Oscillation theorems for the Dirac operator with spectral parameter in the boundary condition

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Abstract. We consider the boundary value problem for the one-dimensional Dirac equation with spectral parameter dependent boundary condition. We give location of the eigenvalues on the real axis, study the oscillation properties of eigenvector-functions and obtain the asymptotic behavior of the eigenvalues and eigenvector-functions of this problem.

Keywords: one dimensional Dirac equation, eigenvalue, eigenvector-function, oscillatory properties, asymptotic behavior of the eigenvalues and eigenfunctions.

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1   Introduction

We consider the following boundary value problem for the one-dimensional Dirac canonical system

$$v' - \{\lambda + p(x)\} u = 0, \qquad u' + \{\lambda + r(x)\} v = 0, \qquad 0 < x < \pi, \tag{1.1}$$

$$v(0) \cos \alpha + u(0) \sin \alpha = 0, \tag{1.2}$$

$$(\lambda \cos \beta + a_1) v(\pi) + (\lambda \sin \beta + b_1) u(\pi) = 0, \tag{1.3}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter, the functions $p(x)$ and $r(x)$ are continuous on the interval $[0, \pi]$, α, β, a_1 and b_1 are real constants such that $0 \le \alpha, \beta < \pi$ and

$$\sigma = a_1 \sin \beta - b_1 \cos \beta > 0. \tag{1.4}$$

λ is called an eigenvalue with corresponding eigenvector-function w if boundary value problem (1.1)–(1.3) under consideration have a non-trivial solution $w(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}$ for λ.

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In the case where $p(x) = V(x) + m$, $r(x) = V(x) - m$, $V(x)$ is a potential function and m is the mass of a particle, (1.1) is called an one-dimensional stationary Dirac system in relativistic quantum theory [15,18].

The basic and comprehensive results (except the oscillation properties) about Dirac operator were given in [15]. The oscillatory properties of the eigenvector-functions of the Dirac operator have been studied in a recent work [5] (see also [4,6]). Direct and inverse problems for Dirac operators were extensively studied in [1,7,10,20,21] (see also the references in these works).

Boundary value problems with spectral parameter in boundary conditions often appear in mathematics, mechanics, physics, and other branches of natural sciences. A bibliography of papers in which such problems were considered in connection with specific physical processes can be found in [3,9,17,19,22]. Eigenvalue-dependent boundary conditions were examined even before the time of Poisson. Eigenvalue problems for ordinary differential operators with spectral parameter contained in the boundary conditions were considered in various settings in numerous papers [2,3,7–9,12–14,16,17,19,21,22].

In [13] and [21] oscillatory properties of eigenvector-functions of the Dirac system with spectral parameter contained in the boundary conditions were studied. It should be noted that these studies did not specify the exact number of zeros of the components of the eigenvector-function corresponding n-th eigenvalue (although for sufficiently large n).

In the present paper, we study the general characteristics of the location of the eigenvalues on the real axis, oscillation properties of eigenvector-functions and asymptotic behavior of eigenvalues and eigenvector-functions of the spectral problem (1.1)–(1.3).

2 Several auxiliary facts and some properties of the solution of problem (1.1), (2.1)

Lemma 2.1. *The eigenvalues of the boundary value problem* (1.1)–(1.3) *are real and simple, and form a countable set without finite limit points.*

Proof. The proof of this lemma is similar to that of [15, Lemma 10.2 and Lemma 11.2]. □

Consider the boundary condition

$$v(\pi) \cos \gamma + u(\pi) \sin \gamma = 0, \tag{2.1}$$

where $\gamma \in (0, \pi)$.

In order to study the location of the eigenvalues on the real axis and the oscillation properties of eigenvector-functions of the problem (1.1)–(1.3) alongside with this problem we shall consider the following boundary value problem

$$v' - \{\lambda + \mu p(x)\} u = 0, \qquad u' + \{\lambda + \mu r(x)\} v = 0, \qquad 0 < x < \pi,$$
$$v(0) \cos \alpha + u(0) \sin \alpha = 0, \tag{2.2}$$
$$v(\pi) \cos \gamma + u(\pi) \sin \gamma = 0.$$

where $\mu \in [0, 1]$. It is known (see [15, Ch. 1, §11]) that the eigenvalues of this problem are real, simple and the values range is from $-\infty$ to $+\infty$, and they can be enumerated in the following increasing order

$$\cdots < \eta_{-k}(\mu) < \cdots < \eta_{-1}(\mu) < \eta_0(\mu) < \eta_1(\mu) < \cdots < \eta_k(\mu) < \cdots,$$

such that (see [5])

$$\eta_k(0) = k + (\alpha - \beta)/\pi, \qquad k \in \mathbb{Z}.$$

Remark 2.2. From continuous dependence of solutions of a system of differential equations on the parameter we obtain that the eigenvalues $\eta_k(\mu)$, $k \in \mathbb{Z}$, of problem (2.2) depends continuously on the parameter $\mu \in [0, 1]$. Hence the map $\eta_k(\mu)$ continuously transforms $\eta_k(0)$ to $\eta_k(1)$ for any $k \in \mathbb{Z}$ (see [11, § 6–7]).

We denote by $s(g)$ the number of zeros of the function $g \in C([0, \pi]; \mathbb{R})$ in the interval $(0, \pi)$.

For problem (2.2) with a suitable interpretation (see Remark 2.2) the following oscillation theorem holds.

Theorem 2.3. ([5, Theorem 3.1]) *The eigenvector-functions* $w_{k,\mu}(x) = \begin{pmatrix} u_{k,\mu}(x) \\ v_{k,\mu}(x) \end{pmatrix}$, *corresponding to the eigenvalues* $\eta_k(\mu)$, $k \in \mathbb{Z}$, *have the following oscillation properties (except for* $k = 0$ *the cases* $\alpha = \beta = 0$ *and* $\alpha = \beta = \pi/2$):

$$\begin{pmatrix} s(u_{k,\mu}) \\ s(v_{k,\mu}) \end{pmatrix} = \begin{pmatrix} |k| - 1 + H\left((\alpha - \pi/2)\,\omega_{\alpha,\gamma}(k)\right) + H\left((\pi/2 - \gamma)\,\omega_{\alpha,\gamma}(k)\right) \\ |k| - 1 + \operatorname{sgn}\alpha\, H\left(\omega_{\alpha,\beta}(k)\right) + \operatorname{sgn}\gamma\, H\left(-\omega_{\alpha,\gamma}(k)\right) \end{pmatrix} \qquad (2.3)$$

where $H(x)$ *is the Heaviside function, i.e.*

$$H(x) = \begin{cases} 0 & \text{if } x \le 0, \\ 1 & \text{if } x > 0, \end{cases}$$

and $\omega_{\alpha,\gamma}(x)$, $x \in \mathbb{R}$, *is defined as follows:*

$$\omega_{\alpha,\gamma}(x) = \begin{cases} -1, & \text{if } x < 0 \text{ or } x = 0, \ \alpha < \gamma, \\ 1, & \text{if } x > 0 \text{ or } x = 0, \ \alpha \ge \gamma. \end{cases} \qquad (2.4)$$

Moreover, the functions $u_{k,\mu}(x)$ *and* $v_{k,\mu}(x)$ *have only nodal zeros in the interval* $(0, \pi)$ *(by a nodal zero we mean a function that changes its sign at the zero).*

We denote by $\tau_k(\mu)$ and $\nu_k(\mu)$, $k \in \mathbb{Z}$, the eigenvalues of problem (2.2) for $\gamma = 0$ and $\gamma = \pi/2$, respectively. By virtue of [5, formula (3.1)] the eigenvalues of problem (2.2) have the following location on the real axis: if $\gamma \in (0, \pi/2)$, then

$$\begin{aligned} \ldots &< \tau_{-2}(\mu) < \nu_{-1}(\mu) < \eta_{-1}(\mu) < \tau_{-1}(\mu) < \nu_0(\mu) < \eta_0(\mu) \\ &< \tau_0(\mu) < \nu_1(\mu) < \eta_1(\mu) < \tau_1(\mu) < \cdots, \end{aligned} \qquad (2.5)$$

and if $\gamma \in (\pi/2, \pi)$, then

$$\begin{aligned} \ldots &< \tau_{-2}(\mu) < \eta_{-1}(\mu) < \nu_{-1}(\mu) < \tau_{-1}(\mu) < \eta_0(\mu) < \nu_0(\mu) \\ &< \tau_0(\mu) < \eta_1(\mu) < \nu_1(\mu) < \tau_1(\mu) < \cdots. \end{aligned} \qquad (2.6)$$

One can readily show that there exists a unique solution $w(x, \lambda) = \begin{pmatrix} u(x,\lambda) \\ v(x,\lambda) \end{pmatrix}$ of system (1.1) satisfying the initial condition

$$u(0, \lambda) = \cos\alpha, \qquad v(0, \lambda) = -\sin\alpha; \qquad (2.7)$$

moreover, for each fixed $x \in [0, \pi]$, the functions $u(x, \lambda)$ and $v(x, \lambda)$ are entire functions of the argument λ. The proof of this assertion repeats that of Theorem 1.1 in [15, Ch. 1, §1] with obvious modifications.

By (2.7) the functions $u(x, \lambda)$ and $v(x, \lambda)$ satisfy the boundary condition (1.2), so that to find the eigenvalues of the boundary value problem (1.1)–(1.3) we have to insert the functions $u(x, \lambda)$ and $v(x, \lambda)$ in the boundary condition (1.3) and find the roots of this equation. Thus, the eigenvalues of problem (1.1)–(1.3) are the roots of the following equation

$$(\lambda \cos \beta + a_1) \, v(\pi, \lambda) + (\lambda \sin \beta + b_1) \, u \, (\pi, \lambda) = 0. \tag{2.8}$$

Obviously, the eigenvalues $\eta_k(1)$, $k \in \mathbb{Z}$, of problem (1.1), (1.2), (2.1) (or (2.2) for $\mu = 1$) are zeros of the entire function $v(\pi, \lambda) \cos \gamma + u(\pi, \lambda) \sin \gamma = 0$.

We set $\tau_k = \tau_k(1)$ and $\nu_k = \nu_k(1)$, $k \in \mathbb{Z}$. Note that the function

$$F(\lambda) = \frac{u \, (\pi, \lambda)}{v(\pi, \lambda)}$$

is defined for

$$\lambda \in D \equiv (\mathbb{C} \backslash \mathbb{R}) \bigcup \left(\bigcup_{k=-\infty}^{+\infty} (\tau_{k-1}, \tau_k) \right)$$

and is meromorphic function of finite order, and τ_k and ν_k, $k \in \mathbb{Z}$, are poles and zeros of this function, respectively.

Lemma 2.4. *The following formula holds:*

$$\frac{\partial}{\partial \lambda} \left(\frac{u \, (\pi, \lambda)}{v(\pi, \lambda)} \right) = -\frac{\int_0^\pi \{u^2(x, \lambda) + v^2(x, \lambda)\} \, dx}{v^2(\pi, \lambda)}, \qquad \lambda \in D. \tag{2.9}$$

Proof. By (1.1) for any μ, $\lambda \in \mathbb{Z}$, we obtain

$$\frac{d}{dx} \{v(x, \mu) \, u \, (x, \lambda) - u(x, \mu) \, v(x, \lambda)\} = (\mu - \lambda) \{u \, (x, \mu) \, u \, (x, \lambda) + v(x, \mu) \, v(x, \lambda)\} \,.$$

Integrating this relation from 0 to π and taking into account condition (1.2) we find

$$v(\pi, \mu) \, u \, (\pi, \lambda) - u \, (\pi, \mu) \, v(\pi, \lambda) = (\mu - \lambda) \int_0^\pi \{u \, (x, \mu) \, u \, (x, \lambda) + v(x, \mu) \, v(x, \lambda)\} \, dx \tag{2.10}$$

By (2.10) for any μ, $\lambda \in D$ we have

$$-\left(\frac{u \, (\pi, \mu)}{v(\pi, \mu)} - \frac{u \, (\pi, \lambda)}{v(\pi, \lambda)} \right) = (\mu - \lambda) \frac{\int_0^\pi \{u \, (x, \mu) \, u \, (x, \lambda) + v(x, \mu) \, v(x, \lambda)\} \, dx}{v(\pi, \mu) \, v(\pi, \lambda)}.$$

Dividing this equality by $(\mu - \lambda)$ and passing to the limit as $\mu \to \lambda$ we obtain (2.9). □

Corollary 2.5. *The function $F(\lambda)$ is continuous and strictly decreasing on each interval (τ_{k-1}, τ_k), $k \in \mathbb{Z}$.*

By $m \, (\lambda)$ and $n \, (\lambda)$, $\lambda \in \mathbb{R}$, we denote the number of zeros in the interval $(0, \pi)$ of the functions $u(x, \lambda)$ and $v(x, \lambda)$, respectively. We define numbers m_k and n_k, $k \in \mathbb{Z}$, as follows:

$$m_k = |k| - 1 + H \left(\left(\alpha - \frac{\pi}{2} \right) \omega_{\alpha, \frac{\pi}{2}}(k) \right),$$
$$n_k = |k| - 1 + \operatorname{sgn} \alpha \, H(\omega_{\alpha, 0}(k)),$$

where the function $\omega_{\alpha,\gamma}$ is defined by formula (2.4).

Then, by Theorem 2.3 (see (2.3)), we have

$$m(\nu_k) = m_k \quad \text{and} \quad n(\tau_k) = n_k.$$

Let

$$h_{-1} = \max \left\{ k \in \mathbb{Z} : \eta_{k,1} + p(x) < 0, \ \eta_{k,1} + r(x) < 0, \ x \in [0,\pi] \right\},$$
$$h_1 = \min \left\{ k \in \mathbb{Z} : \eta_{k,1} + p(x) > 0, \ \eta_{k,1} + r(x) > 0, \ x \in [0,\pi] \right\}.$$

Theorem 2.6. *If $\lambda \in [\nu_{k-1}, \nu_k)$ for $k < h_{-1}$, then $m(\lambda) = m_{k-1}$, and if $\lambda \in (\nu_{k-1}, \nu_k]$ for $k > h_1$, then $m(\lambda) = m_k$. If $\lambda \in [\tau_{k-1}, \tau_k)$ for $k < h_{-1}$, then $n(\lambda) = n_{k-1}$, and if $\lambda \in (\tau_{k-1}, \tau_k]$ for $k > h_1$, then $n(\lambda) = n_k$.*

Proof. Let $\lambda \in [\nu_{k-1}, \nu_k)$ for $k < h_{-1}$. Then, by the definition of number h_{-1}, we have $\lambda < \nu_{h_{-1}}$. It follows from [15, formulas (11.12) and (11.13)] that the number of zeros of $u(x,\lambda)$ on $(0,\pi)$ grows unboundedly as $|\lambda| \to +\infty$. By Corollary 2.1 from [4], the number of zeros of $u(x,\lambda)$ is a nondecreasing function of λ. By [4, Lemma 2.1], the roots of equation $u(x,\lambda) = 0$ continuously depend on λ. On the other hand, by [4, Corollary 2.1], as λ decreases, every zero of u moves to the left but cannot pass through 0, since the number of zeros does not decrease. By [4, Corollary 2.2], zeros enter through the point π. Since $m(\nu_k) = m_k$, $k \in \mathbb{Z}$ and $u(\pi,\lambda) \neq 0$ for $\lambda \in (\nu_{k-1}, \nu_k)$ it follows from these considerations that $m(\lambda) = m_{k-1}$ for $\lambda \in [\nu_{k-1}, \nu_k)$. The remaining cases are considered similarly. $\square$

3 Oscillatory properties of eigenvector-functions of problem (1.1)–(1.3)

For $\beta \neq 0$ let $N = 0$ if $-\frac{b_1}{\sin\beta} = \tau_0$, $N < 0$ be an integer such that $\tau_N \leq -\frac{b_1}{\sin\beta} < \tau_{N+1}$ if $-\frac{b_1}{\sin\beta} < \tau_0$, $N > 0$ be an integer such that $\tau_{N-1} < -\frac{b_1}{\sin\beta} \leq \tau_N$ if $-\frac{b_1}{\sin\beta} > \tau_0$, and for $\beta \neq \frac{\pi}{2}$ let $M = 0$ if $-\frac{a_1}{\cos\beta} = \nu_0$, $M < 0$ be an integer such that $\nu_M \leq -\frac{a_1}{\cos\beta} < \nu_{M+1}$ if $-\frac{a_1}{\cos\beta} < \nu_0$, $M > 0$ be an integer such that $\nu_{M-1} < -\frac{a_1}{\cos\beta} \leq \nu_M$ if $-\frac{a_1}{\cos\beta} > \nu_0$.

By virtue of the properties of the function $F(\lambda)$ (see Lemma 2.4 and Corollary 2.5) and the relations $v(\pi, \tau_k) = 0$, $k \in \mathbb{Z}$, we have

$$\lim_{\lambda \to \tau_{k-1}+0} F(\lambda) = +\infty, \qquad \lim_{\lambda \to \tau_k - 0} F(\lambda) = -\infty;$$

moreover, the function $F(\lambda)$ takes each value in $(-\infty, +\infty)$ at a unique point in the interval (τ_{k-1}, τ_k).

For the function $G(\lambda) = -\frac{\lambda \cos\beta + a_1}{\lambda \sin\beta + b_1}$ we have $G'(\lambda) = \sigma/(\lambda \sin\beta + b_1)^2$. Since $\sigma > 0$ (see (1.4)), it follows that for $\beta = 0$ the function $G(\lambda)$ is strictly increasing in the interval $(-\infty, +\infty)$; for $\beta \in (0, \pi)$ the function $G(\lambda)$ is increasing in both intervals $(-\infty, -b_1/\sin\beta)$ and $(-b_1/\sin\beta, +\infty)$; moreover,

$$\lim_{\lambda \to -b_1/\sin\beta - 0} G(\lambda) = +\infty, \qquad \lim_{\lambda \to -b_1/\sin\beta + 0} G(\lambda) = -\infty.$$

Assume that either $\beta = 0$, or $\beta \neq 0$ and $-\frac{b_1}{\sin\beta} = \tau_0$, or $\beta \neq 0$ and $-\frac{b_1}{\sin\beta} \notin [\tau_k, \tau_{k+1})$ if $k < 0$, $-\frac{b_1}{\sin\beta} \notin (\tau_{k-1}, \tau_k]$ if $k > 0$. It follows from the preceding considerations that in the interval (τ_{k-1}, τ_k), there exists a unique point $\lambda = \lambda_k^*$ such that

$$F(\lambda) = G(\lambda), \tag{3.1}$$

i.e., condition (1.3) is satisfied. Therefore, λ_k^* is an eigenvalue of the boundary value problem (1.1)–(1.3) and $w(x, \lambda_k^*)$ is the corresponding eigenvector-function.

Assume that $\beta \neq 0$ and $-\frac{b_1}{\sin \beta} \in (\tau_k, \tau_{k+1})$ if $k < 0$, $-\frac{b_1}{\sin \beta} \in (\tau_{k-1}, \tau_k)$ if $k > 0$. In a similar way, one can show that in each of the intervals $(-\frac{b_1}{\sin \beta}, \tau_{k+1})$ and $(\tau_k, -\frac{b_1}{\sin \beta})$ if $k < 0$, $(\tau_{k-1}, -\frac{b_1}{\sin \beta})$ and $(-\frac{b_1}{\sin \beta}, \tau_k)$ if $k > 0$, there exists a unique value ($\lambda_{k,1}^*$ and $\lambda_{k,2}^*$, respectively) such that relation (3.1) is valid.

The case in which $\beta \neq 0$ and $-\frac{b_1}{\sin \beta} = \tau_N$ can be considered in a similar way; here one uses the fact that τ_N is also an eigenvalue of the boundary value problem (1.1)–(1.3). In this case, we have $\lambda_{k,1}^* \in (\tau_N, \tau_{N+1})$ if $N < 0$, $\lambda_{k,1}^* \in (\tau_{N-1}, \tau_N)$ if $N > 0$, and $\lambda_{k,2}^* = \tau_N$.

Therefore, it follows from these considerations that there exist an unboundedly decreasing sequence of negative eigenvalues and an unboundedly increasing sequence of nonnegative eigenvalues of the boundary value problem (1.1)–(1.3). Hence, these eigenvalues can be enumerated in increasing order.

Remark 3.1. When numbering the eigenvalues of the problem (1.1)–(1.3) we will proceed from the following consideration: the number zero will be assigned to eigenvalue that is contained in the half-open interval $(\tau_{-1}, \tau_0]$ and is closest to τ_0.

Thus, the following theorem is proved.

Theorem 3.2. *There exists an infinite set of eigenvalues $\{\lambda_k\}_{k \in \mathbb{Z}}$ of problem (1.1)–(1.3) with values ranging from $-\infty$ to $+\infty$ which can be enumerated in increasing order:*

$$\cdots < \lambda_{-k} < \cdots < \lambda_{-1} < \lambda_0 < \lambda_1 < \cdots < \lambda_k < \cdots,$$

where λ_0 is defined in Remark 3.1.

Let $k^* = \max \{ |h_{-1}|, |h_1|, |N| + 1, |M| + 1 \}.$

Theorem 3.3. *The eigenvector-functions $w_k(x) = w(x, \lambda_k) = \begin{pmatrix} u(x, \lambda_k) \\ v(x, \lambda_k) \end{pmatrix} = \begin{pmatrix} u_k(x) \\ v_k(x) \end{pmatrix}$, corresponding to the eigenvalues λ_k of the problem (1.1)–(1.3), for $|k| > k^*$ have the following oscillation properties:*

(a) *if $\beta = 0$, then $m(\lambda_k) = m_k$, $n(\lambda_k) = n_{k+H(k)-1}$;*

(b) *if $\beta \in (0, \frac{\pi}{2})$, then $m(\lambda_k) = m_{k-H(N)+1}$, $n(\lambda_k) = n_{k-H(N)}$;*

(c) *if $\beta = \frac{\pi}{2}$, then $m(\lambda_k) = m_{k+H(k)-H(N)}$, $n(\lambda_k) = n_{k-H(N)}$;*

(d) *if $\beta \in (\frac{\pi}{2}, \pi)$, then $m(\lambda_k) = m_{k-H(N)}$, $n(\lambda_k) = n_{k-H(N)}$.*

Proof. Let $\beta = 0$. In this case it follows from the proof of the Theorem 3.2 and the Remark 3.1 that $\lambda_k \in (\tau_{k-1}, \tau_k)$ for any $k \in \mathbb{Z}$; moreover, $\lambda_k \in (\nu_k, \nu_{k+1})$ for $k < -k^*$, $\lambda_k \in (\nu_{k-1}, \nu_k)$ for $k > k^*$. Hence, by Theorem 2.6 we obtain that $m(\lambda_k) = m_k$ for $|k| > k^*$, $n(\lambda_k) = n_{k-1}$ for $k < -k^*$ and $n(\lambda_k) = n_k$ for $k > k^*$.

Let $\beta \in (0, \frac{\pi}{2})$. Then, again, from the proof of the Theorem 3.2 and the Remark 3.1 it follows that $\lambda_k \in (\tau_k, \tau_{k+1})$ for $k < -k^*$ and $\lambda_k \in (\tau_{k-1}, \tau_k)$ for $k > k^*$ in the case $N \leq 0$; $\lambda_k \in (\tau_{k-1}, \tau_k)$ for $k < -k^*$ and $\lambda_k \in (\tau_{k-2}, \tau_{k-1})$ for $k > k^*$ in the case where $N > 0$; moreover, $\lambda_k \in (\nu_{k+1}, \nu_{k+2})$ for $k < -k^*$ and $\lambda_k \in (\nu_k, \nu_{k+1})$ for $k > k^*$ in the case $N \leq 0$; $\lambda_k \in (\nu_k, \nu_{k+1})$ for $k < -k^*$ and $\lambda_k \in (\nu_{k-1}, \nu_k)$ for $k > k^*$ in the case where $N > 0$. Hence, by virtue of the Theorem 2.6 we have $m(\lambda_k) = m_{k+1}$, $n(\lambda_k) = n_k$ for $|k| > k^*$ in the case $N \leq 0$; $m(\lambda_k) = m_k$, $n(\lambda_k) = n_{k-1}$ for $|k| > k^*$ in the case $N > 0$.

Let $\beta = \frac{\pi}{2}$. Then, $\lambda_k \in (\tau_k, \tau_{k+1})$ for $k < -k^*$ and $\lambda_k \in (\tau_{k-1}, \tau_k)$ for $k > k^*$ in the case $N \leq 0$; $\lambda_k \in (\tau_{k-1}, \tau_k)$ for $k < -k^*$ and $\lambda_k \in (\tau_{k-2}, \tau_{k-1})$ for $k > k^*$ in the case where $N > 0$; moreover, $\lambda_k \in (\nu_k, \nu_{k+1})$ for $k < -k^*$ and $\lambda_k \in (\nu_k, \nu_{k+1})$ for $k > k^*$ in the case $N \leq 0$; $\lambda_k \in (\nu_{k-1}, \nu_k)$ for $k < -k^*$ and $\lambda_k \in (\nu_{k-1}, \nu_k)$ for $k > k^*$ in the case where $N > 0$. Hence, by virtue of the Theorem 2.6 we have $m(\lambda_k) = m_k$ for $k < -k^*$, $m(\lambda_k) = m_{k+1}$ for $k > k^*$, $n(\lambda_k) = n_k$ for $|k| > k^*$ in the case $N \leq 0$; $m(\lambda_k) = m_{k-1}$ for $k < -k^*$, $m(\lambda_k) = m_k$ for $k > k^*$, $n(\lambda_k) = n_{k-1}$ for $|k| > k^*$ in the case $N \leq 0$.

Let $\beta \in \left(\frac{\pi}{2}, \pi\right)$. Then, $\lambda_k \in (\tau_k, \tau_{k+1})$ for $k < -k^*$ and $\lambda_k \in (\tau_{k-1}, \tau_k)$ for $k > k^*$ in the case $N \leq 0$; $\lambda_k \in (\tau_{k-1}, \tau_k)$ for $k < -k^*$ and $\lambda_k \in (\tau_{k-2}, \tau_{k-1})$ for $k > k^*$ in the case where $N > 0$; moreover, $\lambda_k \in (\nu_k, \nu_{k+1})$ for $k < -k^*$ and $\lambda_k \in (\nu_{k-1}, \nu_k)$ for $k > k^*$ in the case $N \leq 0$; $\lambda_k \in (\nu_{k-1}, \nu_k)$ for $k < -k^*$ and $\lambda_k \in (\nu_{k-2}, \nu_{k-1})$ for $k > k^*$ in the case where $N > 0$. Hence, by virtue of the Theorem 2.6 we have $m(\lambda_k) = m_k$, $n(\lambda_k) = n_k$ for $|k| > k^*$ in the case $N \leq 0$; $m(\lambda_k) = m_{k-1}$, $n(\lambda_k) = n_{k-1}$ for $|k| > k^*$ in the case $N > 0$. $\qquad\square$

4 Asymptotic formulas for the eigenvalues and eigenvector-functions of problem (1.1)–(1.3)

By [15, Ch. 1, Lemma 11.1] for $|\lambda| \to +\infty$ the following estimates hold uniformly with respect to x, in x, $x \in [0, \pi]$:

$$u(x, \lambda) = \cos(\xi(x, \lambda) - \alpha) + O(1/\lambda), \tag{4.1}$$

$$v(x, \lambda) = \sin(\xi(x, \lambda) - \alpha) + O(1/\lambda), \tag{4.2}$$

where

$$\xi(x, \lambda) = \lambda x + (1/2) \int_0^x \{p(t) + r(t)\} \, dt. \tag{4.3}$$

Remark 4.1. Note that the formula (11.12) from [15, Ch. 1] has an error. This is due to the fact that in [15, Ch. 1, formula (11.9)] the expression for the function $\beta(x)$ to be of minus sign, whereby the formula (11.12) from [15, Ch. 1] must be of the form (4.3). Moreover, the asymptotic formula (11.18) from [15, Ch. 1] for the eigenvalues of the boundary problem (2.2) as $\mu = 1$ is incorrect, and this formula by [5, formula (3.26)] should be in the following form

$$\eta_k(1) = k + \frac{\alpha - \gamma - (1/2) \int_0^\pi \{p(t) + r(t)\} \, dt}{\pi} + O\left(\frac{1}{k}\right). \tag{4.4}$$

By (2.5) and (2.6) the following location on the real axis of eigenvalues of problem (1.1), (1.2), (2.1) (i.e. of problem (2.2) for $\mu = 1$) is valid: if $\gamma \in (0, \pi/2)$, then

$$\cdots < \tau_{-2} < \nu_{-1} < \eta_{-1}(1) < \tau_{-1} < \nu_0 < \eta_0(1) < \tau_0 < \nu_1 < \eta_1(1) < \tau_1 < \cdots, \tag{4.5}$$

if $\gamma \in (\pi/2, \pi)$, then

$$\cdots < \tau_{-2} < \eta_{-1}(1) < \nu_{-1} < \tau_{-1} < \eta_0(1) < \nu_0 < \tau_0 < \eta_1(1) < \nu_1 < \tau_1 < \cdots. \tag{4.6}$$

Theorem 4.2. *The following asymptotic formulas hold for sufficiently large $|k|$ ($|k| > k^*$)*

$$\lambda_k = k + (1 - H(N)) \operatorname{sgn}\beta - H(k) + \frac{\alpha - \beta - (1/2) \int_0^\pi \{p(t) + r(t)\} \, dt}{\pi} + O\left(\frac{1}{k}\right). \tag{4.7}$$

Proof. Recall that the eigenvalues of problem (1.1)–(1.3) are the roots of the equation (2.8). Substituting $u(\pi, \lambda)$ and $v(\pi, \lambda)$ from the estimates (4.1) and (4.2), we obtain

$$\sin(\xi(\pi, \lambda) - \alpha + \beta) + O(1/\lambda) = 0,$$

which is implied by (4.3) that

$$\sin\left(\lambda\pi - \alpha + \beta + (1/2)\int_0^\pi \{p(t) + r(t)\}\, dt\right) + O\left(\frac{1}{\lambda}\right) = 0. \tag{4.8}$$

It is obvious that for a large $|\lambda|$, Eq. (4.8) has solutions of the form (see [15, p. 57])

$$\lambda_k \pi - \alpha + \beta + (1/2)\int_0^\pi \{p(t) + r(t)\}\, dt = (k + \tau)\pi + \delta_k, \qquad k \in \mathbb{Z},$$

where τ is some integer which dependence of β and $\operatorname{sgn} k$. Inserting these values in (4.8), we see that $\sin \delta_k = O\left(\frac{1}{k}\right)$, so that $\delta_k = O\left(\frac{1}{k}\right)$. Therefore for the eigenvalues of the problem (1.1)–(1.3) we obtain the following asymptotic formula

$$\lambda_k = k + \tau + \frac{\alpha - \beta - (1/2)\int_0^\pi \{p(t) + r(t)\}dt)}{\pi} + O\left(\frac{1}{k}\right). \tag{4.9}$$

By location of eigenvalues of the problem (1.1)–(1.3) for large $|k| > k^*$ (see proof of the Theorem 3.3), relations (4.5) and (4.6) and formula (4.4) it follows that

$$\tau = (1 - H(N))\operatorname{sgn}\beta - H(k). \tag{4.10}$$

Now inserting (4.10) in (4.9), we obtain (4.7). □

By (4.1)–(4.3) for sufficiently large $|k| > k^*$ we obtain the following asymptotic formulas for the components of the eigenvector-functions $\begin{pmatrix} u(x, \lambda_k) \\ v(x, \lambda_k) \end{pmatrix} = \begin{pmatrix} u_k(x) \\ v_k(x) \end{pmatrix}$ of problem (1.1)–(1.3):

$$u_k(x) = \cos\left(\lambda_k x + (1/2)\int_0^x \{p(t) + r(t)\}\, dt - \alpha\right) + O(1/k),$$

$$v_k(x) = \sin\left(\lambda_k x + (1/2)\int_0^x \{p(t) + r(t)\}\, dt - \alpha\right) + O(1/k).$$

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