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EXISTENCE OF POSITIVE SOLUTIONS FOR A CLASS OF HIGHER-ORDER m-POINT BOUNDARY VALUE PROBLEMS

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Abstract. We investigate the existence of positive solutions with respect to a cone for a higher-order nonlinear differential system, subject to some boundary conditions which involve m points.

Keywords: Higher-order differential system, boundary conditions, positive solutions, fixed point theorem.

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1 Introduction

We consider the higher-order nonlinear differential system

$$(S)\qquad \begin{cases} u^{(n)}(t) + \lambda b(t) f(v(t)) = 0, \ \ t \in (0,T), \\ v^{(n)}(t) + \mu c(t) g(u(t)) = 0, \ \ t \in (0,T), \ \ n \geq 2, \end{cases}$$

with the m-point boundary conditions

$$(BC)\qquad \begin{cases} u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \ \ u(T) = \displaystyle\sum_{i=1}^{m-2} a_i u(\xi_i), \\ v(0) = v'(0) = \cdots = v^{(n-2)}(0) = 0, \ \ v(T) = \displaystyle\sum_{i=1}^{m-2} a_i v(\xi_i), \ \ m \geq 3, \end{cases}$$

where $0 < \xi_1 < \cdots < \xi_{m-2} < T$, $a_i > 0$, $i = \overline{1, m-2}$.

In this paper we shall investigate the existence of positive solutions with respect to a cone of (S), (BC), where λ, $\mu > 0$. The existence of positive solutions for (S) with $n = 2$ and the boundary conditions $\beta u(0) - \gamma u'(0) = 0$, $u(T) = \sum_{i=1}^{m-2} a_i u(\xi_i) + b_0$,

$\beta v(0) - \gamma v'(0) = 0$, $v(T) = \sum_{i=1}^{m-2} a_i v(\xi_i) + b_0$ has been investigated in [19] for $b_0 = 0$ and in [18] for $b_0 > 0$ and $\lambda = \mu = 1$. The corresponding discrete case, namely the system with second-order differences

$$\begin{cases} \Delta^2 u_{n-1} + \lambda b_n f(v_n) = 0, & n = \overline{1, N-1} \\ \Delta^2 v_{n-1} + \mu c_n g(u_n) = 0, & n = \overline{1, N-1}, \end{cases}$$

with the $m+1$-point boundary conditions $\beta u_0 - \gamma \Delta u_0 = 0$, $u_N = \sum_{i=1}^{m-2} a_i u_{\xi_i}$, $\beta v_0 - \gamma \Delta v_0 = 0$, $v_N = \sum_{i=1}^{m-2} a_i v_{\xi_i}$, $m \geq 3$ has been studied in [17]. We also mention the paper [15] where the authors investigated the existence of positive solutions to the n-th order m-point boundary value problem $u^{(n)}(t) + f(t, u, u') = 0$, $t \in (0, 1)$, $u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0$, $u(1) = \sum_{i=1}^{m-2} k_i u(\xi_i)$.

Due to applications in different areas of applied mathematics and physics, the existence of positive solutions of multi-point boundary value problems for second-order or higher-order differential or difference equations has been the subject of investigations by many authors (see [1]–[14], [16], [20]–[25]).

In Section 2, we shall present several auxiliary results which investigate a boundary value problem for a n-th order equation (the problem (1), (2) below), some of them from the paper [15]. In Section 3, we shall give sufficient conditions on λ and μ such that positive solutions with respect to a cone for our problem (S), (BC) exist. In Section 4, we shall present an example that illustrates the obtained results. Our main results (Theorem 2 and Theorem 3) are based on the Guo-Krasnoselskii fixed point theorem, presented below.

Theorem 1. *Let X be a Banach space and let $C \subset X$ be a cone in X. Assume Ω_1 and Ω_2 are bounded open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$ and let $\mathcal{A} : C \cap (\overline{\Omega_2} \setminus \Omega_1) \to C$ be a completely continuous operator such that, either*

i) $\|\mathcal{A}u\| \leq \|u\|$, $u \in C \cap \partial\Omega_1$, and $\|\mathcal{A}u\| \geq \|u\|$, $u \in C \cap \partial\Omega_2$, or

ii) $\|\mathcal{A}u\| \geq \|u\|$, $u \in C \cap \partial\Omega_1$, and $\|\mathcal{A}u\| \leq \|u\|$, $u \in C \cap \partial\Omega_2$.

Then $\mathcal{A}$ has a fixed point in $C \cap (\overline{\Omega_2} \setminus \Omega_1)$.

2 Auxiliary results

In this section, we shall study the n-th order differential equation with the boundary conditions

$$u^{(n)}(t) + y(t) = 0, \ \ 0 < t < T \tag{1}$$

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \ \ u(T) = \sum_{i=1}^{m-2} a_i u(\xi_i). \tag{2}$$

We denote by $d = T^{n-1} - \sum_{i=1}^{m-2} a_i \xi_i^{n-1}$.

Lemma 1. *If $d \neq 0$, $0 < \xi_1 < \cdots < \xi_{m-2} < T$ and $y \in C([0,T])$, then the solution of* (1), (2) *is given by*

$$\begin{aligned} u(t) = \frac{t^{n-1}}{d(n-1)!} \int_0^T (T-s)^{n-1} y(s)\, ds - \frac{t^{n-1}}{d(n-1)!} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)^{n-1} y(s)\, ds \\ - \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} y(s)\, ds, \ \ 0 \le t \le T. \end{aligned} \tag{3}$$

Proof. By (1) and the first relations from (2) we deduce

$$u(t) = -\frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} y(s)\, ds + \frac{Ct^{n-1}}{(n-1)!}. \tag{4}$$

From the above relation and the condition $u(T) = \sum_{i=1}^{m-2} a_i u(\xi_i)$ we obtain

$$-\frac{1}{(n-1)!} \int_0^T (T-s)^{n-1} y(s)\, ds + \frac{CT^{n-1}}{(n-1)!} = \sum_{i=1}^{m-2} a_i \left[-\frac{1}{(n-1)!} \int_0^{\xi_i} (\xi_i - s)^{n-1} y(s)\, ds + \frac{C\xi_i^{n-1}}{(n-1)!} \right]$$

or

$$C \left(T^{n-1} - \sum_{i=1}^{m-2} a_i \xi_i^{n-1} \right) = \int_0^T (T-s)^{n-1} y(s)\, ds - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)^{n-1} y(s)\, ds,$$

and so

$$C = \frac{1}{d} \int_0^T (T-s)^{n-1} y(s)\, ds - \frac{1}{d} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)^{n-1} y(s)\, ds.$$

Therefore from (4) and the above expression for C we obtain the relation (3). □

Lemma 2. *If $d \neq 0$, $0 < \xi_1 < \cdots < \xi_{m-2} < T$ then the Green's function for the* *boundary value problem* (1), (2) *is given by*

$$G(t,s)=\begin{cases}\dfrac{t^{n-1}}{d(n-1)!}\left[(T-s)^{n-1}-\displaystyle\sum_{i=j+1}^{m-2}a_i(\xi_i-s)^{n-1}\right]-\dfrac{1}{(n-1)!}(t-s)^{n-1},\\ \qquad if\ \ \xi_j\le s<\xi_{j+1},\ \ s\le t,\\ \dfrac{t^{n-1}}{d(n-1)!}\left[(T-s)^{n-1}-\displaystyle\sum_{i=j+1}^{m-2}a_i(\xi_i-s)^{n-1}\right],\\ \qquad if\ \ \xi_j\le s<\xi_{j+1},\ \ s\ge t,\ \ j=\overline{0,m-3},\\ \dfrac{t^{n-1}}{d(n-1)!}(T-s)^{n-1}-\dfrac{1}{(n-1)!}(t-s)^{n-1},\ \ if\ \ \xi_{m-2}\le s\le T,\ \ s\le t,\\ \dfrac{t^{n-1}}{d(n-1)!}(T-s)^{n-1},\ \ if\ \ \xi_{m-2}\le s\le T,\ \ s\ge t,\ \ (\xi_0=0).\end{cases}$$

Proof. Using the relation (3) we obtain

$$u(t)=\frac{t^{n-1}}{d(n-1)!}\sum_{j=0}^{m-2}\int_{\xi_j}^{\xi_{j+1}}(T-s)^{n-1}y(s)\,ds-\frac{t^{n-1}}{d(n-1)!}\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}(\xi_i-s)^{n-1}y(s)\,ds$$
$$-\frac{1}{(n-1)!}\int_0^t(t-s)^{n-1}y(s)\,ds$$
$$=\frac{t^{n-1}}{d(n-1)!}\sum_{j=0}^{m-2}\int_{\xi_j}^{\xi_{j+1}}(T-s)^{n-1}y(s)\,ds-\frac{t^{n-1}}{d(n-1)!}\left[\int_0^{\xi_1}\sum_{i=1}^{m-2}a_i(\xi_i-s)^{n-1}y(s)\,ds\right.$$
$$\left.+\int_{\xi_1}^{\xi_2}\sum_{i=2}^{m-2}a_i(\xi_i-s)^{n-1}\,y(s)\,ds+\cdots+\int_{\xi_{m-3}}^{\xi_{m-2}}a_{m-2}(\xi_{m-2}-s)^{n-1}y(s)\,ds\right]$$
$$-\frac{1}{(n-1)!}\int_0^t(t-s)^{n-1}y(s)\,ds$$
$$=\frac{t^{n-1}}{d(n-1)!}\sum_{j=0}^{m-2}\int_{\xi_j}^{\xi_{j+1}}(T-s)^{n-1}y(s)\,ds-\frac{t^{n-1}}{d(n-1)!}\sum_{j=0}^{m-3}\int_{\xi_j}^{\xi_{j+1}}\sum_{i=j+1}^{m-2}a_i(\xi_i-s)^{n-1}y(s)\,ds$$
$$-\frac{1}{(n-1)!}\int_0^t(t-s)^{n-1}y(s)\,ds,$$

where we denoted $\xi_0=0$ and $\xi_{m-1}=T$.

Therefore, we obtain

$$\begin{aligned}u(t)=&\sum_{j=0}^{m-3}\frac{t^{n-1}}{d(n-1)!}\left[\int_{\xi_j}^{\xi_{j+1}}(T-s)^{n-1}y(s)\,ds-\int_{\xi_j}^{\xi_{j+1}}\sum_{i=j+1}^{m-2}a_i(\xi_i-s)^{n-1}y(s)\,ds\right]\\&+\frac{t^{n-1}}{d(n-1)!}\int_{\xi_{m-2}}^{T}(T-s)^{n-1}y(s)\,ds-\frac{1}{(n-1)!}\int_0^t(t-s)^{n-1}y(s)\,ds.\end{aligned}\tag{5}$$

By (5) we have $u(t)=\displaystyle\int_0^T G(t,s)y(s)\,ds$, where G is of the form given in the statement of this lemma. □

Lemma 3. *If $a_i > 0$ for all $i = \overline{1, m-2}$, $0 < \xi_1 < \cdots < \xi_{m-2} < T$, $d > 0$ and $y \in C([0,T])$, $y(t) \geq 0$ for all $t \in [0,T]$, then the unique solution u of problem* (1), (2) *satisfies $u(t) \geq 0$ for all $t \in [0,T]$.*

Proof. We first show that $u(T) \geq 0$. Indeed we have

$$u(T) = \frac{T^{n-1}}{d(n-1)!}\int_0^T (T-s)^{n-1}y(s)\,ds - \frac{T^{n-1}}{d(n-1)!}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)^{n-1}y(s)\,ds$$

$$-\frac{1}{(n-1)!}\int_0^T (T-s)^{n-1}y(s)\,ds$$

$$= \frac{T^{n-1}-d}{d(n-1)!}\int_0^T (T-s)^{n-1}y(s)\,ds - \frac{T^{n-1}}{d(n-1)!}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)^{n-1}y(s)\,ds$$

$$= \frac{\sum_{i=1}^{m-2} a_i\xi_i^{n-1}}{d(n-1)!}\int_0^T (T-s)^{n-1}y(s)\,ds - \frac{T^{n-1}}{d(n-1)!}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)^{n-1}y(s)\,ds$$

$$= \frac{1}{d(n-1)!}\left[\sum_{i=1}^{m-2} a_i\xi_i^{n-1}\left(\int_0^{\xi_i} (T-s)^{n-1}y(s)\,ds + \int_{\xi_i}^T (T-s)^{n-1}y(s)\,ds\right)\right.$$

$$\left.-T^{n-1}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)^{n-1}y(s)\,ds\right]$$

$$= \frac{1}{d(n-1)!}\left\{\int_0^{\xi_i}\sum_{i=1}^{m-2} a_i\left[\xi_i^{n-1}(T-s)^{n-1} - T^{n-1}(\xi_i-s)^{n-1}\right]y(s)\,ds\right.$$

$$\left.+\sum_{i=1}^{m-2} a_i\xi_i^{n-1}\int_{\xi_i}^T (T-s)^{n-1}y(s)\,ds\right\} \geq 0,$$

because for $s \in [0,\xi_i]$ we have $\xi_i^{n-1}(T-s)^{n-1} - T^{n-1}(\xi_i - s)^{n-1} = (\xi_i T - \xi_i s)^{n-1} - (\xi_i T - Ts)^{n-1} > 0$.

Using a result from [6] (see also Theorem 1.1 from [15]), we deduce that $u(t) \geq 0$ for all $t \in [0,T]$. □

Lemma 4. ([15]) *If $d > 0$, $a_i > 0$ for all $i = \overline{1, m-2}$, $0 < \xi_1 < \cdots < \xi_{m-2} < T$, then $G(t,s) \geq 0$ for all t, $s \in [0,T]$.*

Remark 1. Under the assumptions of Lemma 3, by using Lemma 4 and the expression of $u(t) = \int_0^T G(t,s)y(s)\,ds$, we can also deduce that $u(t) \geq 0$ for all $t \in [0,T]$.

Lemma 5. *If $a_i > 0$ for all $i = \overline{1, m-2}$, $0 < \xi_1 < \cdots < \xi_{m-2} < T$, $d > 0$, $y \in C([0,T])$, $y(t) \geq 0$ for all $t \in [0,T]$, then the solution of problem* (1), (2) *satisfies*

$$\begin{cases} u(t) \leq \dfrac{T^{n-1}}{d(n-1)!}\displaystyle\int_0^T (T-s)^{n-1}y(s)\,ds, \ \ \forall t \in [0,T], \\ u(\xi_j) \geq \dfrac{\xi_j^{n-1}}{d(n-1)!}\displaystyle\int_{\xi_{m-2}}^T (T-s)^{n-1}y(s)\,ds, \ \ \forall\, j = \overline{1, m-2}. \end{cases} \tag{6}$$

Proof. By (3) we have

$$u(t) \le \frac{t^{n-1}}{d(n-1)!}\int_0^T (T-s)^{n-1}y(s)\,ds \le \frac{T^{n-1}}{d(n-1)!}\int_0^T (T-s)^{n-1}y(s)\,ds,$$

for all $t \in [0,T]$.

Then

$$u(\xi_j) = \int_0^T G(\xi_j,s)y(s)\,ds \ge \int_{\xi_{m-2}}^T G(\xi_j,s)y(s)\,ds = \frac{\xi_j^{n-1}}{d(n-1)!}\int_{\xi_{m-2}}^T (T-s)^{n-1}y(s)\,ds,$$

for all $j = \overline{1,m-2}$. □

From the proof of Lemma 2.2 in [15] we obtain the following result.

Lemma 6. *We assume that* $0 < \xi_1 < \cdots < \xi_{m-2} < T$, $a_i > 0$ *for all* $i = \overline{1,m-2}$, $d > 0$ *and* $y \in C([0,T])$, $y(t) \ge 0$ *for all* $t \in [0,T]$. *Then the solution of problem* (1), (2) *satisfies* $\inf\limits_{t\in[\xi_{m-2},T]} u(t) \ge \gamma\|u\|$, *where*

$$\gamma = \begin{cases} \min\left\{\dfrac{a_{m-2}(T-\xi_{m-2})}{T-a_{m-2}\xi_{m-2}}, \dfrac{a_{m-2}\xi_{m-2}^{n-1}}{T^{n-1}}\right\}, & if \ \sum\limits_{i=1}^{m-2} a_i < 1, \\ \min\left\{\dfrac{a_1\xi_1^{n-1}}{T^{n-1}}, \dfrac{\xi_{m-2}^{n-1}}{T^{n-1}}\right\}, & if \ \sum\limits_{i=1}^{m-2} a_i \ge 1 \end{cases}$$

and $\|u\| = \sup\limits_{t\in[0,T]} |u(t)|$.

Remark 2. From the above expression for γ, we see that $\gamma < 1$.

3 The existence of positive solutions

In this section we shall give sufficient conditions on λ and μ such that positive solutions with respect to a cone for problem (S), (BC) exist.

We present the assumptions that we shall use in the sequel.

(H1) $0 < \xi_1 < \cdots < \xi_{m-2} < T$, $a_i > 0$, $i = \overline{1,m-2}$, $d = T^{n-1} - \sum\limits_{i=1}^{m-2} a_i\xi_i^{n-1} > 0$.

(H2) The functions b, $c : [0,T] \to [0,\infty)$ are continuous and each does not vanish identically on any subinterval of $[0,T]$.

(H3) The functions f, $g : [0,\infty) \to [0,\infty)$ are continuous and the limits

$$f_0 = \lim_{x\to0^+}\frac{f(x)}{x},\ \ g_0 = \lim_{x\to0^+}\frac{g(x)}{x},\ \ f_\infty = \lim_{x\to\infty}\frac{f(x)}{x},\ \ g_\infty = \lim_{x\to\infty}\frac{g(x)}{x}$$

exist and are positive numbers.

Using the Green's function given in Lemma 2, a pair $(u(t), v(t))$, $t \in [0,T]$ is a solution of the eigenvalue problem (S), (BC) if and only if

$$\begin{cases} u(t) = \lambda \int_0^T G(t,s)b(s)f\left(\mu \int_0^T G(s,\tau)c(\tau)g(u(\tau))\,d\tau\right) ds, & 0 \le t \le T, \\ v(t) = \mu \int_0^T G(t,s)c(s)g(u(s))\,ds, & 0 \le t \le T. \end{cases}$$

We consider the Banach space $X = C([0,T])$ with supremum norm $\|\cdot\|$ and define the cone $C \subset X$ by

$$C = \{u \in X,\ u(t) \ge 0,\ \forall\, t \in [0,T] \text{ and } \inf_{t\in[\xi_{m-2},T]} u(t) \ge \gamma\|u\|\},$$

where γ is defined in Lemma 6.

For our first result we define the positive numbers L_1 and L_2 by

$$L_1 = \max\left\{ \left(\frac{\gamma^2 \xi_{m-2}^{n-1}}{d(n-1)!} \int_{\xi_{m-2}}^T (T-s)^{n-1} b(s) f_\infty \, ds \right)^{-1}, \left(\frac{\gamma^2 \xi_{m-2}^{n-1}}{d(n-1)!} \int_{\xi_{m-2}}^T (T-s)^{n-1} c(s) g_\infty \, ds \right)^{-1} \right\},$$

$$L_2 = \min\left\{ \left(\frac{T^{n-1}}{d(n-1)!} \int_0^T (T-s)^{n-1} b(s) f_0 \, ds \right)^{-1}, \left(\frac{T^{n-1}}{d(n-1)!} \int_0^T (T-s)^{n-1} c(s) g_0 \, ds \right)^{-1} \right\}.$$

Theorem 2. *Assume that (H1)–(H3) hold and $L_1 < L_2$. Then for each λ and μ satisfying λ, $\mu \in (L_1, L_2)$, there exist a positive solution with respect to a cone, $(u(t), v(t))$, $t \in [0,T]$, of problem (S), (BC).*

Proof. Let λ, $\mu \in (L_1, L_2)$ and we choose a positive number ε such that $\varepsilon < f_\infty$, $\varepsilon < g_\infty$,

$$\max\left\{ \left(\frac{\gamma^2 \xi_{m-2}^{n-1}}{d(n-1)!} \int_{\xi_{m-2}}^T (T-s)^{n-1} b(s)(f_\infty - \varepsilon) \, ds \right)^{-1}, \left(\frac{\gamma^2 \xi_{m-2}^{n-1}}{d(n-1)!} \int_{\xi_{m-2}}^T (T-s)^{n-1} c(s)(g_\infty - \varepsilon) \, ds \right)^{-1} \right\} \le \min(\lambda, \mu)$$

and

$$\max(\lambda,\mu) \le \min\left\{ \left(\frac{T^{n-1}}{d(n-1)!} \int_0^T (T-s)^{n-1} b(s)(f_0 + \varepsilon) \, ds \right)^{-1}, \left(\frac{T^{n-1}}{d(n-1)!} \int_0^T (T-s)^{n-1} c(s)(g_0 + \varepsilon) \, ds \right)^{-1} \right\}.$$

We now define the operator $\mathcal{A} : C \to X$, by

$$\mathcal{A}(u)(t) = \lambda \int_0^T G(t,s)b(s)f\left(\mu \int_0^T G(s,\tau)c(\tau)g(u(\tau))\,d\tau\right) ds, \quad 0 \le t \le T, \quad u \in C.$$

By Lemma 6, we have $\mathcal{A}(C) \subset C$. By using the Arzela-Ascoli theorem we deduce that the operator $\mathcal{A}$ is completely continuous (compact and continuous). By definitions of f_0 and g_0 there exists $K_1 > 0$ such that

$$f(x) \le (f_0+\varepsilon)x \quad \text{and} \quad g(x) \le (g_0+\varepsilon)x, \;\; 0 < x \le K_1.$$

Using (H3) we have $f(0) = g(0) = 0$ and the above inequalities are also valid for $x = 0$.

Let $u \in C$ with $\|u\| = K_1$. Because $v(t) = \mu \int_0^T G(t,s)c(s)g(u(s))\,ds,\;\; t \in [0,T]$ satisfies the problem (1), (2) with $y(t) = \mu c(t)g(u(t)),\;\; t \in [0,T]$, then by (6) and the above property of g, we deduce for $t \in [0,T]$

$$v(t) = \mu \int_0^T G(t,s)c(s)g(u(s))\,ds \le \frac{\mu T^{n-1}}{d(n-1)!}\int_0^T (T-s)^{n-1}c(s)g(u(s))\,ds$$
$$\le \frac{\mu T^{n-1}}{d(n-1)!}\int_0^T (T-s)^{n-1}c(s)(g_0+\varepsilon)u(s)\,ds$$
$$\le \frac{\mu T^{n-1}}{d(n-1)!}\int_0^T (T-s)^{n-1}c(s)(g_0+\varepsilon)\|u\|\,ds \le \|u\| = K_1.$$

By using once again Lemma 5 (relations (6)) and the properties of the function f, we have

$$\mathcal{A}(u)(t) = \lambda \int_0^T G(t,s)b(s)f\left(\mu \int_0^T G(s,\tau)c(\tau)g(u(\tau))\,d\tau\right) ds$$
$$\le \frac{\lambda T^{n-1}}{d(n-1)!}\int_0^T (T-s)^{n-1}b(s)f\left(\mu \int_0^T G(s,\tau)c(\tau)g(u(\tau))\,d\tau\right) ds$$
$$\le \frac{\lambda T^{n-1}}{d(n-1)!}\int_0^T (T-s)^{n-1}b(s)(f_0+\varepsilon)\left(\mu \int_0^T G(s,\tau)c(\tau)g(u(\tau))\,d\tau\right) ds$$
$$\le \frac{\lambda T^{n-1}}{d(n-1)!}\int_0^T (T-s)^{n-1}b(s)(f_0+\varepsilon)K_1\,ds \le K_1 = \|u\|, 0 \le t \le T.$$

Then $\|\mathcal{A}(u)\| \le \|u\|$, for all $u \in C$ with $\|u\| = K_1$. If we denote by $\Omega_1 = \{u \in C,\;\; \|u\| < K_1\}$, then we obtain $\|\mathcal{A}(u)\| \le \|u\|$ for all $u \in C \cap \partial\Omega_1$.

Next, by the definitions of f_∞ and g_∞, there exists $\bar{K}_2 > 0$ such that

$$f(x) \ge (f_\infty-\varepsilon)x \quad \text{and} \quad g(x) \ge (g_\infty-\varepsilon)x, \;\; x \ge \bar{K}_2.$$

We consider now $K_2 = \max\left\{2K_1, \bar{K}_2/\gamma\right\}$. For $u \in C$ with $\|u\| = K_2$, we obtain by using Lemma 6, that

$$u(t) \ge \inf_{s\in[\xi_{m-2},T]} u(s) \ge \gamma\|u\| = \gamma K_2 \ge \bar{K}_2, \;\; \forall\, t \in [\xi_{m-2},T].$$

Then, by using (6), Lemma 6, and the above relations, we obtain for $t \ge \xi_{m-2}$

$$v(t) = \mu \int_0^T G(t,s)c(s)g(u(s))\,ds \ge \gamma\|v\| \ge \gamma v(\xi_{m-2})$$
$$= \gamma\mu \int_0^T G(\xi_{m-2},s)c(s)g(u(s))\,ds$$
$$\ge \frac{\gamma\mu\xi_{m-2}^{n-1}}{d(n-1)!}\int_{\xi_{m-2}}^T (T-s)^{n-1}c(s)g(u(s))\,ds$$
$$\ge \frac{\gamma\mu\xi_{m-2}^{n-1}}{d(n-1)!}\int_{\xi_{m-2}}^T (T-s)^{n-1}c(s)(g_\infty-\varepsilon)u(s)\,ds$$
$$\ge \frac{\gamma^2\mu\xi_{m-2}^{n-1}}{d(n-1)!}\int_{\xi_{m-2}}^T (T-s)^{n-1}c(s)(g_\infty-\varepsilon)\|u\|\,ds \ge \|u\| = K_2$$

and

$$\mathcal{A}(u)(\xi_{m-2})$$
$$\geq \lambda \int_{\xi_{m-2}}^{T} \frac{\xi_{m-2}^{n-1}}{d(n-1)!}(T-s)^{n-1}b(s)f\left(\mu \int_0^T G(s,\tau)c(\tau)g(u(\tau))\,d\tau\right) ds$$
$$\geq \frac{\lambda\xi_{m-2}^{n-1}}{d(n-1)!}\int_{\xi_{m-2}}^{T}(T-s)^{n-1}b(s)(f_\infty-\varepsilon)\left(\mu\int_0^T G(s,\tau)c(\tau)g(u(\tau))\,d\tau\right) ds$$
$$\geq \frac{\lambda\xi_{m-2}^{n-1}}{d(n-1)!}\int_{\xi_{m-2}}^{T}(T-s)^{n-1}b(s)(f_\infty-\varepsilon)K_2\,ds$$
$$\geq \frac{\gamma^2\lambda\xi_{m-2}^{n-1}}{d(n-1)!}\int_{\xi_{m-2}}^{T}(T-s)^{n-1}b(s)(f_\infty-\varepsilon)K_2\,ds \geq K_2 = \|u\|.$$

Therefore $\|\mathcal{A}(u)\| \geq \mathcal{A}(u)(\xi_{m-2}) \geq \|u\|$, for all $u \in C$ with $\|u\| = K_2$. We denote by $\Omega_2 = \{u \in C,\ \|u\| < K_2\}$. Then $\|\mathcal{A}(u)\| \geq \|u\|$, for all $u \in C \cap \partial\Omega_2$.

We now apply Theorem 1 i) and we deduce that $\mathcal{A}$ has a fixed point $u \in C \cap (\bar{\Omega}_2 \setminus \Omega_1)$. This element together with $v(t) = \mu \int_0^T G(t,s)c(s)g(u(s))\,ds,\ t \in [0,T]$ represent a positive solution of (S), (BC) with respect to cone C, for the given λ and μ. □

Remark 3. The condition $L_1 < L_2$ from Theorem 2 is equivalent to
$$\frac{d(n-1)!}{\gamma^2\xi_{m-2}^{n-1}}\left(\min\left\{\int_{\xi_{m-2}}^{T}(T-s)^{n-1}b(s)f_\infty\,ds, \int_{\xi_{m-2}}^{T}(T-s)^{n-1}c(s)g_\infty\,ds\right\}\right)^{-1}$$
$$< \frac{d(n-1)!}{T^{n-1}}\left(\max\left\{\int_0^T (T-s)^{n-1}b(s)f_0\,ds, \int_0^T (T-s)^{n-1}c(s)g_0\,ds\right\}\right)^{-1}$$
or
$$\frac{\max\left\{\int_0^T (T-s)^{n-1}b(s)f_0\,ds, \int_0^T (T-s)^{n-1}c(s)g_0\,ds\right\}}{\min\left\{\int_{\xi_{m-2}}^{T}(T-s)^{n-1}b(s)f_\infty\,ds, \int_{\xi_{m-2}}^{T}(T-s)^{n-1}c(s)g_\infty\,ds\right\}} < \frac{\gamma^2\xi_{m-2}^{n-1}}{T^{n-1}}.$$

In what follows we shall present another existence result for (S), (BC). Let us consider positive numbers
$$L_3 = \max\left\{\left(\frac{\gamma\xi_{m-2}^{n-1}}{d(n-1)!}\int_{\xi_{m-2}}^{T}(T-s)^{n-1}b(s)f_0\,ds\right)^{-1}, \left(\frac{\gamma\xi_{m-2}^{n-1}}{d(n-1)!}\int_{\xi_{m-2}}^{T}(T-s)^{n-1}c(s)g_0\,ds\right)^{-1}\right\},$$
$$L_4 = \min\left\{\left(\frac{T^{n-1}}{d(n-1)!}\int_0^T (T-s)^{n-1}b(s)f_\infty\,ds\right)^{-1}, \left(\frac{T^{n-1}}{d(n-1)!}\int_0^T (T-s)^{n-1}c(s)g_\infty\,ds\right)^{-1}\right\}.$$

Theorem 3. *Assume the assumptions (H1)–(H3) hold and $L_3 < L_4$. Then for each λ and μ satisfying λ, $\mu \in (L_3, L_4)$, there exists a positive solution with respect to a cone, $(u(t), v(t))$, $t \in [0,T]$, of (S), (BC).*

Proof. Let λ and μ with λ, $\mu \in (L_3, L_4)$. We select a positive number ε such that

$\varepsilon < f_0$, $\varepsilon < g_0$ and

$$\max\left\{\left(\frac{\gamma\xi_{m-2}^{n-1}}{d(n-1)!}\int_{\xi_{m-2}}^{T}(T-s)^{n-1}b(s)(f_0-\varepsilon)\,ds\right)^{-1},\right.$$
$$\left.\left(\frac{\gamma\xi_{m-2}^{n-1}}{d(n-1)!}\int_{\xi_{m-2}}^{T}(T-s)^{n-1}c(s)(g_0-\varepsilon)\,ds\right)^{-1}\right\}\le\min(\lambda,\mu)$$

and

$$\max(\lambda,\mu)\le\min\left\{\left(\frac{T^{n-1}}{d(n-1)!}\int_0^T(T-s)^{n-1}b(s)(f_\infty+\varepsilon)\,ds\right)^{-1},\right.$$
$$\left.\left(\frac{T^{n-1}}{d(n-1)!}\int_0^T(T-s)^{n-1}c(s)(g_\infty+\varepsilon)\,ds\right)^{-1}\right\}.$$

We also consider the operator $\mathcal{A}$ defined in the proof of Theorem 2. From the definitions of f_0 and g_0, we deduce that there exists $\bar{K}_3>0$ such that

$$f(x)\ge(f_0-\varepsilon)x\quad\text{and}\quad g(x)\ge(g_0-\varepsilon)x,\;\;0<x\le\bar{K}_3.$$

Using the properties of f and g the above inequalities are also valid for $x=0$.

In addition, because g is a continuous function with $g_0>0$, then $g(0)=0$ and there exists $K_3\in(0,\bar{K}_3)$ such that

$$g(x)\le\frac{\bar{K}_3}{\dfrac{\mu T^{n-1}}{d(n-1)!}\displaystyle\int_0^T(T-s)^{n-1}c(s)\,ds},\;\;0<x\le K_3.$$

For $u\in C$ with $\|u\|=K_3$, by (6) and the above inequality, we deduce that for all $t\in[0,T]$

$$v(t)=\mu\int_0^T G(t,s)c(s)g(u(s))\,ds\le\frac{\mu T^{n-1}}{d(n-1)!}\int_0^T(T-s)^{n-1}c(s)g(u(s))\,ds$$
$$\le\frac{\mu T^{n-1}}{d(n-1)!}\int_0^T(T-s)^{n-1}c(s)\frac{\bar{K}_3}{\dfrac{\mu T^{n-1}}{d(n-1)!}\displaystyle\int_0^T(T-\tau)^{n-1}c(\tau)\,d\tau}\,ds=\bar{K}_3.$$

By using (6), Lemma 6 and the properties of f, g we then obtain

$$\mathcal{A}(u)(\xi_{m-2})\ge\lambda\int_{\xi_{m-2}}^T\frac{\xi_{m-2}^{n-1}}{d(n-1)!}(T-s)^{n-1}b(s)f\left(\mu\int_0^T G(s,\tau)c(\tau)g(u(\tau))\,d\tau\right)ds$$
$$\ge\frac{\lambda\xi_{m-2}^{n-1}}{d(n-1)!}\int_{\xi_{m-2}}^T(T-s)^{n-1}b(s)(f_0-\varepsilon)\left(\mu\int_0^T G(s,\tau)c(\tau)g(u(\tau))\,d\tau\right)ds$$
$$\ge\frac{\lambda\xi_{m-2}^{n-1}}{d(n-1)!}\int_{\xi_{m-2}}^T(T-s)^{n-1}b(s)(f_0-\varepsilon)\gamma\|v\|\,ds$$
$$\ge\frac{\lambda\gamma\xi_{m-2}^{n-1}}{d(n-1)!}\int_{\xi_{m-2}}^T(T-s)^{n-1}b(s)(f_0-\varepsilon)v(\xi_{m-2})\,ds$$
$$\ge\left(\frac{\lambda\gamma\xi_{m-2}^{n-1}}{d(n-1)!}\int_{\xi_{m-2}}^T(T-s)^{n-1}b(s)(f_0-\varepsilon)\,ds\right)$$
$$\times\left(\frac{\mu\xi_{m-2}^{n-1}}{d(n-1)!}\int_{\xi_{m-2}}^T(T-s)^{n-1}c(s)g(u(s))\,ds\right)$$

$$\geq \left(\frac{\lambda\gamma\xi_{m-2}^{n-1}}{d(n-1)!} \int_{\xi_{m-2}}^{T} (T-s)^{n-1} b(s)(f_0-\varepsilon)\, ds \right)$$
$$\times \left(\frac{\mu\xi_{m-2}^{n-1}}{d(n-1)!} \int_{\xi_{m-2}}^{T} (T-s)^{n-1} c(s)(g_0-\varepsilon) u(s)\, ds \right)$$
$$\geq \left(\frac{\lambda\gamma\xi_{m-2}^{n-1}}{d(n-1)!} \int_{\xi_{m-2}}^{T} (T-s)^{n-1} b(s)(f_0-\varepsilon)\, ds \right)$$
$$\times \left(\frac{\mu\gamma\xi_{m-2}^{n-1}}{d(n-1)!} \int_{\xi_{m-2}}^{T} (T-s)^{n-1} c(s)(g_0-\varepsilon)\, ds \right) \|u\| \geq \|u\|.$$

Hence, $\|\mathcal{A}(u)\| \geq \mathcal{A}(u)(\xi_{m-2}) \geq \|u\|$, for $u \in C$ with $\|u\| = K_3$. We denote by $\Omega_3 = \{u \in C,\ \|u\| < K_3\}$, and then we have $\|\mathcal{A}(u)\| \geq \|u\|$ for all $u \in C \cap \partial\Omega_3$.

We now consider the functions $f^*,\ g^* : [0,\infty) \to [0,\infty)$ defined by $f^*(x) = \sup_{0\leq y\leq x} f(y)$, $g^*(x) = \sup_{0\leq y\leq x} g(y)$. By (H2) we obtain for f^* and g^* the relations $\lim_{x\to\infty} \frac{f^*(x)}{x} = f_\infty$, $\lim_{x\to\infty} \frac{g^*(x)}{x} = g_\infty$.
We also have $f(x) \leq f^*(x)$, $g(x) \leq g^*(x)$, for all $x \geq 0$. Then there exists $\bar{K}_4 > 0$ such that

$$f^*(x) \leq (f_\infty+\varepsilon)x,\ \ g^*(x) \leq (g_\infty+\varepsilon)x, \text{ for all } x \geq \bar{K}_4.$$

Let $K_4 > \max\{2K_3, \bar{K}_4\}$. Then for u with $\|u\| = K_4$ we obtain

$$\mathcal{A}(u)(t) \leq \frac{T^{n-1}}{d(n-1)!} \int_0^T (T-s)^{n-1} b(s)\lambda f\left(\mu \int_0^T G(s,\tau)c(\tau)g(u(\tau))\, d\tau \right) ds$$
$$\leq \frac{\lambda T^{n-1}}{d(n-1)!} \int_0^T (T-s)^{n-1} b(s) f^*\left(\mu \int_0^T G(s,\tau)c(\tau)g(u(\tau))\, d\tau \right) ds$$
$$\leq \frac{\lambda T^{n-1}}{d(n-1)!} \int_0^T (T-s)^{n-1} b(s) f^*\left(\frac{\mu T^{n-1}}{d(n-1)!} \int_0^T (T-\tau)^{n-1} c(\tau)g(u(\tau))\, d\tau \right) ds$$
$$\leq \frac{\lambda T^{n-1}}{d(n-1)!} \int_0^T (T-s)^{n-1} b(s) f^*\left(\frac{\mu T^{n-1}}{d(n-1)!} \int_0^T (T-\tau)^{n-1} c(\tau)g^*(u(\tau))\, d\tau \right) ds$$
$$\leq \frac{\lambda T^{n-1}}{d(n-1)!} \int_0^T (T-s)^{n-1} b(s) f^*\left(\frac{\mu T^{n-1}}{d(n-1)!} \int_0^T (T-\tau)^{n-1} c(\tau)g^*(K_4)\, d\tau \right) ds$$
$$\leq \frac{\lambda T^{n-1}}{d(n-1)!} \int_0^T (T-s)^{n-1} b(s) f^*\left(\frac{\mu T^{n-1}}{d(n-1)!} \int_0^T (T-\tau)^{n-1} c(\tau)(g_\infty+\varepsilon)K_4\, d\tau \right) ds$$
$$\leq \frac{\lambda T^{n-1}}{d(n-1)!} \int_0^T (T-s)^{n-1} b(s) f^*(K_4)\, ds$$
$$\leq \frac{\lambda T^{n-1}}{d(n-1)!} \int_0^T (T-s)^{n-1} b(s)(f_\infty+\varepsilon)K_4\, ds \leq K_4 = \|u\|.$$

So $\|\mathcal{A}(u)\| \leq \|u\|$, for all $u \in C$ with $\|u\| = K_4$. If we denote by $\Omega_4 = \{u \in C,\ \|u\| < K_4\}$, then we obtain $\|\mathcal{A}(u)\| \leq \|u\|$, for all $u \in C \cap \partial\Omega_4$.

By Theorem 1 ii) we deduce that $\mathcal{A}$ has a fixed point $u \in C \cap (\bar{\Omega}_4 \setminus \Omega_3)$, which together with $v(t) = \mu \int_0^T G(t,s)c(s)g(u(s))\, ds$, $t \in [0,T]$ give us a positive solution of (S), (BC) with respect to cone C, for the chosen values λ and μ. □

Remark 4. The condition $L_3 < L_4$ is equivalent to

$$\frac{d(n-1)!}{\gamma\xi_{m-2}^{n-1}}\left(\min\left\{\int_{\xi_{m-2}}^T (T-s)^{n-1}b(s)f_0\,ds,\ \int_{\xi_{m-2}}^T (T-s)^{n-1}c(s)g_0\,ds\right\}\right)^{-1}$$

$$< \frac{d(n-1)!}{T^{n-1}}\left(\max\left\{\int_0^T (T-s)^{n-1}b(s)f_\infty\,ds,\ \int_0^T (T-s)^{n-1}c(s)g_\infty\,ds\right\}\right)^{-1}$$

or

$$\frac{\max\left\{\int_0^T (T-s)^{n-1}b(s)f_\infty\,ds,\ \int_0^T (T-s)^{n-1}c(s)g_\infty\,ds\right\}}{\min\left\{\int_{\xi_{m-2}}^T (T-s)^{n-1}b(s)f_0\,ds,\ \int_{\xi_{m-2}}^T (T-s)^{n-1}c(s)g_0\,ds\right\}} < \frac{\gamma\xi_{m-2}^{n-1}}{T^{n-1}}.$$

4 An example

As in Example 4.1 in [12], let us consider the functions

$$\begin{cases} f(x) = p_2|\sin x| + p_1 x e^{-1/x}, & x \in [0,\infty), \\ g(x) = q_2|\sin x| + q_1 x e^{-1/x}, & x \in [0,\infty), \end{cases}$$

with $p_1,\ p_2,\ q_1,\ q_2 > 0$.

We have $\lim_{x\to 0^+}\frac{f(x)}{x} = p_2,\ \lim_{x\to\infty}\frac{f(x)}{x} = p_1,\ \lim_{x\to 0^+}\frac{g(x)}{x} = q_2,\ \lim_{x\to\infty}\frac{g(x)}{x} = q_1$.

Let $T = 1,\ n = 3,\ m = 4,\ b(t) = b_0 t,\ c(t) = c_0 t,\ t \in [0,1]$, with $b_0,\ c_0 > 0$ and $\xi_1 = \frac{1}{3},\ \xi_2 = \frac{2}{3},\ a_1 = 1,\ a_2 = \frac{1}{2}$.

We consider the third-order differential system

$$(S_0)\qquad \begin{cases} u'''(t) + \lambda b_0 t\left[p_2|\sin v(t)| + p_1 v(t)e^{-1/v(t)}\right] = 0, & t \in (0,1) \\ v'''(t) + \mu c_0 t\left[q_2|\sin u(t)| + q_1|u(t)|e^{-1/u(t)}\right] = 0, & t \in (0,1), \end{cases}$$

with the boundary conditions

$$(BC_0)\qquad \begin{cases} u(0) = u'(0) = 0, & u(1) = u(\tfrac{1}{3}) + \tfrac{1}{2}u(\tfrac{2}{3}) \\ v(0) = v'(0) = 0, & v(1) = v(\tfrac{1}{3}) + \tfrac{1}{2}v(\tfrac{2}{3}). \end{cases}$$

We also have $d = 1 - \sum_{i=1}^{2} a_i\xi_i^2 = \frac{2}{3} > 0,\ \sum_{i=1}^{2} a_i = \frac{3}{2} > 1$ and $\gamma = \min\{a_1\xi_1^2, \xi_2^2\} = \frac{1}{9}$.

The condition $L_1 < L_2$ or the equivalent form given in Remark 3 is

$$\frac{\max\left\{\int_0^1 (1-s)^2 b_0 s p_2\,ds,\ \int_0^1 (1-s)^2 c_0 s q_2\,ds\right\}}{\min\left\{\int_{2/3}^1 (1-s)^2 b_0 s p_1\,ds,\ \int_{2/3}^1 (1-s)^2 c_0 s q_1\,ds\right\}} < \frac{4}{729}$$

or

$$\frac{\max\{b_0p_2, c_0q_2\}}{\min\{b_0p_1, c_0q_1\}} < \frac{4}{6561}.$$

Therefore if the above condition is verified, then by Theorem 2 we deduce that for all numbers λ, $\mu \in (L_1, L_2)$ the problem (S_0), (BC_0) has positive solutions.

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