On reducibility of linear quasiperiodic systems with bounded solutions [1]

Victor I. Tkachenko

Institute of Mathematics National Academy of Sciences of Ukraine
Tereshchenkivs'ka str. 3, Kiev, Ukraine
e-mail: vitk@imath.kiev.ua

Abstract

It is proved that nonreducible systems form a dense G_δ subset in the space of systems of linear differential equations with quasiperiodic skew-symmetric matrices and fix frequency module. There exists an open set of nonreducible systems in this space.

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1. Introduction.

We consider a linear quasiperiodic system of differential equations

$$\frac{dx}{dt} = A(\varphi \cdot t)x, \tag{1}$$

where $x \in \mathbb{R}^n$, $\varphi \in \mathbb{T}_m$, $\mathbb{T}_m = \mathbb{R}^m/2\pi\mathbb{Z}^m$ is an m-dimensional torus, $A(\varphi)$ is a continuous function $\mathbb{T}_m \to o(n)$, $o(n)$ is the set of nth-order skew-symmetric matrices, $\varphi \cdot t$ denotes an irrational twist flow on the torus $\mathbb{T}_m$

$$\varphi \cdot t = \omega t + \varphi, \ \varphi \in \mathbb{T}_m. \tag{2}$$

$\omega = (\omega_1, ..., \omega_m)$ is a constant vector with rationally independent coordinates.

Fix a flow (2) on the torus $\mathbb{T}_m$ and consider a set of all quasiperiodic systems (1) with continuous skew-symmetric matrices $A(\varphi)$. The distance between two systems (1) is introduced in terms of uniform norm of matrix-functions $A(\varphi)$ on the torus. The aim of this paper is to prove that C^0-generic subsets of systems (1) are not reducible to systems with constant matrices.

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There exists an open subset of nonreducible systems in the space of systems (1).

We note that near system (1) on $\mathbb{T}_m \times \mathbb{R}^3$ with constant coefficients the reducible and uniquely ergodic (end, hence, nonreducible) analytic systems are mixed: analytic nonreducible systems form a dense G_δ set but there is a dense set of reducible systems (see [2], [3], [5]).

2. Main results.

For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we define the norm $\|x\| = (\sum_{j=1}^n x_j^2)^{1/2}$. The corresponding norm $\|A\|$ for n-dimensional matrix A is defined as follows: $\|A\| = \sup\{\|Ax\|, x \in \mathbb{R}^n, \|x\| = 1\}$. Thus $\|Ax\| = 1$ for $A \in SO(n)$, where $SO(n)$ is the set of all orthogonal matrices of dimension n with determinant equal to 1.

Let $C(\mathbb{T}_m, o(n))$ be the space of continuous functions on the torus $\mathbb{T}_m$ with values in the group $o(n)$. In the space $C(\mathbb{T}_m, o(n))$, we introduce the ordinary norm

$$\|a(\varphi)\|_0 = \sup_{\varphi \in \mathbb{T}_m} \|a(\varphi)\|.$$

Let $\Phi(t, \varphi), \Phi(0, \varphi) = I (I$ is the identity matrix) be the fundamental solution for system (1). It forms a cocycle

$$\Phi(t_1 + t_2, \varphi) = \Phi(t_2, \varphi \cdot t_1)\Phi(t_1, \varphi). \tag{3}$$

If $A(\varphi) \in o(n)$ then $\Phi(t, \varphi) \in SO(n)$.

Define a quasiperiodic skew-product flow on $\mathbb{T}_m \times SO(n)$ as follows

$$(\varphi, X) \cdot t = (\varphi \cdot t, \Phi(t, \varphi)X), \ t \in \mathbb{R}, \tag{4}$$

where $(\varphi, X) \in \mathbb{T}_m \times SO(n)$.

Let $X = cls\{(\varphi_0, I) \cdot t : t \in \mathbb{R}\} \subseteq \mathbb{T}_m \times SO(n)$ be the closure of the trajectory with the initial point (φ_0, I) of the flow (4). The set X is minimal and distal. Let π be the projector onto the first component, i.e., $\pi : X \to \mathbb{T}_m$. As shown in [1], $\pi^{-1}(\varphi_0)$ forms a compact group. Denote it by G. For all $\varphi \in \mathbb{T}_m$, the preimage $\pi^{-1}(\varphi)$ is the uniform space of the group $G \subseteq SO(n)$.

Definition 1 *System (1) is said to be reducible if there is a linear change of variables $x = P(\varphi)y$ that transforms (1) to a system with a constant matrix where $P(\varphi)$ is a continuous map $P : \mathbb{T}_m \to SO(n)$ the map $t \to P(\varphi \cdot t) : \mathbb{R} \to SO(n)$ is continuously differentiable and $\varphi \to (d/dt)P(\varphi \cdot t)|_{t=0} : \mathbb{T}_m \to SO(n)$ is continuous.*

The flow (4) preserves the product Haar measure $\mu \times \nu$ on $\mathbb{T}_m \times SO(n)$. It preserves distances both in the $\mathbb{T}_m$-direction and in the $SO(n)$-direction but it is not isometry. In the reducible case this measure is not ergodic and there are invariant measures supported on each invariant torus.

Theorem 1 *The functions $A(\varphi)$ corresponding to systems (1) whose trajectories $(\varphi, I) \cdot t$ is dense in $\mathbb{T}_m \times SO(n)$ form a dense G_δ subset of the space $C(\mathbb{T}_m, o(n))$. These systems have unique invariant measure and are not reducible.*

The proof of theorem is preceded by the two lemmas.

Lemma 1 [10] *Suppose that a continuous function $A(\varphi) : \mathbb{T}_m \to SO(n)$ satisfies*

$$\sup_{\varphi \in \mathsf{T}_m} \|A(\varphi) - I\| \leq \varepsilon \leq \frac{1}{2}. \tag{5}$$

Then there exists a real continuous logarithm of the function $A(\varphi)$ defined on the torus T_m such that

$$\sup_{\varphi \in \mathsf{T}_m} \| \ln A(\varphi)\| \leq \frac{4\sqrt{2}\varepsilon}{1 - 2\varepsilon}. \tag{6}$$

Lemma 2 *Assume that the mapping $F(t, \varphi) : [0, 1] \times \mathbb{T}_m \to SO(n)$ is continuous in t and φ, continuously differentiable with respect to t, and such that $F(0, \varphi) = I$ and $F(1, \varphi) = a(\varphi)$. Then, for $\varepsilon \leq \alpha < 1/2$ and $b(\varphi) \in C(\mathbb{T}_m, SO(n))$, such that $\|a(\varphi) - b(\varphi)\|_0 < \varepsilon$ there exists a mapping $G(t, \varphi) : [0, 1] \times \mathbb{T}_m \to SO(n)$ continuous in t and φ, continuously differentiable with respect to t, and such that $G(0, \varphi) = I, G(1, \varphi) = b(\varphi)$, and*

$$\|G(t, \varphi) - F(t, \varphi)\|_0 < \varepsilon, \qquad \left\| \frac{\partial G(t, \varphi)}{\partial t} - \frac{\partial F(t, \varphi)}{\partial t} \right\|_0 < K\varepsilon, \tag{7}$$

where $K > 0$ is a constant depending only on the function $F(t, \varphi)$.

Proof. We have

$$\|a^*(\varphi)b(\varphi) - I\|_0 \leq \|a^*(\varphi)\|_0 \cdot \|a(\varphi) - b(\varphi)\|_0 < \varepsilon.$$

For sufficiently small $\varepsilon > 0$, there exists a logarithm

$$\ln(a^*(\varphi)b(\varphi)) = \frac{1}{2\pi i}\int_\Gamma (\lambda I - a^*(\varphi)b(\varphi))^{-1}\ln\lambda d\lambda, \tag{8}$$

continuous on the torus $\mathbb{T}_m$; here, Γ is the boundary of a simply connected domain in the complex plane which contains the closure of the set of eigenvalues of the matrices $a^*(\varphi)b(\varphi), \varphi \in \mathbb{T}_m$, and does not contain zero. By lemma 1, we get

$$\|\ln(a^*(\varphi)b(\varphi))\|_0 \leq \frac{4\sqrt{2}\varepsilon}{1-2\varepsilon}.$$

The function $H(t,\varphi) = \exp[t\ln(a^*(\varphi)b(\varphi))]$ realizes the homotopy of $a^*(\varphi)b(\varphi)$ to the identity matrix and

$$\left\|\frac{\partial H(t,\varphi)}{\partial t}\right\|_0 \leq \|H(\varphi)\|_0 \cdot \|\ln(a^*(\varphi)b(\varphi))\|_0 \leq \frac{4\sqrt{2}\varepsilon}{1-2\varepsilon}.$$

The required function $G(t,\varphi)$ has the form $G(t,\varphi) = F(t,\varphi)H(t,\varphi)$. Taking the last inequality into account, we arrive at estimates (7). Indeed,

$$\|G - F\|_0 \leq \|H - I\|_0 < \varepsilon,$$

$$\left\|\frac{\partial G}{\partial t} - \frac{\partial F}{\partial t}\right\|_0 \leq \left\|\frac{\partial F}{\partial t}(H-I)\right\|_0 + \left\|F\frac{\partial H}{\partial t}\right\|_0 \leq \varepsilon\left(\left\|\frac{\partial F}{\partial t}\right\|_0 + 8\right) = K\varepsilon.$$

The lemma is proved.

Proof of theorem 1. We consider the torus $\mathbb{T}_m$ as a product $\mathbb{T}_m = \mathbb{T}_{m-1} \times \mathbb{T}_1$ of the $(m-1)$-dimensional torus $\mathbb{T}_{m-1}$ and of the circle $\mathbb{T}_1$. Then $\varphi = (\psi, \xi)$, $\psi \in \mathbb{T}_{m-1}$, $\xi \in \mathbb{T}_1$, and $\Phi(t,\varphi) = \Phi(t,\psi,\xi)$.

For an open nonempty subset W of the set $\mathbb{T}_{m-1} \times SO(n)$, we consider a set

$$E(W) = \{A(\varphi) : \ A(\varphi) \in C(\mathbb{T}_m, o(n)), \ O'(\varphi_0, A) \cap W \neq \oslash\},$$

where $O'(\varphi_0, A)$ is a trajectory of system (1) in $\mathbb{T}_m \times SO(n)$ which passes through the point $(\varphi_0, I) = (\psi_0, \xi_0, I)$.

Using continuous dependence of solutions for (1) on parameters, we get that $E(W)$ is open in $C(\mathbb{T}_m, o(n))$.

Let us prove that this set is dense. We fix an arbitrary number $\varepsilon > 0$ and a continuous function $A(\varphi) : \mathbb{T}_m \to o(n)$.

Equation (1) is associated with the linear discrete system

$$y_{n+1} = a(\psi \cdot n) y_n, \quad n \in \mathbb{Z}, \tag{9}$$

where $\psi \in \mathbb{T}_{m-1}$, $a(\psi) = \Phi_A(2\pi/\omega_m, \psi, \xi_0)$, $\psi \cdot n = \bar{\omega} n + \psi$, $\bar{\omega} = (\omega_1, ..., \omega_{m-1})$.

Analogously to Lemma 2, [4], we prove that in uniform on $\mathbb{T}_{m-1}$ topology the ε-neighborhood of the function $a(\psi)$ contains a function $a_1(\psi) \in C(\mathbb{T}_{m-1}, SO(n))$ such that $a_1(\psi) \in E'(W)$, where

$$E'(W) = \{a(\psi) : \ a(\psi) \in C(\mathbb{T}_{m-1}, SO(n)), O'(\psi_0, a) \cap W \neq \oslash\},$$

where $O'(\varphi_0, a))$ is a trajectory of system (9) in $\mathbb{T}_{m-1} \times SO(n)$ which passes through the point (ψ_0, I). For sufficiently small ε, the function $a_1(\psi)$ is homotopic to $a(\psi)$, and, hence, to the identity matrix.

Let us show that $a_1(\psi)$ is associated with quasiperiodic system

$$\frac{dx}{dt} = A_1(\varphi \cdot t)x, \tag{10}$$

such that the difference between the matrices $A(\varphi)$ and $A_1(\varphi)$ is small. For $\varphi = (\psi, \xi_0), t \in [0, 2\pi/\omega_m]$, the matrix function $\Phi_A(t, \psi, \xi_0)$ defines a homotopy of $a(\psi) = \Phi_A(2\pi/\omega_m, \psi, \xi_0)$ to the identity matrix. Suppose that the function $F(t, \psi) : [0, 2\pi/\omega_m] \times \mathbb{T}_{m-1} \to SO(n)$ defines the homotopy of $a_1(\varphi)$ to the identity matrix. By virtue of Lemma 2, the function $F(t, \psi)$ can be chosen so that

$$\|F(t, \psi) - \Phi_A(t, \psi)\|_0 < \varepsilon, \quad \left\| \frac{\partial F(t, \psi)}{\partial t} - \frac{\partial \Phi_A(t, \psi, \xi_0)}{\partial t} \right\|_0 < K\varepsilon$$

where $K > 0$ is a constant depending only on the right-hand side of system (1). The function $F(t, \psi)$ is the solution of system (10) for $\varphi = (\psi, \xi_0), t \in [0, 2\pi/\omega_m]$. For $\varphi = (\psi, \xi_0), t \in \mathbb{R}$, the solution of system (10) is given by the cocycle formula

$$\Phi_{A_1}(t, \varphi) = \Phi_{A_1}(t - [t\omega_m/2\pi], \varphi \cdot [t\omega_m/2\pi]) \Phi_{A_1}([t\omega_m/2\pi], \varphi),$$

where $[t]$ is the integer part of the number t. The map $\varphi \cdot [t\omega_m/2\pi]$ don't change the m-coordinate of the point $\varphi \in \mathbb{T}_m$. Therefore,

$$\Phi_{A_1}(t - [t\omega_m/2\pi], \varphi \cdot [t\omega_m/2\pi]) = F(t - [t\omega_m/2\pi], \psi \cdot [t\omega_m/2\pi])$$

for $\varphi = (\psi, \xi_0)$ and $\Phi_{A_1}([t\omega_m/2\pi], \varphi)$ can be found if we know the solution of the discrete equation $x_{n+1} = a_1(\psi \cdot n)x_n$. For other values of $\varphi \in \mathbb{T}_m$, the function $\Phi_{A_1}(t, \varphi)$ are determined by using the property of cocycle (3).

The matrix $A_1(\varphi)$ in system (10) can be found by formula

$$A_1(\varphi) = \frac{\partial F(t, \psi)}{\partial t} F^*(t, \psi),$$

where $\varphi = (\psi, \xi_0), t \in [0, 2\pi/\omega_m]$. Note that, for each point $\varphi \in \mathbb{T}_m$, there exists a mapping $\varphi = (\psi, \xi_0) \cdot t, t \in [0, 2\pi/\omega_m]$. Taking into account inequalities (7), we get

$$\|A(\varphi) - A_1(\varphi)\|_0 < 3K\varepsilon.$$

Thus, the systems whose solutions pass through the set W form an open everywhere dense subset of the set of all systems (1).

For any $\delta > 0$, there exists a bounded covering of the compact set $\mathbb{T}_{m-1} \times SO(n)$ by open sets $W_i^0, i = 1, ..., n_0$, such that the diameter of each set W_i^0 is less than δ. Then the set $\bigcap_{i=1}^{n_0} E(W_i^0)$ is open and dense in $C(\mathbb{T}_m, o(n))$. The elements $B \in \bigcap_{i=1}^{n_0} E(W_i^0)$ satisfy the condition that the trajectory of corresponding equation $\dot{x} = B(\varphi \cdot t)x$ passes through all balls with diameter 2δ in the space $\mathbb{T}_{m-1} \times SO(n)$.

Consider a covering of the set $\mathbb{T}_{m-1} \times SO(n)$ by open sets W_i^1 with diameter $\delta/2$. Assume that, for any W_i^1 there exists a set W_j^0 such that $W_i^1 \subset W_j^0$, $i = 1, ..., n_1$.

By analogy, we consider the covering of the set $\mathbb{T}_{m-1} \times SO(n)$ by the sets W_i^j with diameter $\delta/2^j$. The elements $A(\varphi)$ of the set $E_j = \bigcap_{i=1}^{n_j} E(W_i^j)$ the trajectory of corresponding equation $\dot{x} = A(\varphi \cdot t)x$ passes through all balls with diameter $\delta/2^{j-1}$ in the space $\mathbb{T}_{m-1} \times SO(n)$. The sets E_j are open and dense in $C(\mathbb{T}_m, o(n))$ by construction.

The set $F = \bigcap_{i=0}^{\infty} E_i$ is dense G_δ in $C(\mathbb{T}_m, o(n))$. The elements $A(\varphi) \in F$ correspond to Eqn. (1) whose trajectories are dense in $\mathbb{T}_{m-1} \times SO(n)$. For these equations, the set $\pi^{-1}(\varphi)$ coincides with $SO(n)$.

Therefore, the systems with $G = SO(n)$ form a dense G_δ subset in the set of systems (1).

Now we prove that systems (1) with $G = SO(n)$ are unique ergodic. We use ideas of [2].

Let $f(\varphi, X)$ be a continuous function on $\mathbb{T}_m \times SO(n)$ and let

$$g(\varphi, X) = \lim_{T\to\infty} \frac{1}{T} \int\limits_0^T f(\varphi \cdot t, \Phi(t,\varphi)X)dt. \tag{11}$$

Since the product Haar measure $\mu \times \nu$ is invariant under the flow (4), the limit (11) exists for a. e. (φ, X) and is measurable. Since the flow preserves distances in the $SO(n)$ direction, the function $g(\varphi, X)$ exists for a.e. $\varphi \in \mathbb{T}_m$ and all $X \in SO(n)$ and $X \to g(\varphi, X)$ is equicontinuous. Hence, for a.e. $\varphi \in \mathbb{T}_m$ and for all $t \in \mathbb{R}$ and $X \in SO(n)$:

$$g(\varphi, X) = g(\varphi \cdot t, \Phi(t,\varphi)X).$$

It is easily seen that if $\varphi \cdot t_n \to \varphi$ for all $\varphi \in \mathbb{T}_m$ then

$$g(\varphi \cdot t_n, X) \to g(\varphi, X)$$

for some subsequence $\{n = n_i\}$, for a.e. $\varphi \in \mathbb{T}_m$ and all $X \in SO(n)$.

Let $\varphi \in \mathbb{T}_m$ be a point for which the function $g(\varphi, X)$ exists. Since $G = SO(n)$, for any $B \in SO(n)$, there exists a sequence t_n such that $\varphi \cdot t_n \to \varphi$ and $\Phi(t_n, \varphi) \to B$ as $n \to \infty$. Hence, for some subsequence $\{n = n_i\}$,

$$g(\varphi, X) = \lim_{n\to\infty} g(\varphi \cdot t_n, \Phi(t_n,\varphi)X) = g(\varphi, BX).$$

Therefore, $g(\varphi, X)$ is independent of X for a.e. $\varphi \in \mathbb{T}_m$ and invariant under $\varphi \to \varphi \cdot t$.

Similarly to [2], p. 18, we prove that the system under consideration has a unique invariant Borel measure. This system is nonreducible.

Theorem 2 *There exists a system (1) such that all the systems from some neighborhood of it (in topology $C(\mathbb{T}_m, o(n))$) have no nontrivial almost periodic solutions and, hence, they are not reducible.*

Proof We consider system (1) in $\mathbb{T}_2 \times \mathbb{R}^3$. Let $\varphi = (\theta, \psi) \in \mathbb{T}_2$. We consider the fundamental solution $\Phi(t, \theta, \psi)$ with following properties:

i) $\Phi(t, \theta, 0) = G(g(t), \theta)$ for all $\theta \in \mathbb{T}_1$ and $t \in [0, 2\pi]$, where $g : [0, 2\pi] \to [0, 2\pi]$ is a continuously differentiable function which is zero near 0 and 2π

near 2π. The function $G(\tau, \theta) : [0, 2\pi] \times \mathbb{T}_1 \to SO(3)$ is continuously differentiable and

$$G(0, \theta) = I, \quad G(2\pi, \theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{12}$$

ii) for all values of t, θ, and φ, the function $\Phi(t, \theta, 0)$ is extended by cocycle formula (3).

The function $\Phi(t, \theta, \psi)$ satisfies the following system of differential equations

$$\frac{d\theta}{dt} = \omega, \quad \frac{d\psi}{dt} = 1, \quad \frac{dx}{dt} = B(\theta, \psi)x, \tag{13}$$

where

$$B(\theta, \psi) = \frac{\partial\Phi(0, \theta, \psi)}{\partial t}$$

and ω is irrational number.

System (13) has two invariant bundles γ_1 and γ_2 defined by projectors

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{14}$$

on the circle $(\theta, 0)$ of torus $\mathbb{T}_2$. By construction, the bundle γ_1 is nontrivial.

Let $P(\theta) = \{p_{ij}(\theta)\}_{i,j=1}^3$ be another projector on the circle $(\theta, 0)$ defining an invariant bundle of the system (13). Rewrite the projector $P(\theta)$ in the form $P(\theta) = \{p_{ij}(\theta)\}_{i,j=1}^2$, where p_{11}, p_{12}, p_{21}, and p_{22} are 2×2, 2×1, 1×2, and 1×1 matrices, respectively. Then

$$G(2\pi, \theta)P(\theta) = P(\theta + 2\pi\omega)G(2\pi, \theta), \tag{15}$$

hence

$$\begin{pmatrix} p_{11}(\theta + 2\pi\omega) & p_{12}(\theta + 2\pi\omega) \\ p_{21}(\theta + 2\pi\omega) & p_{22}(\theta + 2\pi\omega) \end{pmatrix} = \begin{pmatrix} O_2(\theta)p_{11}(\theta)O_2^*(\theta) & O_2(\theta)p_{12}(\theta) \\ p_{21}(\theta)O_2^*(\theta) & p_{22}(\theta) \end{pmatrix},$$

where

$$O_2(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Since $P(\theta)$ is 2π-periodic, vector-functions $p_{12}(\theta)$ and $p_{21}(\theta)$ are identically zero. Otherwise, the equation

$$p(\theta + 2\pi\omega) = O_2(\theta)p(\theta), \quad p \in \mathbb{R}^2, \tag{16}$$

has a nontrivial 2π-periodic solution $p(\theta)$. For periodic function $p(\theta)$, the sequence $p_k = p(\theta + 2\pi\omega k), k \in \mathbb{Z}$, is almost periodic (see [7], p. 198). By (16), p_k has the following explicit form

$$p_k = O_2(k\theta + k(k+1)\omega/2)P(\theta).$$

By definition of almost periodic sequence, for $\varepsilon > 0$, there exists a positive integer $q = q(\varepsilon)$ such that $\|p_{k+q} - p_k\| < \varepsilon$ for all $k \in \mathbb{Z}$. Hence

$$\left\| O_2\left(q\theta + \frac{q(q-1)\omega}{2} + \frac{(2q-1)k\omega}{2} \right) - I \right\| = \tag{17}$$

$$= \|p_k^{-1}(p_{k+q} - p_k)\| \leq \varepsilon.$$

The set

$$\left\{ \frac{(2q-1)\omega k}{2} (\mathrm{mod} 2\pi), k \in \mathbb{Z} \right\}$$

is dense on the circle $\mathbb{T}_1$, therefore we can select k_0 such that

$$\frac{3\pi}{4} \leq \left(q\theta + \frac{q(q-1)\omega}{2} + \frac{k_0(2q-1)\omega}{2} \right) (\mathrm{mod} 2\pi) \leq \frac{5\pi}{4}.$$

Then

$$\left\| O_2\left(q\theta + \frac{q(q-1)\omega}{2} + \frac{k_0(2q-1)\omega}{2} \right) - I \right\| > 1.$$

Choosing $\varepsilon < 1$ we have contradiction. Therefore, system (13) has only invariant bundles γ_1 and γ_2.

Fix $\varepsilon > 0$. We consider another system in $\mathbb{T}_2 \times \mathbb{R}^3$

$$\frac{d\theta}{dt} = \omega, \quad \frac{d\psi}{dt} = 1, \quad \frac{dx}{dt} = \tilde{B}(\theta, \psi)x, \tag{18}$$

which fundamental solution $\tilde{\Phi}(t, \theta, \psi)$ satisfies condition

$$\|\tilde{\Phi}(t, \theta, \psi) - \Phi(t, \theta, \psi)\| \leq \varepsilon \tag{19}$$

for $t \in [0, 2\pi]$, $(\theta, \psi) \in \mathbb{T}_2$. Then

$$\tilde{\Phi}(2\pi, \theta, 0) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} (I + U), \tag{20}$$

where $\|U\| \le \varepsilon$.

We consider a projector $\tilde{P}(\theta)$ on the circle $(\theta, 0)$ defining an invariant bundle $\tilde{\gamma}$ of system (18). Denote $\tilde{P}(\theta) = \{\tilde{p}_{ij}(\theta)\}_{i,j=1}^{2}$, where $\tilde{p}_{11}, \tilde{p}_{12}, \tilde{p}_{21}$, and $\tilde{p}_{22}$ are 2×2, 2×1, 1×2, and 1×1 matrices, respectively.

The projector $\tilde{P}(\theta)$ satisfies the equation

$$\tilde{P}(\theta + 2\pi\omega) = \tilde{\Phi}(t, \theta, 0)\tilde{P}(\theta)\tilde{\Phi}(t, \theta, 0). \tag{21}$$

Using (19), we get

$$\|\tilde{P}(\theta + \omega) - G(2\pi, \theta)\tilde{P}(\theta)G^*(2\pi, \theta)\| < 2p\varepsilon, \tag{22}$$

where $p = \sup \|\tilde{P}(\theta)\|$. Hence

$$\left\| \begin{pmatrix} \tilde{p}_{11}(\theta + 2\pi\omega) & \tilde{p}_{12}(\theta + 2\pi\omega) \\ \tilde{p}_{21}(\theta + 2\pi\omega) & \tilde{p}_{22}(\theta + 2\pi\omega) \end{pmatrix} - \right.$$

$$\left. - \begin{pmatrix} O_2(\theta)\tilde{p}_{11}(\theta)O_2^*(\theta) & O_2(\theta)\tilde{p}_{12}(\theta) \\ \tilde{p}_{21}(\theta)O_2^*(\theta) & \tilde{p}_{22}(\theta) \end{pmatrix} \right\| < 2p\varepsilon, \tag{23}$$

There exists $\varepsilon_0 > 0$ such that the bundle $\tilde{\gamma}$ is homotopic to the bundle γ_i if $\|\tilde{P}(\theta) - P_i\| < \varepsilon_0$, $i = 1, 2$.

It can be shown that inequality $\|p(\theta + 2\pi\omega) - O_2(\theta)p(\theta)\| \le \varepsilon$ for continuous 2π-periodic vector-function $p(\theta)$ implies $\|p(\theta)\| \le a_0\varepsilon$, $\theta \in [0, 2\pi]$, where constant a_0 is independent from ε.

Therefore, by (23), we get $\|p_{12}(\theta)\| < 2pa_0\varepsilon$, $\|p_{21}(\theta)\| < 2pa_0\varepsilon$. Hence, for $\varepsilon > 0$ satisfying inequality $2pa_0\varepsilon < \varepsilon_0$, the bundle $\tilde{\gamma}$ is homotopic to the bundle γ_1 or to the bundle γ_2 in accordance with $\mathrm{rank}\tilde{P}(\theta) = 2$ or $\mathrm{rank}\tilde{P}(\theta) = 1$ for all $\theta \in [0, 2\pi]$. If $P(\theta)$ defines an invariant bundle then $I - P(\theta)$ defines an invariant bundle too. Therefore, if system (18) has an invariant bundle, it has an invariant bundle of rank 2. By proved above, this bundle is homotopic to the nontrivial bundle γ_1 and, hence, is nontrivial.

If system (18) has almost periodic fundamental solution and, hence, it is reducible then the space $\mathbb{T}_2 \times \mathbb{R}^3$ is the Whitney sum of three one-dimensional trivial invariant bundles over the torus $\mathbb{T}_2$ or the Whitney sum of one-dimensional and two-dimensional trivial invariant bundles over the torus $\mathbb{T}_2$ [6], [9]. Therefore, system (18) is not reducible if $\varepsilon < \varepsilon_0/2pa_0$. The theorem is proved.

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