mathcal{N}^-$), $\langle J'(u), u\rangle = 0$, and $\langle \psi'(u), u\rangle \ne 0$. Hence, $\lambda = 0$. Thus, the proof is complete. □

3 Proofs of theorems

First, we also give some lemmas, which are necessary for our results.

Lemma 3.1. *Every (PS)$_c$-sequence $\{u_n\} \subset \mathcal{N}^+$ (or $\mathcal{N}^-$) for J on X has a strongly convergent subsequence.*

Proof. Assume that $\{u_n\} \subset \mathcal{N}^+$ (or $\mathcal{N}^-$) such that $J(u_n) \to c$, $J'(u_n) \to 0$ as $n \to \infty$. By the proof of Lemma 2.7, we obtain that $\{u_n\} \subset \mathcal{N}^+$ (or $\mathcal{N}^-$) for J on X is bounded, and by Lemma 2.4, going to a subsequence if necessary, we have

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } X,\\ u_n &\to u \quad \text{in } L^p(\mathbb{R}),\ p \in [2,\infty). \end{aligned}$$

Note that

$$\begin{aligned} \langle J'(u_n) - J'(u), u_n - u\rangle &= \langle J'(u_n), u_n - u\rangle - \langle J'(u), u_n - u\rangle \\ &\ge \|u_n - u\|^2 - \int_{\mathbb{R}} f(x)(|u_n|^{q-2}u_n - |u|^{q-2}u)(u_n - u)dx \\ &\quad - \int_{\mathbb{R}} g(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)dx, \end{aligned}$$

then we can deduce that $\|u_n - u\| \to 0$ as $n \to \infty$. Indeed, from the boundedness of $\{u_n\}$ in X and Lemma 2.4, $\{u_n\}$ is bounded in $L^p(\mathbb{R})$, $p \in [2,\infty)$. By Hölder's inequality, one obtains

$$\begin{aligned} &\left|\int_{\mathbb{R}} f(x)(|u_n|^{q-2}u_n - |u|^{q-2}u)(u_n - u)dx\right| \\ &\quad \le \left(\int_{\mathbb{R}} |f|^{q^*}dx\right)^{1/q^*} \left(\int_{\mathbb{R}} \left||u_n|^{q-2}u_n - |u|^{q-2}u\right|^{p/q} |u_n - u|^{p/q}dx\right)^{q/p} \\ &\quad \le C|f|_{q^*}\left(|u_n|_p^{q-1} + |u|_p^{q-1}\right)|u_n - u|_p \to 0, \end{aligned}$$

as $n \to \infty$, where C is a positive constant. Similarly, we have

$$\int_{\mathbb{R}} g(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)dx \to 0,$$

as $n \to \infty$. From $\langle J'(u_n) - J'(u), u_n - u\rangle \to 0$, as $n \to \infty$, we have $\|u_n - u\| \to 0$ as $n \to \infty$. This completes the proof. □

Lemma 3.2. *If $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0, \sigma)$, then the minimization problem $c_1 = \inf_{\mathcal{N}^+} J(u)$ is solved at a point $u_1 \in \mathcal{N}^+$. That is, u_1 is a critical point of J.*

Proof. First, we prove the minimizing sequence $\{u_n\} \subset \mathcal{N}^+$ is a $(\mathrm{PS})_{c_1}$-sequence on X. Indeed, by Lemma 2.2 and the Ekeland variational principle (see [20]) on $\mathcal{N}^+ \cup \mathcal{N}^0$, there exists a minimizing sequence $\{u_n\} \subset \mathcal{N}^+ \cup \mathcal{N}^0$ such that

$$\inf_{u \in \mathcal{N}^+ \cup \mathcal{N}^0} J(u) \le J(u_n) < \inf_{u \in \mathcal{N}^+ \cup \mathcal{N}^0} J(u) + \frac{1}{n}, \tag{3.1}$$

$$J(u_n) - \frac{1}{n}\|v - u_n\| \le J(v), \quad \forall v \in \mathcal{N}^+ \cup \mathcal{N}^0. \tag{3.2}$$

From Lemma 2.6, we obtain that for each $u \in X\backslash\{0\}$, there exists a unique t^+ such that $t^+u \in \mathcal{N}^+$, then $\inf_{u\in\mathcal{N}^+} J(u) \le J(t^+u)$. Now, we prove that for each $u \in \mathcal{N}^+$, $J(u) < 0$. Indeed, for each $u \in \mathcal{N}^+$, $N_u''(1) > 0$. From (2.7), we have

$$(p - q)\int_{\mathbb{R}} f(x)|u|^q dx > (p - 2)\|u\|^2,$$

then for each $u \in \mathcal{N}^+$,

$$\begin{aligned} J(u) &= J(u) - \frac{1}{p}\langle J'(u), u\rangle \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^2 + \left(\frac{1}{p} - \frac{1}{q}\right)\int_{\mathbb{R}} f(x)|u|^q dx \\ &< \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^2 - \frac{p-2}{pq}\|u\|^2 \\ &= \frac{(p-2)(q-2)}{2pq}\|u\|^2 < 0. \end{aligned}$$

From the inequality above, we have $\inf_{u\in\mathcal{N}^+} J(u) < 0$. Since $J(0) = 0$, we have

$$\inf_{u\in\mathcal{N}^+ \bigcup \mathcal{N}^0} J(u) = \inf_{u\in\mathcal{N}^+} J(u) = c_1.$$

Thus we may assume $\{u_n\} \subset \mathcal{N}^+$, $J(u_n) \to c_1 < 0$. By Lemma 2.9, for $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0, \sigma)$, we can find $\delta_n > 0$ and differentiable function $\varphi_{1n} = \varphi_{1n}(w) > 0$ such that $\varphi_{1n}(w)(u_n - w) \in \mathcal{N}^+$, $\forall w \in (-\delta_n, \delta_n)$. By the continuity of $\varphi_{1n}(w)$ and $\varphi_{1n}(0) = 1$, without loss of generality, we can assume δ_n is sufficiently small such that $\frac{1}{2} \le \varphi_{1n}(w) \le \frac{3}{2}$, for $|w| \le \delta_n$. From $\varphi_{1n}(w)(u_n - w) \in \mathcal{N}^+$ and (3.2), we have

$$J(\varphi_{1n}(w)(u_n - w)) \ge J(u_n) - \frac{1}{n}\|\varphi_{1n}(w)(u_n - w) - u_n\|,$$

which implies

$$\langle J'(u_n), \varphi_{1n}(w)(u_n - w) - u_n\rangle + o(\|\varphi_{1n}(w)(u_n - w) - u_n\|) \ge \frac{1}{n}\|\varphi_{1n}(w)(u_n - w) - u_n\|.$$

Consequently,

$$\varphi_{1n}(w)\langle J'(u_n), w\rangle + (1-\varphi_{1n}(w))\langle J'(u_n), u_n\rangle \\ \leq \frac{1}{n}\|(\varphi_{1n}(w)-1)u_n - \varphi_{1n}(w)w\| + o(\|\varphi_{1n}(w)(u_n - w) - u_n\|).$$

By the choice of δ_n and $\frac{1}{2} \leq \varphi_{1n}(w) \leq \frac{3}{2}$, we infer that there exists $C_3 > 0$ such that

$$|\langle J'(u_n), w\rangle| \leq \frac{1}{n}\|\langle \varphi'_{1n}(0), w\rangle u_n\| + \frac{C_3}{n}\|w\| + o\left(|\langle \varphi'_{1n}(0), w\rangle(\|u_n\| + \|w\|)|\right).$$

Then, we prove that for $\{u_n\} \subset \mathcal{N}^+$, $\inf_n \|u_n\| \geq C_1$, where C_1 is a constant. Indeed, if not, then we have $J(u_n) \to 0$, which contradicts $J(u_n) \to c_1 < 0$. Moreover, by Lemma 2.7, we know that J is coercive on $\mathcal{N}^+$, $\{u_n\}$ is bounded in X. Thus, there exists $C_2 > 0$ such that $0 < C_1 \leq \|u_n\| \leq C_2$. From Lemma 2.9, $|\langle \varphi'_{1n}(0), w\rangle| \leq C\|w\|$, so

$$|\langle J'(u_n), w\rangle| \leq \frac{C}{n}\|w\| + \frac{C}{n}\|w\| + o(\|w\|),$$
$$\|J'(u_n)\| = \sup_{w \in X\setminus\{0\}} \frac{|\langle J'(u_n), w\rangle|}{\|w\|} \leq \frac{C}{n} + o(1),$$

then $\|J'(u_n)\| \to 0$ as $n \to \infty$. Thus, $\{u_n\} \subset \mathcal{N}^+$ is a $(\mathrm{PS})_{c_1}$-sequence for J on X. From Lemma 3.1, there is a strongly convergent subsequence $\{u_n\}$, we will denote by $\{u_n\}$, such that $u_n \to u_1$ as $n \to \infty$ in X. From the above, we obtain that there exist C_1, $C_2 > 0$, such that $0 < C_1 \leq \|u_n\| \leq C_2$, then $< C_1 \leq \|u_1\| \leq C_2$, thus $u_1 \neq 0$.

Finally, we prove $u_1 \in \mathcal{N}^+$. Indeed, by (2.6), it follows that $N''_{u_n}(1) \to N''_{u_1}(1)$. From $N''_{u_n}(1) > 0$, we have $N''_{u_1}(1) \geq 0$. by Lemma 2.5, we have $N''_{u_1}(1) > 0$. Thus $u_1 \in \mathcal{N}^+$, $J(u_1) = \lim_{n\to\infty} J(u_n) = \inf_{u\in\mathcal{N}^+} J(u)$. □

Lemma 3.3. *If $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0, \sigma)$, then the minimization problem $c_2 = \inf_{\mathcal{N}^-} J(u)$ is solved at a point $u_2 \in \mathcal{N}^-$. That is, u_2 is a critical point of J.*

Proof. From Lemma 2.8, $\mathcal{N}^-$ is closed in X. By Lemma 2.7, we know J is conceive on $\mathcal{N}^-$, so we use Ekland variational principle on $\mathcal{N}^-$ and then obtain a minimizing sequence $\{u_n\} \subset \mathcal{N}^-$ such that

$$\inf_{u\in\mathcal{N}^-} J(u) \leq J(u_n) < \inf_{u\in\mathcal{N}^-} J(u) + \frac{1}{n},$$

$$J(u_n) - \frac{1}{n}\|v - u_n\| \leq J(v), \qquad \forall v \in \mathcal{N}^-.$$

By (2.14) and Lemma 2.7, one obtains that there exist C_1, $C_2 > 0$ such that $0 < C_1 \leq \|u_n\| \leq C_2$. Hence, by Lemma 2.10, similar to Lemma 3.2, there exists a minimizing sequence $\{u_n\} \subset \mathcal{N}^-$ is the $(\mathrm{PS})_{c_2}$-sequence on X. From Lemma 3.1, we know that there is a strongly convergent subsequence, still denotes by $\{u_n\}$, $u_n \to u_2$ in X. By Lemma 2.8, the set $\mathcal{N}^-$ is closed, we know $u_2 \in \mathcal{N}^-$, thus $J(u_2) = \lim_{n\to\infty} J(u_n) = \inf_{u\in\mathcal{N}^-} J(u)$. □

Proof of Theorem 1.1. From Lemma 3.2 and Lemma 3.3, we know if $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0, \sigma)$, then problem (1.1) has at least two nontrivial solutions u_1 and u_2, and by Lemma 3.2, the solution $u_1 \in \mathcal{N}^+$ with $J(u_1) < 0$; by Lemma 3.3, the solution $u_2 \in \mathcal{N}^-$ with $J(u_2) > 0$. The proof is completed. □

Proof of Theorem 1.3. First, for $0 < \sigma^* := \frac{q}{2}\sigma < \sigma$, then if $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0,\sigma^*)$, by Theorem 1.1, the problem (1.1) has at least two nontrivial solutions $u_1 \in \mathcal{N}^+$ with $J(u_1) < 0$ and $u_2 \in \mathcal{N}^-$ with $J(u_2) > 0$. Next, we will prove that u_1 is a ground state solution of (1.1). If $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0,\sigma^*)$, then by (2.14), we can infer that

$$\begin{aligned}
J(u) &= J(u) - \frac{1}{p}\langle J'(u), u\rangle \\
&= \left(\frac{1}{2}-\frac{1}{p}\right)\|u\|^2 + \left(\frac{1}{p}-\frac{1}{q}\right)\int_{\mathbb{R}} f(x)|u|^q dx \\
&\geq \left(\frac{1}{2}-\frac{1}{p}\right)\|u\|^2 - \left(\frac{1}{q}-\frac{1}{p}\right)|f|_{q^*}S_p^{-q/2}\|u\|^q \\
&= \|u\|^q\left[\left(\frac{1}{2}-\frac{1}{p}\right)\|u\|^{2-q} - \left(\frac{1}{q}-\frac{1}{p}\right)|f|_{q^*}S_p^{-q/2}\right] \\
&\geq \left(\frac{(2-q)S_p^{p/2}}{(p-q)|g|_\infty}\right)^{q/(p-2)}\left[\left(\frac{1}{2}-\frac{1}{p}\right)\left(\frac{(2-q)S_p^{p/2}}{(p-q)|g|_\infty}\right)^{(2-q)/(p-2)} - \left(\frac{1}{q}-\frac{1}{p}\right)|f|_{q^*}S_p^{-q/2}\right] \\
&> 0.
\end{aligned}$$

That is, for $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0,\sigma^*)$, $J(u) > 0$ for $\forall u \in \mathcal{N}^-$, then $J(u_1) = \inf_{u\in\mathcal{N}} J(u)$, u_1 is a ground state solution. This completes the proof. □

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