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Differentiability of solutions with respect to the delay function in functional differential equations

Dedicated to Professor Tibor Krisztin on the occasion of his 60th birthday

Ferenc Hartung ✉

University of Pannonia, H-8201 Veszprém, P.O. Box 158, Hungary

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Abstract. In this paper we consider a class of functional differential equations with time-dependent delay. We show continuous differentiability of the solution with respect to the time delay function for each fixed time value assuming natural conditions on the delay function. As an application of the differentiability result, we give a numerical study to estimate the time delay function using the quasilinearization method.

Keywords: delay differential equation, time-dependent delay, differentiability with respect to parameters.

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1 Introduction

In this paper we consider a class of functional differential equations (FDEs) with a time-dependent delay of the form

$$\dot{x}(t) = f(t, x(t), x(t - \tau(t))), \qquad t \geq 0, \tag{1.1}$$

where the associated initial condition is

$$x(t) = \varphi(t), \qquad t \in [-r, 0]. \tag{1.2}$$

Here and throughout the manuscript $r > 0$ is a fixed constant, and $0 \leq \tau(t) \leq r$ for all $t \geq 0$.

In this paper we consider the delay function τ as parameter in the initial value problem (IVP) (1.1)-(1.2), and we denote the corresponding solution by $x(t, \tau)$. The main goal of this paper is to discuss the differentiability of the solution $x(t, \tau)$ with respect to (wrt) τ. By differentiability we mean Fréchet-differentiability throughout this paper. Differentiability of solutions of FDEs wrt to other parameters is studied, e.g., in the monograph [6]. The first

✉ Email: hartung.ferenc@uni-pannon.hu

paper which discussed and proved the differentiability of solutions of FDEs wrt constant delay was [7]. The result was formulated for the class of FDEs of the form

$$\dot{x}(t) = g(x(t), x(t-\eta)), \tag{1.3}$$

where $g\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable function. It was shown that if $\alpha > 0$ is such that the solutions $x(t, \eta)$ of (1.3) are defined for $t \in [0, \alpha]$ and $\eta \in (\delta_1, \delta_2)$ with some $0 < \delta_1 < \delta_2$, then the map

$$\mathbb{R} \supset (\delta_1, \delta_2) \ni \eta \mapsto x(\cdot, \eta) \in W^{1,1}([0, \alpha], \mathbb{R}^n)$$

is continuously differentiable. Here $W^{1,1}([0, \alpha], \mathbb{R}^n)$ is the space of absolutely continuous functions of finite norm

$$\|\psi\|_{W^{1,1}([0,\alpha],\mathbb{R}^n)} = \int_0^\alpha (|\psi(s)| + |\dot{\psi}(s)|)ds.$$

Differentiability of the solution $x(t, \tau)$ wrt τ at a fixed time t was an open question, but in many applications this stronger sense of differentiability is needed. This problem was investigated later in [11] and recently in [12]. We note that in both papers the proofs are incorrect.

In this paper we prove, under natural conditions, that the solution $x(t, \tau)$ of the Equation (1.1) is differentiable wrt the time delay function τ for each fixed time t (see Theorem 4.4 below). The proof uses the method developed in [9] to show differentiability of solutions wrt parameters in FDEs with state-dependent delays. As a consequence of our main result, we get the differentiability of the solutions $x(t, \eta)$ of (1.3) wrt the constant delay η (see Corollary 4.5).

As an application of the differentiability results, we give a numerical study where we estimate the time delay function using the method of quasilinearization. This method uses point evaluations of the derivatives of the solution wrt the delay function τ.

This paper is organized as follows. Section 2 introduces notations and some preliminary results, Section 3 discusses the well-posedness of the IVP (1.1)–(1.2), Section 4 studies differentiability of the solution wrt the delay function, and Section 5 presents a numerical study for the parameter estimation of the delay function τ using the quasilinearization method.

2 Notations and preliminaries

In this section we introduce some basic notations which will be used throughout this paper, and recall two results from the literature which will be important in our proofs.

A fixed norm on $\mathbb{R}^n$ and its induced matrix norm on $\mathbb{R}^{n\times n}$ are both denoted by $|\cdot|$. For a fixed $\alpha > 0$, $C_\alpha := C([-r, \alpha], \mathbb{R}^n)$ denotes the Banach space of continuous functions $\psi : [-r, \alpha] \to \mathbb{R}^n$ equipped with the norm $\|\psi\|_{C_\alpha} := \sup\{|\psi(s)| : s \in [-r, \alpha]\}$. $L^\infty_\alpha := L^\infty([-r, \alpha], \mathbb{R}^n)$ denotes the space of Lebesgue-measurable functions which are essentially bounded, where the norm is defined by $\|\psi\|_{L^\infty_\alpha} := \operatorname{ess\,sup}\{|\psi(s)| : s \in [-r, \alpha]\}$. $W^{1,\infty}_\alpha := W^{1,\infty}([-r, \alpha], \mathbb{R}^n)$ denotes the Banach space of absolutely continuous functions $\psi\colon [-r, \alpha] \to \mathbb{R}^n$ of finite norm defined by $\|\psi\|_{W^{1,\infty}_\alpha} := \max\left\{\|\psi\|_{C_\alpha}, \|\dot{\psi}\|_{L^\infty_\alpha}\right\}$. For $\alpha = 0$ we use te notations C, L^∞ and $W^{1,\infty}$ instead of C_0, L^∞_0 and $W^{1,\infty}_0$. We note that $W^{1,\infty}$ is equal to the space of Lipschitz-continuous functions from $[-r, 0]$ to $\mathbb{R}^n$. We also use the notations $C_{\alpha,1} := C([0, \alpha], \mathbb{R})$ and $W^{1,\infty}_{\alpha,1} := W^{1,\infty}([0, \alpha], \mathbb{R})$.

$\mathcal{L}(X, Y)$ denotes the space of bounded linear operators from X to Y, where X and Y are normed linear spaces. An open ball in the normed linear space $(X, \|\cdot\|_X)$ centered at a point $x \in X$ with radius δ is denoted by $\mathcal{B}_X(x;\delta) := \{y \in X : \|x-y\|_X < \delta\}$. An open neighbourhood of a set $M \subset X$ with radius δ is denoted by $\mathcal{B}_X(M;\delta) := \{y \in X :$ there exists $x \in M$ s.t. $\|x-y\|_X < \delta\}$.

The partial derivatives of a function $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ wrt the second and third variables will be denoted by $D_2 f$ and $D_3 f$, respectively. Then $D_i f(t,u,v) \in \mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)$ for $t \in \mathbb{R}$, $u, v \in \mathbb{R}^n$ and $i = 2,3$, which will be identified by its $n \times n$ matrix-valued representation.

We recall the following result from [4], which was essential to prove differentiability wrt parameters in SD-DDEs in [9]. Note that the second part of the lemma was stated in [4] under the assumption $|u_k - u|_{W^{1,\infty}_{\alpha,1}} \to 0$ as $k \to \infty$, but this stronger assumption on the convergence is not needed in the proof.

Lemma 2.1 ([4])**.** *Let $p \in [1,\infty)$, $g \in L^p([-r,\alpha],\mathbb{R}^n)$, $\varepsilon > 0$, and $u \in \mathcal{A}(\varepsilon)$, where*

$$\mathcal{A}(\varepsilon) := \big\{ v \in W^{1,\infty}([0,\alpha],[-r,\alpha]) \,:\, \dot{v}(s) \geq \varepsilon \text{ for a.e. } s \in [0,\alpha] \big\}.$$

Then

$$\int_0^\alpha |g(u(s))|^p \, ds \leq \frac{1}{\varepsilon} \int_{-r}^\alpha |g(s)|^p \, ds.$$

Moreover, if the sequence $u_k \in \mathcal{A}(\varepsilon)$ is such that $|u_k - u|_{C_{\alpha,1}} \to 0$ as $k \to \infty$, then

$$\lim_{k\to\infty} \int_0^\alpha \big|g(u_k(s)) - g(u(s))\big|^p \, ds = 0.$$

Let $\mathbb{R}_+ := [0,\infty)$. We recall the following result from [9], which is a simple consequence of Gronwall's lemma.

Lemma 2.2 ([9])**.** *Suppose $a \geq 0$, $b\colon [0,\alpha] \to \mathbb{R}_+$ and $g\colon [-r,\alpha] \to \mathbb{R}^n$ are continuous functions such that $|g(s)| \leq a$ for $-r \leq s \leq 0$, and*

$$|g(t)| \leq a + \int_0^t b(s) \max_{s-r\leq\theta\leq s} |g(\theta)| \, ds, \qquad t \in [0,\alpha].$$

Then

$$|g(t)| \leq \max_{t-r\leq\theta\leq t} |g(\theta)| \leq a e^{\int_0^\alpha b(s)\,ds}, \qquad t \in [0,\alpha].$$

3 Well-posedness

Consider the nonlinear FDE with time-dependent delay

$$\dot{x}(t) = f(t, x(t), x(t-\tau(t))), \qquad t \geq 0, \tag{3.1}$$

and the corresponding initial condition

$$x(t) = \varphi(t), \qquad t \in [-r, 0]. \tag{3.2}$$

It is known (see, e.g., [6]) that if $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\tau : \mathbb{R}_+ \to [0,r]$ and $\varphi \in C$ are continuous functions, and f is Lipschitz-continuous in its second and third variables, then the IVP (3.1)–(3.2) has a unique noncontinuable solution on an interval $[-r,T)$ for some finite $T > 0$ or for $T = \infty$. If we want to emphasize the dependence of this solution on the delay function τ, we will use the notation $x(t,\tau)$.

Throughout the rest of the manuscript we assume

(H) $f \in C(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, and it is continuously differentiable wrt its second and third variables, and $\varphi \in W^{1,\infty}$.

The next result shows that, assuming the condition (H) and $0 < \tau(t) < r$ for $t \geq 0$, the solution $x(t,\tau)$ depends Lipschitz-continuously on τ.

Lemma 3.1. *Suppose (H). Then for every $\hat\tau \in C(\mathbb{R}_+, (0,r))$ there exists a unique noncontinuable solution $x(t,\hat\tau)$ of the IVP (3.1)–(3.2) defined on the interval $[-r,T)$ for some $T > 0$ or $T = \infty$. Then for every $\alpha \in (0,T)$ there exist a radius $\hat\delta > 0$, a compact set $M \subset \mathbb{R}^n$, and a Lipschitz-constant $L > 0$ such that $\varphi(t) \in M$ for $t \in [-r,0]$, and for every $\tau \in \mathcal{B}_{C_{\alpha,1}}\Big(\hat\tau|_{[0,\alpha]};\, \hat\delta\Big)$ a unique solution $x(t,\tau)$ of the IVP (3.1)–(3.2) exists for $t \in [-r,\alpha]$, and*

$$0 < \tau(t) < r \quad \text{and} \quad x(t,\tau) \in M \quad \text{for } t \in [0,\alpha], \tag{3.3}$$

and

$$|x(t,\tau) - x(t,\bar\tau)| \leq L\|\tau - \bar\tau\|_{C_{\alpha,1}} \quad \text{for } t \in [0,\alpha] \text{ and } \tau,\bar\tau \in \mathcal{B}_{C_{\alpha,1}}\Big(\hat\tau|_{[0,\alpha]};\, \hat\delta\Big). \tag{3.4}$$

Proof. Let $\hat\tau \in C(\mathbb{R}_+, (0,r))$ be fixed, and let $\hat x(t) := x(t,\hat\tau)$ be the unique noncontinuable solution of the corresponding IVP (3.1)–(3.2) on $[-r,T)$, where T is possibly equal to ∞. Let $0 < \alpha < T$ be fixed, and define the set

$$M_0 := \{\hat x(t) \colon t \in [0,\alpha]\} \cup \{\varphi(t) \colon t \in [-r,0]\}.$$

Clearly, M_0 is a compact subset of $\mathbb{R}^n$. Fix $\rho > 0$, and let $\mathcal{B}_{\mathbb{R}^n}(M_0,\rho)$ be the neighbourhood of M_0 with radius ρ, and its closure is denoted by $M := \overline{\mathcal{B}_{\mathbb{R}^n}(M_0,\rho)}$. Define the constant L_1 by

$$L_1 := \max_{i=2,3}\Big\{\max\{|D_i f(t,u,v)| \colon t \in [0,\alpha],\ u,v \in M\}\Big\}. \tag{3.5}$$

Then the Mean Value Theorem yields

$$|f(t,u,v) - f(t,\bar u,\bar v)| \leq L_1(|u - \bar u| + |v - \bar v|), \qquad t \in [0,\alpha],\ u,v,\bar u,\bar v \in M. \tag{3.6}$$

The constants $m_1 := \min\{\hat\tau(t) \colon t \in [0,\alpha]\}$ and $m_2 := \max\{\hat\tau(t) \colon t \in [0,\alpha]\}$ satisfy $0 < m_1 \leq m_2 < r$. Define $\delta_1 := \min\{m_1, r - m_2\}$. Then for $\tau \in \mathcal{B}_{C_{\alpha,1}}(\hat\tau;\, \delta_1)$ it follows $0 < \tau(t) < r$ for $t \in [0,\alpha]$.

Let $\tau \in \mathcal{B}_{C_{\alpha,1}}\Big(\hat\tau|_{[0,\alpha]};\, \delta_1\Big)$, and let $x(t) := x(t,\tau)$ be the corresponding unique noncontinuable solution of the IVP (3.1)–(3.2) which is defined on the interval $[-r, T_\tau)$ for some $T_\tau \in (0,\alpha)$, or on $[-r, T_\tau]$ with $T_\tau = \alpha$. Since $\varphi(0) \in M_0$, it follows that $x(t) \in M$ for small positive t. We introduce

$$\beta_\tau := \sup\Big\{t \in (0,T_\tau) \colon x(s) \in M \text{ and } x(s - \tau(s)) \in M \text{ for } s \in [0,t]\Big\}.$$

We note that $x(\beta_\tau) \in M$, since M is compact. Then $0 < \beta_\tau \leq \alpha$. We show that $\beta_\tau = \alpha$ if τ is close enough to $\hat\tau$. We have

$$x(t) = \varphi(0) + \int_0^t f(s, x(s), x(s - \tau(s)))\, ds, \qquad t \in [0,\beta_\tau]$$

$$\hat x(t) = \varphi(0) + \int_0^t f(s, \hat x(s), \hat x(s - \hat\tau(s)))\, ds, \qquad t \in [0,\alpha].$$

Hence, for $t \in [0, \beta_\tau]$, (3.6) implies

$$
\begin{aligned}
|x(t) - \hat{x}(t)| &\le \int_0^t \big| f(s, x(s), x(s-\tau(s))) - f(s, \hat{x}(s), \hat{x}(s-\hat{\tau}(s)))\big|\, ds \\
&\le L_1 \int_0^t (|x(s) - \hat{x}(s)| + |x(s-\tau(s)) - \hat{x}(s-\hat{\tau}(s))|)\, ds \\
&\le L_1 \int_0^t \Big(|x(s) - \hat{x}(s)| + |x(s-\tau(s)) - \hat{x}(s-\tau(s))| \\
&\qquad + |\hat{x}(s-\tau(s)) - \hat{x}(s-\hat{\tau}(s))| \Big)\, ds. \qquad (3.7)
\end{aligned}
$$

We define

$$N := \max\Big\{ |\dot{\varphi}|_{L^\infty}, \max\{|f(t,u,v)| \colon t \in [0,\alpha],\ u, v \in M\} \Big\}. \qquad (3.8)$$

Then (3.1), (3.8) and the Mean Value Theorem yield

$$|\hat{x}(s-\tau(s)) - \hat{x}(s-\hat{\tau}(s))| \le N|\tau(s) - \hat{\tau}(s)| \le N\|\tau - \hat{\tau}\|_{C_{\alpha,1}}, \quad s \in [0, \beta_\tau]. \qquad (3.9)$$

Therefore, it follows from (3.7) that

$$|x(t) - \hat{x}(t)| \le \alpha L_1 N \|\tau - \hat{\tau}\|_{C_{\alpha,1}} + 2L_1 \int_0^t \max_{s-r \le \theta \le s} |x(\theta) - \hat{x}(\theta)|\, ds, \quad t \in [0, \beta_\tau]. \qquad (3.10)$$

Hence, Lemma 2.2 gives

$$|x(t) - \hat{x}(t)| \le L\|\tau - \hat{\tau}\|_{C_{\alpha,1}}, \qquad t \in [0, \beta_\tau], \qquad (3.11)$$

where $L := \alpha L_1 N e^{2L_1\alpha}$. Fix $0 < \rho_1 < \rho$, and define

$$\hat{\delta} := \min\left\{ \delta_1, \frac{\rho_1}{L} \right\}.$$

Then (3.11) implies $|x(t) - \hat{x}(t)| \le L\hat{\delta} \le \rho_1 < \rho$ for $t \in [0, \beta_\tau]$ and $\tau \in \mathcal{B}_{C_{\alpha,1}}\Big(\hat{\tau}|_{[0,\alpha]}; \hat{\delta}\Big)$. Suppose $\beta_\tau < \alpha$. Then $x(\beta_\tau)$ is in the interior of M, and hence x has a continuation to the right of β_τ with values in M. This contradicts to the definition of β_τ, hence $\beta_\tau = \alpha$ holds for $\tau \in \mathcal{B}_{C_{\alpha,1}}\Big(\hat{\tau}|_{[0,\alpha]}; \hat{\delta}\Big)$.

Let $\tau, \bar{\tau} \in \mathcal{B}_{C_{\alpha,1}}\Big(\hat{\tau}_{[0,\alpha]}; \hat{\delta}\Big)$. Then, similarly to (3.10), we get

$$|x(t) - \bar{x}(t)| \le \alpha L_1 N \|\tau - \bar{\tau}\|_{C_{\alpha,1}} + 2L_1 \int_0^t \max_{s-r \le \theta \le s} |x(\theta) - \bar{x}(\theta)|\, ds, \qquad t \in [0, \alpha].$$

Therefore Lemma 2.2 yields (3.4). □

4 Differentiability with respect to the delay

In this section we study the differentiability of the solution $x(t, \tau)$ of the IVP (3.1)–(3.2) wrt the delay function τ.

We define the parameter set

$$
\begin{aligned}
P := \Big\{ \tau \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}) \colon 0 < \tau(t) < r,\ t \in \mathbb{R}_+, \quad &\text{and for every } \alpha > 0 \\
\text{there exists} \quad 0 \le \kappa < 1 \ \text{ s.t. } |\dot{\tau}(t)| \le \kappa \quad &\text{for a.e. } t \in [0, \alpha] \Big\}, \qquad (4.1)
\end{aligned}
$$

and for $\alpha > 0$

$$P_\alpha := \Big\{ \tau \in W^{1,\infty}_{\alpha,1} \colon 0 < \tau(t) < r,\ t \in [0,\alpha], \quad \text{and there exists} \quad 0 \le \kappa < 1 \ \text{ s.t.}$$
$$|\dot\tau(t)| \le \kappa \quad \text{for a.e. } t \in [0,\alpha] \Big\}. \tag{4.2}$$

Clearly, if $\tau \in P$, then for any $\alpha > 0$ it follows $\tau|_{[0,\alpha]} \in P_\alpha$. Next we show that P_α is an open subset of $W^{1,\infty}_{\alpha,1}$.

Lemma 4.1. *P_α is an open subset of $W^{1,\infty}_{\alpha,1}$.*

Proof. Let $\bar\tau \in P_\alpha$. Then for some $0 \le \bar\kappa < 1$ it follows $|\dot{\bar\tau}(t)| \le \bar\kappa$ for a.e. $t \in [0,\alpha]$. Let $\gamma_1 := \min\{\bar\tau(t) : t \in [0,\alpha]\}$, $\gamma_2 := \max\{\bar\tau(t) : t \in [0,\alpha]\}$, and fix $\bar\kappa < \kappa < 1$. Let $\delta := \min\{\gamma_1, r-\gamma_2, \kappa-\bar\kappa\}$. Then for $\tau \in \mathcal{B}_{W^{1,\infty}_{\alpha,1}}(\bar\tau;\delta)$ it follows $0 \le \gamma_1 - \delta < \tau(t) = \bar\tau(t) + \tau(t) - \bar\tau(t) < \gamma_2 + \delta \le r$, $t \in [0,\alpha]$, and $|\dot\tau(t)| \le |\dot{\bar\tau}(t)| + |\dot\tau(t) - \dot{\bar\tau}(t)| \le \bar\kappa + \delta \le \kappa < 1$ for a.e. $t \in [0,\alpha]$, hence $\tau \in P_\alpha$. □

Let $\tau \in C(\mathbb{R}_+, (0,r))$ be fixed, and $x(t) = x(t,\tau)$ be the corresponding solution of the IVP (3.1)–(3.2) for $t \in [-r,\alpha]$ for some $\alpha > 0$. To simplify the notation, we introduce the $n \times n$ matrix-valued functions

$$A(t) := D_2 f(t, x(t), x(t-\tau(t))) \quad \text{and} \quad B(t) := D_3 f(t, x(t), x(t-\tau(t))), \quad t \in [0,\alpha]. \tag{4.3}$$

Then for $h \in C_{\alpha,1}$ we define the variational equation associated to $x(\cdot) = x(\cdot,\tau)$ as

$$\dot z(t) = A(t) z(t) + B(t) z(t-\tau(t)) - B(t)\dot x(t-\tau(t)) h(t), \qquad \text{a.e. } t \in [0,\alpha], \tag{4.4}$$
$$z(t) = 0, \qquad t \in [-r,0]. \tag{4.5}$$

It is easy to see that the IVP (4.4)–(4.5) has a unique solution on $[-r,\alpha]$, which we denote by $z(t,\tau,h)$. Clearly, both maps

$$C_{\alpha,1} \ni h \mapsto z(\cdot,\tau,h) \in C_\alpha$$
$$W^{1,\infty}_{\alpha,1} \ni h \mapsto z(\cdot,\tau,h) \in C_\alpha$$

are linear. Part (i) of the next lemma yields that both maps are also bounded.

Lemma 4.2. *Assume (H) and $\hat\tau \in P$, and let $x(t,\hat\tau)$ be the corresponding noncontinuable solution of the IVP (3.1)–(3.2) defined on the interval $[-r,T)$. Fix any $\alpha \in (0,T)$, and let the radius $\delta > 0$ be defined by Lemma 3.1. For $\tau \in \mathcal{B}_{C_{\alpha,1}}\big(\hat\tau|_{[0,\alpha]};\delta\big)$ and $h \in C_{\alpha,1}$ let $z(t,\tau,h)$ be the corresponding solution of the IVP (4.4)–(4.5) for $t \in [-r,\alpha]$. Then*

(i) there exists $N_1 \ge 0$ such that

$$|z(t,\tau,h)| \le N_1 \|h\|_{C_{\alpha,1}}, \qquad t \in [0,\alpha],\ \tau \in \mathcal{B}_{C_{\alpha,1}}\big(\hat\tau|_{[0,\alpha]};\delta\big),\ h \in C_{\alpha,1}; \tag{4.6}$$

(ii) there exists $N_2 \ge 0$ such that

$$|\dot z(t,\tau,h)| \le N_2 \|h\|_{C_{\alpha,1}}, \qquad \tau \in \mathcal{B}_{C_{\alpha,1}}\big(\hat\tau|_{[0,\alpha]};\delta\big),\ h \in C_{\alpha,1},\ \text{and a.e. } t \in [0,\alpha]. \tag{4.7}$$

Proof. (i) Let $\tau \in \mathcal{B}_{C_{\alpha,1}}\left(\hat{\tau}|_{[0,\alpha]};\, \delta\right)$ and $h \in C_{\alpha,1}$, and let $x(t) = x(t,\tau)$ and $z(t) = z(t,\tau,h)$ be the corresponding solutions of the IVP (3.1)–(3.2) and (4.4)–(4.5), respectively, for $t \in [-r,\alpha]$, and let A and B defined by (4.3). Let the compact set $M \subset \mathbb{R}^n$ be defined by Lemma 3.1, and L_1 and N be defined by (3.5) and (3.8), respectively, corresponding to α and M. Then we get

$$|A(t)| \le L_1 \quad \text{and} \quad |B(t)| \le L_1, \qquad t \in [0,\alpha]. \tag{4.8}$$

Hence (4.4), (4.5) and (4.8) yield

$$\begin{aligned}|z(t)| &\le \int_0^t \Big(|A(s)||z(s)| + |B(s)||z(s-\tau(s))| + |B(s)||\dot{x}(s-\tau(s))||h(s)|\Big)\, ds\\ &\le L_1 N\alpha \|h\|_{C_{\alpha,1}} + 2L_1 \int_0^t \max_{s-r\le\theta\le s} |z(\theta)|\, ds, \qquad t \in [0,\alpha].\end{aligned}$$

Therefore Lemma 2.2 implies (4.6) with $N_1 := L_1 N\alpha e^{2L_1\alpha}$.

(ii) To prove (4.7), we use (3.8), (4.4), (4.6) and (4.8) to get

$$|\dot{z}(t)| \le (2L_1N_1 + L_1N)\|h\|_{C_{\alpha,1}}, \qquad \text{for a.e. } t \in [0,\alpha]. \qquad \square$$

Next we show that the map $z(t,\tau,\cdot)$ is continuous in t and τ.

Lemma 4.3. *Suppose (H) and $\hat{\tau} \in P$. Let $x(t,\hat{\tau})$ be the corresponding unique noncontinuable solution of the IVP (3.1)–(3.2) defined on the interval $[-r,T)$. Then for every finite $\alpha \in (0,T)$ there exists an open neighbourhood $U \subset W^{1,\infty}_{\alpha,1}$ of $\hat{\tau}|_{[0,\alpha]}$ such that the map*

$$\mathbb{R} \times W^{1,\infty}_{\alpha,1} \supset [0,\alpha] \times U \ni (t,\tau) \mapsto z(t,\tau,\cdot) \in \mathcal{L}(C_{\alpha,1},\mathbb{R}^n)$$

is continuous.

Proof. Fix $\alpha \in (0,T)$, and let the radius $\hat{\delta} > 0$, the compact set $M \subset \mathbb{R}^n$ and the Lipschitz-constant L be defined by Lemma 3.1, the constants L_1 and N be defined by (3.5) and (3.8), respectively. Then the IVP (3.1)–(3.2) has a unique solution on $[-r,\alpha]$ for any $\tau \in \mathcal{B}_{C_{\alpha,1}}\left(\hat{\tau}|_{[0,\alpha]};\, \hat{\delta}\right)$, and (3.3), (3.4) and (3.6) hold. Let $0 < \delta_1 \le \hat{\delta}$ be such that $U := \mathcal{B}_{W^{1,\infty}_{\alpha,1}}\left(\hat{\tau}|_{[0,\alpha]};\, \delta_1\right) \subset P_\alpha$.

Fix $\tau \in U$, and let $\delta > 0$ be such that $\mathcal{B}_{W^{1,\infty}_{\alpha,1}}(\tau;\delta) \subset U$, and $h_k \in W^{1,\infty}_{\alpha,1}$ ($k \in \mathbb{N}$) be a sequence with $0 < \|h_k\|_{W^{1,\infty}_{\alpha,1}} \le \delta$ for $k \in \mathbb{N}$ and $\|h_k\|_{W^{1,\infty}_{\alpha,1}} \to 0$ as $k \to \infty$. Fix $h \in C_{\alpha,1}$, and let $x(t) = x(t,\tau)$, $x_k(t) = x(t,\tau+h_k)$, $z(t) = z(t,\tau,h)$ and $z_k(t) := z(t,\tau+h_k,h)$ be the corresponding solutions of the IVP (3.1)–(3.2) and (4.4)–(4.5), respectively, for $t \in [-r,\alpha]$.

Let A and B defined by (4.3), and introduce

$$A_k(t) := D_2 f(t, x_k(t), x_k(t-\tau(t)-h_k(t))) \quad \text{and} \quad B_k(t) := D_3 f(t, x_k(t), x_k(t-\tau(t)-h_k(t)))$$

for $t \in [0,\alpha]$. Then (4.8) and

$$|A_k(t)| \le L_1 \quad \text{and} \quad |B_k(t)| \le L_1, \qquad t \in [0,\alpha], \quad k \in \mathbb{N} \tag{4.9}$$

hold.

The functions z_k and z satisfy

$$\begin{aligned}z_k(t) = \int_0^t \Big(&A_k(s)z_k(s) + B_k(s)z_k(s-\tau(s)-h_k(s))\\ &- B_k(s)\dot{x}_k(s-\tau(s)-h_k(s))h(s)\Big)\, ds, \qquad t \in [0,\alpha],\end{aligned}$$

$$z(t) = \int_0^t \Big(A(s)z(s) + B(s)z(s-\tau(s)) - B(s)\dot{x}(s-\tau(s))h(s)\Big)\, ds, \qquad t \in [0,\alpha].$$

Therefore it follows for $t \in [0,\alpha]$

$$
\begin{aligned}
&|z_k(t)-z(t)| \\
&\quad\le \int_0^t \Big(|A_k(s)-A(s)||z(s)| + |B_k(s)-B(s)|(|z(s-\tau(s))| + |\dot{x}(s-\tau(s))||h(s)|)\Big)ds \\
&\qquad + \int_0^t |B_k(s)|\Big(|z(s-\tau(s)-h_k(s)) - z(s-\tau(s))| \\
&\qquad\qquad + |\dot{x}_k(s-\tau(s)-h_k(s)) - \dot{x}(s-\tau(s))||h(s)|\Big)ds \\
&\qquad + \int_0^t \Big(|A_k(s)||z_k(s)-z(s)| \\
&\qquad\qquad + |B_k(s)||z_k(s-\tau(s)-h_k(s)) - z(s-\tau(s)-h_k(s))|\Big)ds.
\end{aligned} \tag{4.10}
$$

We define the function

$$
\begin{aligned}
\Omega_f(\varepsilon) := \sup\Big\{ &\max\Big(|D_2f(t,u,v)-D_2f(t,\tilde{u},\tilde{v})|, |D_3f(t,u,v)-D_3f(t,\tilde{u},\tilde{v})|\Big): \\
&|u-\tilde{u}| + |v-\tilde{v}| \le \varepsilon,\ \ t\in[0,\alpha],\ u,\tilde{u},v,\tilde{v}\in M\Big\}.
\end{aligned} \tag{4.11}
$$

Note that Ω_f is well-defined and $\Omega_f(\varepsilon) \to 0$ as $\varepsilon \to 0$, since M is compact, and D_2f and D_3f are uniformly continuous on $[0,\alpha]\times M\times M$.

Relations (3.3), (3.4), (3.8) and the Mean Value Theorem yield

$$
\begin{aligned}
&|x_k(s)-x(s)| + |x_k(s-\tau(s)-h_k(s)) - x(s-\tau(s))| \\
&\quad\le |x_k(s)-x(s)| + |x_k(s-\tau(s)-h_k(s)) - x(s-\tau(s)-h_k(s))| \\
&\qquad + |x(s-\tau(s)-h_k(s)) - x(s-\tau(s))| \\
&\quad\le (2L+N)\|h_k\|_{C_{\alpha,1}}, \qquad s\in[0,\alpha],
\end{aligned} \tag{4.12}
$$

so we have from (4.11)

$$
\begin{aligned}
|A_k(s)-A(s)| &\le \big|D_2f(t,x_k(t),x_k(t-\tau(s)-h_k(s))) - D_2f(t,x(t),x(t-\tau(s)-h_k(s)))\big| \\
&\le \Omega_f\Big((2L+N)\|h_k\|_{C_{\alpha,1}}\Big), \qquad s\in[0,\alpha].
\end{aligned} \tag{4.13}
$$

Similarly, we get

$$
|B_k(s)-B(s)| \le \Omega_f\Big((2L+N)\|h_k\|_{C_{\alpha,1}}\Big), \qquad s\in[0,\alpha]. \tag{4.14}
$$

Relation (4.7) and the initial condition (4.5) imply

$$
|z(s-\tau(s)-h_k(s)) - z(s-\tau(s))| \le N_2\|h_k\|_{C_{\alpha,1}}\|h\|_{C_{\alpha,1}}, \qquad s\in[0,\alpha]. \tag{4.15}
$$

Combining (4.6), (4.9), (4.13), (4.14) and (4.15), we get from (4.10)

$$
|z_k(t)-z(t)| \le (\bar{a}_k+\bar{b}_k)\|h\|_{C_{\alpha,1}} + 2L_1\int_0^t \max_{s-r\le\theta\le s}|z_k(\theta)-z(\theta)|\,ds, \qquad t\in[0,\alpha], \tag{4.16}
$$

where $\bar{a}_k$ and $\bar{b}_k$ are defined by

$$
\bar{a}_k := \Omega_f\Big((2L+N)\|h_k\|_{C_{\alpha,1}}\Big)(2N_1+N)\alpha + L_1N_2\|h_k\|_{C_{\alpha,1}}
$$

and

$$\bar{b}_k := L_1 \int_0^t |\dot{x}_k(s-\tau(s)-h_k(s)) - \dot{x}(s-\tau(s))|\, ds.$$

Then Lemma 2.2 gives

$$|z_k(t) - z(t)| \le (\bar{a}_k + \bar{b}_k) e^{2L_1\alpha} \|h\|_{C_{\alpha,1}}, \qquad t \in [0,\alpha]. \tag{4.17}$$

The assumed continuity of $D_2 f$ and $D_3 f$ yields $\bar{a}_k \to 0$ as $k \to \infty$. We have

$$\begin{aligned} &|\dot{x}_k(s-\tau(s)-h_k(s)) - \dot{x}(s-\tau(s))| \\ &\quad \le |\dot{x}_k(s-\tau(s)-h_k(s)) - \dot{x}(s-\tau(s)-h_k(s))| \\ &\qquad + |\dot{x}(s-\tau(s)-h_k(s)) - \dot{x}(s-\tau(s))|. \end{aligned} \tag{4.18}$$

To estimate the first term of the last inequality, first note that

$$|\dot{x}_k(s-\tau(s)-h_k(s)) - \dot{x}(s-\tau(s)-h_k(s))| = 0, \qquad \text{if } s-\tau(s)-h_k(s) \le 0$$

and φ is differentiable at $s-\tau(s)-h_k(s)$. Suppose s is such that $s-\tau(s)-h_k > 0$. Then for such s we have

$$\begin{aligned} &|\dot{x}_k(s-\tau(s)-h_k(s)) - \dot{x}(s-\tau(s)-h_k(s))| \\ &\quad \le \Big| f(s-\tau(s)-h_k(s), x_k(s-\tau(s)-h_k(s)), x_k(s-2\tau(s)-2h_k(s))) \\ &\qquad - f(s-\tau(s)-h_k(s), x(s-\tau(s)-h_k(s)), x(s-2\tau(s)-h_k(s))) \Big| \\ &\quad \le L_1 \Big(|x_k(s-\tau(s)-h_k(s)) - x(s-\tau(s)-h_k(s))| \\ &\qquad + |x_k(s-2\tau(s)-2h_k(s)) - x(s-2\tau(s)-h_k(s))| \Big) \\ &\quad \le L_1 \Big(L\|h_k\|_{C_{\alpha,1}} + |x_k(s-2\tau(s)-2h_k(s)) - x(s-2\tau(s)-2h_k(s))| \\ &\qquad + |x(s-2\tau(s)-2h_k(s)) - x(s-2\tau(s)-h_k(s))| \Big) \\ &\quad \le L_1(2L+N)\|h_k\|_{C_{\alpha,1}}. \end{aligned}$$

Hence (4.18) yields

$$\bar{b}_k \le L_1^2(2L+N)\alpha\|h_k\|_{C_{\alpha,1}} + L_1 \int_0^t |\dot{x}(s-\tau(s)-h_k(s)) - \dot{x}(s-\tau(s))|\, ds.$$

We note that $\tau + h_k \in P_\alpha$ for all $k \in \mathbb{N}$, so $\frac{d}{ds}(s-\tau(s)-h_k(s)) \ge \varepsilon$ for some $\varepsilon > 0$ and for a.e. $s \in [0,\alpha]$. Therefore Lemma 2.1 gives $\bar{b}_k \to 0$ as $k \to \infty$. But then (4.17) gives the continuity of $z(t,\tau,\cdot)$ wrt τ.

The continuity of $z(t,\tau,\cdot)$ wrt t follows from (4.7), since

$$|z(t,\tau,h) - z(\bar{t},\tau,h)| \le N_2|t-\bar{t}|\|h\|_{C_{\alpha,1}}, \qquad t,\bar{t} \in [0,\alpha],\ \tau \in U,\ h \in C_{\alpha,1}.$$

This concludes the proof. □

Next we prove that for any $\tau \in P$ the solution $x(t,\tau)$ of the IVP (3.1)–(3.2) is continuously differentiable wrt to the time delay function τ on any compact time interval and in a small neighbourhood of τ. We denote this derivative by $D_2x(t,\tau)$.

Theorem 4.4. *Suppose (H) and $\hat{\tau} \in P$. Let $x(t,\hat{\tau})$ be the corresponding unique noncontinuable solution of the IVP (3.1)–(3.2) defined on the interval $[-r,T)$. Then for every finite $\alpha \in (0,T)$ there exists an open neighbourhood $U \subset W^{1,\infty}_{\alpha,1}$ of $\hat{\tau}|_{[0,\alpha]}$ such that the function*

$$\mathbb{R} \times W^{1,\infty}_{\alpha,1} \supset [0,\alpha] \times U \ni (t,\tau) \mapsto x(t,\tau) \in \mathbb{R}^n$$

is well-defined and it is continuously differentiable wrt τ, and

$$D_2 x(t,\tau)h = z(t,\tau,h), \qquad t \in [0,\alpha],\ \tau \in U,\ h \in W^{1,\infty}_{\alpha,1}, \tag{4.19}$$

where $z(t,\tau,h)$ is the solution of the IVP (4.4)–(4.5) for $t \in [0,\alpha]$, $\tau \in U$ and $h \in W^{1,\infty}_{\alpha,1}$.

Proof. Fix $\alpha \in (0,T)$, and let the radius $\hat{\delta} > 0$, the compact set $M \subset \mathbb{R}^n$ and the Lipschitz-constant L be defined by Lemma 3.1, the constants L_1 and N be defined by (3.5) and (3.8), respectively. Then the IVP (3.1)–(3.2) has a unique solution on $[-r,\alpha]$ for any $\tau \in \mathcal{B}_{C_{\alpha,1}}\left(\hat{\tau}|_{[0,\alpha]};\hat{\delta}\right)$, and (3.3), (3.4) and (3.6) hold. Let $0 < \delta_1 \le \hat{\delta}$ be such that $U := \mathcal{B}_{W^{1,\infty}_{\alpha,1}}\left(\hat{\tau}|_{[0,\alpha]};\delta_1\right) \subset P_\alpha$.

Let $\tau \in U$ and $h \in W^{1,\infty}_{\alpha,1}$, and $x(t) = x(t,\tau)$ and $z(t) = z(t,\tau,h)$ be the corresponding solutions of the IVP (3.1)–(3.2) and (4.4)-(4.5), respectively, for $t \in [-r,\alpha]$. Let A and B be defined by (4.3). Then (4.8) holds.

Let $\delta > 0$ be such that $\mathcal{B}_{W^{1,\infty}_{\alpha,1}}(\tau;\delta) \subset U$, and $h_k \in W^{1,\infty}_{\alpha,1}$ ($k \in \mathbb{N}$) be a sequence with $0 < \|h_k\|_{W^{1,\infty}_{\alpha,1}} \le \delta$ for $k \in \mathbb{N}$ and $\|h_k\|_{W^{1,\infty}_{\alpha,1}} \to 0$ as $k \to \infty$. Let $x_k(t) = x(t,\tau+h_k)$ and $z_k(t) := z(t,\tau,h_k)$ be the corresponding solutions of the IVP (3.1)–(3.2) and (4.4)–(4.5), respectively, for $t \in [-r,\alpha]$. We note that the definition of z_k here is different from that of used in the proof of Lemma 4.3.

Then

$$\begin{aligned} x_k(t) &= \varphi(0) + \int_0^t f(s, x_k(s), x_k(s-\tau(s)-h_k(s)))\,ds, \qquad t \in [0,\alpha], \\ x(t) &= \varphi(0) + \int_0^t f(s, x(s), x(s-\tau(s)))\,ds, \qquad t \in [0,\alpha], \end{aligned}$$

and

$$z_k(t) = \int_0^t \Big(A(s)z_k(s) + B(s)z_k(s-\tau(s)) - B(s)\dot{x}(s-\tau(s))h_k(s)\Big)\,ds, \qquad t \in [0,\alpha].$$

We have

$$\begin{aligned} &x_k(t) - x(t) - z_k(t) \\ &\quad = \int_0^t \Big(f(s, x_k(s), x_k(s-\tau(s)-h_k(s))) - f(s, x(s), x(s-\tau(s))) \\ &\qquad\qquad - A(s)z_k(s) - B(s)z_k(s-\tau(s)) + B(s)\dot{x}(s-\tau(s))h_k(s)\Big) ds. \end{aligned} \tag{4.20}$$

We define

$$\omega_f(t,\bar{u},\bar{v},u,v) := f(t,u,v) - f(t,\bar{u},\bar{v}) - D_2 f(t,\bar{u},\bar{v})(u-\bar{u}) - D_3 f(t,\bar{u},\bar{v})(v-\bar{v}) \tag{4.21}$$

for $t \in \mathbb{R}_+$, $\bar{u}, u, \bar{v}, v \in \mathbb{R}^n$. The definition of ω_f and simple manipulations yield for $s \in [0, \alpha]$

$$\begin{aligned}
& f(s, x_k(s), x_k(s-\tau(s)-h_k(s))) - f(s, x(s), x(s-\tau(s))) - A(s)z_k(s) \\
&\quad - B(s)z_k(s-\tau(s)) + B(s)\dot{x}(s-\tau(s))h_k(s) \\
&\quad = A(s)(x_k(s)-x(s)) + B(s)\Big(x_k(s-\tau(s)-h_k(s)) - x(s-\tau(s))\Big) \\
&\qquad + \omega_f(s, x(s), x(s-\tau(s)), x_k(s), x_k(s-\tau(s)-h_k(s))) - A(s)z_k(s) \\
&\qquad - B(s)z_k(s-\tau(s)) + B(s)\dot{x}(t-\tau(s))h_k(s) \\
&\quad = A(s)(x_k(s)-x(s)-z_k(s)) \\
&\qquad + B(s)\Big(x_k(s-\tau(s)-h_k(s)) - x(s-\tau(s)-h_k(s)) - z_k(s-\tau(s)-h_k(s))\Big) \\
&\qquad + B(s)\Big(x(s-\tau(s)-h_k(s)) - x(s-\tau(s)) + \dot{x}(t-\tau(s))h_k(s)\Big) \\
&\qquad + B(s)\Big(z_k(s-\tau(s)-h_k(s)) - z_k(s-\tau(s))\Big) \\
&\qquad + \omega_f(s, x(s), x(s-\tau(s)), x_k(s), x_k(s-\tau(s)-h_k(s))).
\end{aligned} \tag{4.22}$$

Using (4.8), we get from (4.20) and (4.22)

$$\begin{aligned}
& |x_k(t) - x(t) - z_k(t)| \\
&\quad \leq a_k + b_k + c_k + 2L_1 \int_0^t \max_{s-r\leq\theta\leq s} |x_k(\theta) - x(\theta) - z_k(\theta)|\, ds, \quad t \in [0,\alpha],
\end{aligned} \tag{4.23}$$

where

$$\begin{aligned}
a_k &:= L_1 \int_0^\alpha |x(s-\tau(s)-h_k(s)) - x(s-\tau(s)) + \dot{x}(t-\tau(s))h_k(s)|\, ds, \\
b_k &:= L_1 \int_0^\alpha |z_k(s-\tau(s)-h_k(s)) - z_k(s-\tau(s))|\, ds, \\
c_k &:= \int_0^\alpha |\omega_f(s, x(s), x(s-\tau(s)), x_k(s), x_k(s-\tau(s)-h_k(s)))|\, ds.
\end{aligned}$$

Hence Lemma 2.2 yields

$$|x_k(t) - x(t) - z_k(t)| \leq (a_k + b_k + c_k)e^{2L_1\alpha}, \qquad t \in [0,\alpha]. \tag{4.24}$$

To get (4.19), it is enough to show that

$$\lim_{k\to\infty} \frac{a_k}{\|h_k\|_{W^{1,\infty}_{\alpha,1}}} = 0, \qquad \lim_{k\to\infty} \frac{b_k}{\|h_k\|_{W^{1,\infty}_{\alpha,1}}} = 0 \quad \text{and} \quad \lim_{k\to\infty} \frac{c_k}{\|h_k\|_{W^{1,\infty}_{\alpha,1}}} = 0. \tag{4.25}$$

(i) Now we prove the first relation of (4.25). We get by using simple manipulations and Fubini's theorem that

$$\begin{aligned}
& \int_0^\alpha |x(s-\tau(s)-h_k(s)) - x(s-\tau(s)) + \dot{x}(s-\tau(s))h_k(s)|\, ds \\
&\quad = \int_0^\alpha \left| \int_{s-\tau(s)}^{s-\tau(s)-h_k(s)} \Big(\dot{x}(v) - \dot{x}(s-\tau(s))\Big) dv \right| ds \\
&\quad = \int_0^\alpha \left| \int_0^1 \Big(\dot{x}(s-\tau(s)-\nu h_k(s)) - \dot{x}(s-\tau(s))\Big)(-h_k(s))\, d\nu \right| ds \\
&\quad \leq \|h_k\|_{W^{1,\infty}_{\alpha,1}} \int_0^\alpha \int_0^1 \Big|\dot{x}(s-\tau(s)-\nu h_k(s)) - \dot{x}(s-\tau(s))\Big|\, d\nu\, ds \\
&\quad = \|h_k\|_{W^{1,\infty}_{\alpha,1}} \int_0^1 \int_0^\alpha \Big|\dot{x}(s-\tau(s)-\nu h_k(s)) - \dot{x}(s-\tau(s))\Big|\, ds\, d\nu.
\end{aligned}$$

Lemma 2.1 is applicable since, for a.e. $s \in [0,\alpha]$ and for all $\nu \in [0,1]$, it follows $\frac{d}{ds}(s-\tau(s)-\nu h_k(s)) \geq \varepsilon$ for some $\varepsilon > 0$ and large enough k. Therefore

$$\lim_{k\to\infty}\int_0^\alpha \Big|\dot{x}(s-\tau(s)-\nu h_k(s)) - \dot{x}(s-\tau(s))\Big|\,ds = 0, \quad \nu \in [0,1],$$

hence we conclude the first relation of (4.25) by using the Lebesgue's Dominated Convergence Theorem.

(ii) The second relation of (4.25) follows from (4.7), since we have

$$b_k = L_1\int_0^\alpha |z_k(s-\tau(s)-h_k(s)) - z_k(s-\tau(s))|\,ds \leq \alpha L_1 N_2 \|h_k\|^2_{W^{1,\infty}_{\alpha,1}}.$$

(iii) Finally, we show the third relation of (4.25). It follows from the definition of ω_f that

$$\omega_f(t,\bar{u},\bar{v},u,v) = \int_0^1 \Big[\Big(D_2 f(t,\bar{u}+\nu(u-\bar{u}),\bar{v}+\nu(v-\bar{v})) - D_2 f(t,\bar{u},\bar{v})\Big)(u-\bar{u}) + \Big(D_3 f(t,\bar{u}+\nu(u-\bar{u}),\bar{v}+\nu(v-\bar{v})) - D_3 f(t,\bar{u},\bar{v})\Big)(v-\bar{v})\Big]\,d\nu,$$

therefore

$$|\omega_f(t,\bar{u},\bar{v},u,v)| \leq \sup_{0<\nu<1}\Big(\big|D_2 f(t,\bar{u}+\nu(u-\bar{u}),\bar{v}+\nu(v-\bar{v})) - D_2 f(t,\bar{u},\bar{v})\big||u-\bar{u}| + \big|D_3 f(t,\bar{u}+\nu(u-\bar{u}),\bar{v}+\nu(v-\bar{v})) - D_3 f(t,\bar{u},\bar{v})\big||v-\bar{v}|\Big) \tag{4.26}$$

for $t \in \mathbb{R}_+$, $\bar{u}, u, \bar{v}, v \in \mathbb{R}^n$. We define the function Ω_f by (4.11). Then (4.12), (4.26) and the definition of Ω_f imply

$$\int_0^\alpha |\omega_f(s,x(s),x(s-\tau(s)),x_k(s),x_k(s-\tau(s)-h_k(s)))|\,ds \leq \alpha\Omega_f\Big((N+2L)\|h_k\|_{C_{\alpha,1}}\Big)(N+2L)\|h_k\|_{C_{\alpha,1}},$$

which proves the third relation of (4.25), since $\Omega_f\Big((N+2L)\|h_k\|_{C_{\alpha,1}}\Big) \to 0$ as $k \to \infty$.

Therefore all relations of (4.25) hold, hence (4.24) yields that $x(t,\tau)$ is differentiable wrt τ, and we get (4.19). The continuity of $D_2 x(t,\tau)$ follows from Lemma 4.3. This completes the proof. □

We remark that the result of Theorem 4.4 can be easily extended for FDEs with multiple time delays of the form

$$\dot{x}(t) = f(t,x(t),x(t-\tau_1(t)),\ldots,x(t-\tau_m(t))). \tag{4.27}$$

Now consider the delay equation

$$\dot{x}(t) = f(t,x(t),x(t-\eta)), \qquad t \geq 0, \tag{4.28}$$

where $0 < \eta < r$ is a constant delay. We associate the initial condition

$$x(t) = \varphi(t), \qquad t \in [-r,0]. \tag{4.29}$$

We observe that constant functions belong to the parameter sets P and P_α, so Theorem 4.4 has the following consequence.

Corollary 4.5. *Suppose (H) and $\hat{\eta} \in (0, r)$. Let $x(t, \hat{\eta})$ be the corresponding unique noncontinuable solution of the IVP (4.28)–(4.29) defined on the interval $[-r, T)$. Then for every finite $\alpha \in (0, T)$ there exists $\delta > 0$ such that the solution $x(t, \eta)$ of the IVP (4.28)–(4.29) exists for $t \in [0, \alpha]$ and $\tau \in (\hat{\eta} - \delta, \hat{\eta} + \delta)$, and the function*

$$\mathbb{R} \times \mathbb{R} \supset [0, \alpha] \times (\hat{\eta} - \delta, \hat{\eta} + \delta) \ni (t, \eta) \mapsto x(t, \eta) \in \mathbb{R}^n$$

is continuously differentiable wrt η, and

$$D_2 x(t, \eta)h = z(t, \eta, h), \qquad t \in [0, \alpha],\ \eta \in (\hat{\eta} - \delta, \hat{\eta} + \delta),\ h \in \mathbb{R}, \tag{4.30}$$

where $z(t, \eta, h)$ is the solution of the IVP

$$\dot{z}(t) = A(t)z(t) + B(t)z(t - \eta) - B(t)\dot{x}(t - \eta)h, \qquad a.e.\ t \in [0, \alpha], \tag{4.31}$$

$$z(t) = 0, \qquad t \in [-r, 0]. \tag{4.32}$$

5 Estimation of the time delay function by quasilinearization

In this section we present a numerical study to estimate the delay function in FDEs with the quasilinearization method. This method relies on the computation of the derivative ot the solution wrt the time delay function.

We assume that the parameter $\tau \in P$ in the IVP (3.1)–(3.2) is unknown, but there are measurements $X_0, X_1, \ldots, X_l$ of the solution at the points $t_0, t_1, \ldots, t_l \in [0, \alpha]$. Our goal is to find a parameter value which minimizes the least square cost function

$$J(\tau) := \sum_{i=0}^{l} (x(t_i, \tau) - X_i)^2. \tag{5.1}$$

The method of quasilinearization for parameter estimation was introduced for ODEs in [1] and was applied to estimate finite dimensional parameters in FDEs in [2] and [3], and for FDEs with state-dependent delays in [8] and [10]. Following [8], we formulate this method to estimate the delay function in the IVP (3.1)–(3.2). First we take finite dimensional approximation $\tau_N \in \Gamma_N$ of the delay function τ. Here Γ_N is a finite-dimensional subspace of $C_{\alpha,1}$. In our example below we will use linear spline approximation of the delay function, so Γ_N will be the space of N-dimensional linear spline functions with equidistant mesh points defined on the interval $[0, \alpha]$. We consider the corresponding IVP

$$\dot{x}_N(t) = f(t, x_N(t), x_N(t - \tau_N(t))), \qquad t \in [0, \alpha] \tag{5.2}$$

$$x_N(t) = \varphi(t), \qquad t \in [-r, 0]. \tag{5.3}$$

Then we minimize the least square cost function

$$J_N(\tau_N) := \sum_{i=0}^{l} (x_N(t_i; \tau_N) - X_i)^2, \qquad \tau_N \in \Gamma_N$$

by a gradient-based method. Note that this requires the computation of the derivative of J_N with respect to the delay function τ_N, for which we have to compute the derivative of the solution wrt the delay function.

The quasilinearization algorithm can be formulated as follows: Fix a basis $\{e_1^N, \ldots, e_N^N\}$ of the finite dimensional subspace Γ_N of $C_{\alpha,1}$, and let $c = (c_1, \ldots, c_N)^T$ be the coordinates of the parameter $\tau_N \in \Gamma_N$ with respect to this basis, i.e., $\tau_N = \sum_{i=1}^N c_i e_i^N$. Then we identify τ_N with the column vector $c \in \mathbb{R}^N$, and simply write $x_N(t;c)$ instead of $x_N(t;\tau_N)$. We approximate the parameter vector c by the fixed point iteration described by the following equations:

$$c^{(k+1)} = g(c^{(k)}), \qquad k = 0, 1, \dots, \tag{5.4}$$

$$g(c) = c - (D(c))^{-1} b(c) \tag{5.5}$$

$$D(c) = \sum_{i=0}^{l} M^T(t_i;c) M(t_i;c) \tag{5.6}$$

$$b(c) = \sum_{i=0}^{l} M^T(t_i;c)(x_N(t_i;c) - X_i) \tag{5.7}$$

$$M(t;c) = (M_1(t;c), \dots, M_N(t;c)) \tag{5.8}$$

$$M_i(t;c) = D_2 x_N(t;c) e_i^N, \qquad i = 1, \dots, N. \tag{5.9}$$

This is exactly the same scheme that was used in [2] and [3] except that there the parameter space was finite dimensional, and the set $\{e_1^N, \dots, e_N^N\}$ was the canonical basis of $\mathbb{R}^N$. In our example below we will us the usual hat functions as the basis functions in the space of linear spline functions, i.e., let $\Delta s := \alpha/(N-1)$, $s_i := (i-1)\Delta s$ for $i = 1, \dots, N$, and let $e_i^N(s_j) = 1$ for $j = i$ and $e_i^N(s_j) = 0$ for $j \neq i$.

In our case $D_2 x_N$ is a linear functional defined on $C_{\alpha,1}$, and $D_2 x_N(t;c) e_i^N$ denotes the value of the linear functional applied to the function e_i^N. For the derivation of this method and for the proof of its local convergence we refer to [1] for the finite dimensional case, to [11] for abstract differential equations, and to [10] for FDEs with state-dependent delays.

Next we apply the quasilinearization method (5.4)–(5.9) for a scalar equation with a single time-dependent delay.

Example 5.1. Consider the scalar nonlinear FDE with time-delay

$$\dot{x}(t) = (0.2\cos t + 0.6)x(t - \tau(t)) - (0.01\sin t + 0.02)x^2(t), \qquad t \in [0,4], \tag{5.10}$$

and the initial condition

$$x(t) = t, \qquad t \le 0. \tag{5.11}$$

Here the delay function τ is a parameter in the IVP. We consider $\bar{\tau}(t) = 0.4\sin(2t) + 2$ as the "true parameter". Note that $\bar{\tau} \in P_4$. We solved the IVP (5.10)–(5.11) using this parameter value, and generated the measurements by evaluating the solutions at the mesh points $t_i = 0.4i$, $i = 0, \dots, 10$, i.e., we consider $X_i = x(t_i, \bar{\tau})$, $i = 0, \dots, 10$. For a fixed $h \in C_{\alpha,1}$ we associate the variational equation to (5.10):

$$\begin{aligned} \dot{z}(t) &= (0.2\cos t + 0.6)z(t - \tau(t)) - (0.2\cos t + 0.6)\dot{x}(t - \tau(t))h(t) \\ &\quad - (0.01\sin t + 0.02)2x(t)z(t), \qquad t \in [0,4] \end{aligned} \tag{5.12}$$

$$z(t) = 0, \qquad t \le 0. \tag{5.13}$$

Then Theorem 4.4 yields that, in a neighbourhood of $\bar{\tau}$, the solution $z(t) = z(t, \tau, h)$ of the IVP (5.12)–(5.13) satisfies $D_2 x(t,\tau)h = z(t,\tau,h)$.

In our numerical study we used these measurements data, the linear spline approximation of the parameter τ with $N = 8$ equidistant mesh points, for the initial value we used the constant delay function $\tau_8^{(0)}(t) = 1.5$, and we generated the first three terms of the quasilinearization sequence defined by (5.4)-(5.9). In the course of the computation, we solved the

IVP (5.10)–(5.11) and also (5.12)–(5.13) by an Euler-type numerical approximation scheme introduced in [5] using the discretization stepsize $h = 0.01$. The numerical results can be seen in Figures 5.1-5.4 and in Table 5.1 below. We observe convergence of the method starting from this initial value, and even in the third step the cost function has a value $J_8(\tau_8^{(3)}) = 0.000031$, which indicates that the parameter is close to the "true" parameter $\bar{\tau}$. Table 5.1 contains the errors $\Delta_i^{(k)} = |\bar{\tau}(s_i) - \tau_8^{(k)}(s_i)|$ at the mesh points of the spline approximation. In Figures 5.1–5.4 the solid blue curve is the delay function $\bar{\tau}$, and the dotted red graph is the linear spline function $\tau_8^{(k)}$ for $k = 0, 1, 2, 3$. The figures show that the method quickly recovers the shape of the "true" time delay function, and the last spline function is a good approximation of $\bar{\tau}$.

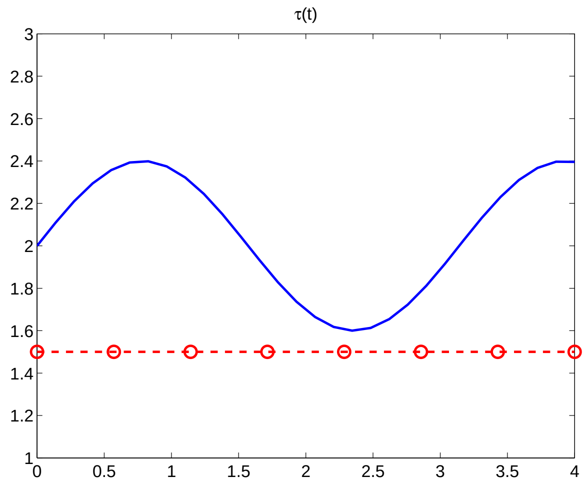

Figure 5.1: Step 0

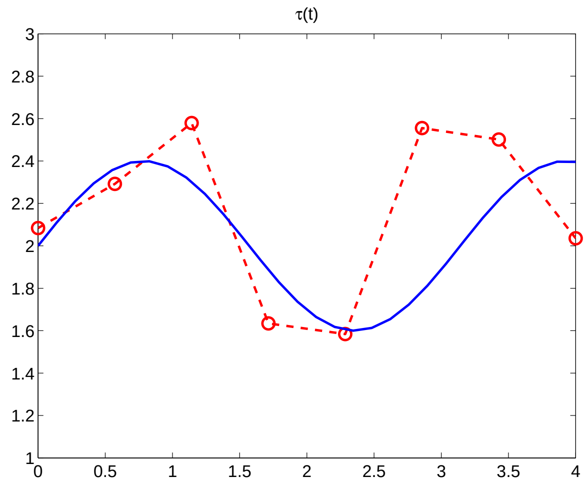

Figure 5.2: Step 1

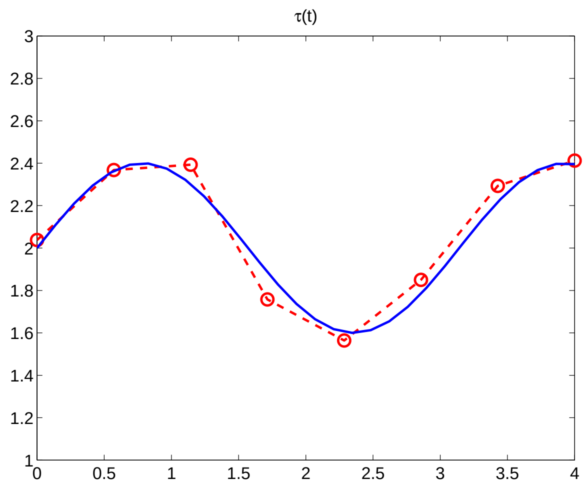

Figure 5.3: Step 2

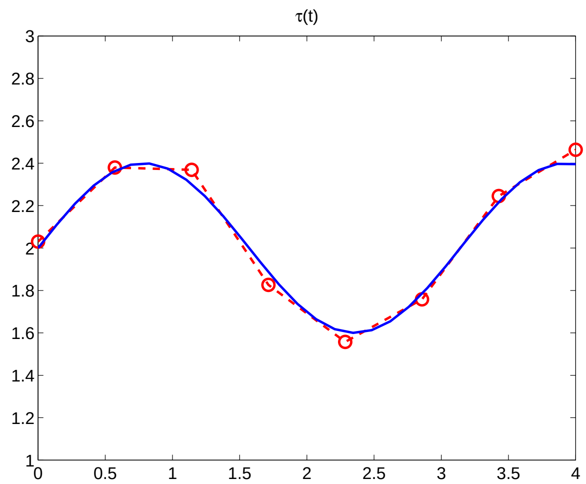

Figure 5.4: Step 3

Acknowledgements

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k	$J(\tau^{(k)})$	$\Delta_1^{(k)}$	$\Delta_2^{(k)}$	$\Delta_3^{(k)}$	$\Delta_4^{(k)}$	$\Delta_5^{(k)}$	$\Delta_6^{(k)}$	$\Delta_7^{(k)}$	$\Delta_8^{(k)}$
0:	10.628249	0.50000	0.86393	0.80206	0.38678	0.10397	0.28452	0.71718	0.89574
1:	0.135430	0.08364	0.07192	0.27676	0.25291	0.02005	0.77079	0.28422	0.36011
2:	0.037887	0.03789	0.00388	0.09066	0.12892	0.04011	0.06543	0.07579	0.01640
3:	0.000031	0.03036	0.01577	0.06691	0.06048	0.04674	0.02636	0.02815	0.06760

Table 5.1: $\tau_8^{(0)}(t) = 1.5$

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