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Existence of solutions to a class of quasilinear Schrödinger systems involving the fractional p-Laplacian

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Abstract. The purpose of this paper is to investigate the existence of solutions to the following quasilinear Schrödinger type system driven by the fractional p-Laplacian

$$\begin{aligned}(-\Delta)_p^s u + a(x)|u|^{p-2}u &= H_u(x,u,v) \quad \text{in } \mathbb{R}^N,\\ (-\Delta)_q^s v + b(x)|v|^{q-2}v &= H_v(x,u,v) \quad \text{in } \mathbb{R}^N,\end{aligned}$$

where $1 < q \le p$, $sp < N$, $(-\Delta)_m^s$ is the fractional m-Laplacian, the coefficients a, b are two continuous and positive functions, and H_u, H_v denote the partial derivatives of H with respect to the second variable and the third variable. By using the mountain pass theorem, we obtain the existence of nontrivial and nonnegative solutions for the above system. The main feature of this paper is that the nonlinearities do not necessarily satisfy the Ambrosetti–Rabinowitz condition.

Keywords: Schrödinger system, fractional p-Laplacian, mountain pass theorem.

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1 Introduction

In this paper we are concerned with the following fractional Schrödinger system

$$\begin{aligned}(-\Delta)_p^s u + a(x)|u|^{p-2}u &= H_u(x,u,v) \quad \text{in } \mathbb{R}^N,\\ (-\Delta)_q^s v + b(x)|v|^{q-2}v &= H_v(x,u,v) \quad \text{in } \mathbb{R}^N,\end{aligned} \tag{1.1}$$

where $N > ps$ with $s \in (0,1)$ and $(-\Delta)_m^s$ is the fractional m-Laplace operator which (up to normalization factors) may be defined along any $\varphi \in C_0^\infty(\mathbb{R}^N)$ as

$$(-\Delta)_m^s \varphi(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|\varphi(x)-\varphi(y)|^{m-2}(\varphi(x)-\varphi(y))}{|x-y|^{N+ms}}\, dy$$

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for $x \in \mathbb{R}^N$, see [14] and the references therein for further details on the fractional m-Laplacian. The nonlinearities H_u and H_v denote the partial derivative of H with respect to the second variable and the third variable, respectively, and H satisfies some hypotheses to be stated in the sequence.

In particular, $(-\Delta)^s_m$ becomes to the fractional Laplacian $(-\Delta)^s$ as $m = 2$, and it is known that $(-\Delta)^s_m$ reduces to the standard m-Laplacian as $s \uparrow 1$, see [14].

In fact, nonlocal and fractional operators arise in a quite natural way in many different applications, such as, continuum mechanics, phase transition phenomena, population dynamics and game theory, as they are the typical outcome of stochastically stabilization of *Lévy* processes, see for instance [3,10]. The literature on nonlocal operators and on their applications is interesting and quite large, we refer the reader to [11,23,27,30–33] and the references therein. For the basic properties of fractional Sobolev spaces, we refer the reader to [14].

On the one hand, in the context of fractional quantum mechanics, nonlinear fractional Schrödinger equation was first proposed by *Laskin* in [18, 19] as a result of expanding the Feynman path integral, from the Brownian-like to the Lévy-like quantum mechanical paths. In the last years, there has been a great interest in the study of the fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^N,$$

where the nonlinearity f satisfies some general conditions. For standing wave solutions of fractional Schrödinger equations in $\mathbb{R}^N$, see, for instance, [12, 15, 24, 29] and the reference therein. It is worthy mentioning that *Autuori* and *Pucci* in [5] studied the following elliptic equations involving fractional Laplacian

$$(-\Delta)^s u + a(x)u = \lambda\omega(x)|u|^{p-2}u - h(x)|u|^{r-2}u \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

where $\lambda \in \mathbb{R}$, $0 < s < 1$, $2s < N$, $2 < q < \min\{r, 2^*_s\}$, $2^*_s = 2N/(N-2s)$, and $(-\Delta)^s$ is the fraction Laplacian operator. The authors obtained the existence and multiplicity of entire solutions of (1.2) by using the direct method in variational methods and the mountain pass theorem. The same nonlinearities were recently considered by *Pucci* and *Saldi* in [25], where the authors established existence and multiplicity of nontrivial nonnegative entire weak solutions of a stationary Kirchhoff eigenvalue problem, involving a general nonlocal integro-differential operator. More precisely, they considered the problem

$$M([u]^2_{s,K})\mathcal{L}_K u = \lambda\omega(x)|u|^{q-2}u - h(x)|u|^{r-2}u \quad \text{in } \mathbb{R}^N,$$
$$[u]_{s,K} = \iint_{\mathbb{R}^{2N}} |u(x)-u(y)|^2 K(x-y)\,dxdy,$$

where $\lambda \in \mathbb{R}$, $0 < s < 1$, $2s < N$, and $\mathcal{L}_K$ is an integro-differential nonlocal operator. The case with variable exponents was treated by *Pucci* and *Zhang* in [28], where the authors studied the one parameter elliptic equation

$$-\mathrm{div}\mathbf{A}(x,\nabla u) + a(x)|u|^{p(x)-2}u = \lambda\omega(x)|u|^{q(x)-2}u - h(x)|u|^{r(x)-2}u \quad \text{in } \mathbb{R}^N,$$

where $\lambda \in \mathbb{R}$ and $\mathbf{A} : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ admits a potential $\mathcal{A}$, with respect to its second variable ξ and satisfies some assumptions listed in the paper.

On the other hand, *Servadei* and *Valdinoci* in [30] studied the following fractional Laplacian problem

$$\begin{cases} (-\Delta)^s u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.3}$$

where Ω is a bounded domain. Since the fractional Laplacian operator is nonlocal, some technical difficulties arise when applying usual variational methods. In this paper, the authors used the mountain pass theorem to obtain the existence of solutions for problem (1.3). After that, many authors devoted to studying the fractional problems by using variational methods and critical point theory. Indeed, a lot of works in the literature involve the following Ambrosetti–Rabinowitz ((AR) in short) condition on the nonlinearity

$$0 < F(x,u) := \int_0^u f(x,t)dt \leq \mu f(x,u)u, \qquad x \in \mathbb{R}^N,\ 0 \neq u \in \mathbb{R}, \tag{1.4}$$

for some constant $\mu > p$, see [2] for more details. In general, the (AR) condition not only ensures that the Euler–Lagrange functional associated with a problem of the form (1.1) has a mountain pass geometry, but also guarantees boundedness of the Palais–Smale sequences corresponding the functional. Although the (AR) condition is very important and widely used to get the existence of solutions for elliptic problems via variational methods, it is not fulfilled by some simple nonlinearities, such as

$$f_1(t) = |t|^{p-2}t(\ln|t|+1) \quad \text{and} \quad f_2(t) = \begin{cases} |t|^{p-2}t - \frac{p-1}{p}|t|^{r-2}t, & |t| \leq 1 \\ |t|^{p-2}t(\ln|t| + \frac{1}{p}), & |t| > 1 \\ p < r < p_s^* := \frac{pN}{N-ps}. \end{cases}$$

In fact, these functions do not satisfy

$$F(x,t) \geq d_1|t|^\mu - d_2, \qquad x \in \mathbb{R}^N,\ t \in \mathbb{R}, \quad \text{where } d_1,\ d_2 > 0 \text{ and } \mu > p$$

which is a consequence of the (AR) condition. Here we would like to sketch some advances about this aspect. For the Laplacian case as $p = q = 2$, the existence of nontrivial weak solutions for nonlinear elliptic problems without assuming (AR) condition was obtained in [21]. For the single p-Laplacian case with $p > 1$, we refer the reader to [20]. For the (p,q)-Laplacian with $1 < q \leq 2 \leq p$ and $a = b = 0$, we refer the reader to [22]. For the fractional Laplacian case, the existence of infinitely many weak solutions for nonlinear elliptic problems without requiring (AR) condition was investigated in [8].

Motivated by the above works, especially [12,13], we are interested in the study of solutions for system (1.1) without the (AR) condition involving the fractional p-Laplacian. For this, let $\mathcal{V}(\mathbb{R}^N)$ be a subset of $C(\mathbb{R}^N)$ and for any $V \in \mathcal{V}(\mathbb{R}^N)$,

(A_1) V is bounded from below by a positive constant;

(A_2) there exists $\kappa > 0$ such that $\lim_{|y|\to\infty} \text{meas}(\{x \in B_\kappa(y) : V(x) \leq c\}) = 0$ for any $c > 0$,

where $B_\kappa(y)$ denotes any open ball of $\mathbb{R}^N$ centered at y and of radius $\kappa > 0$. Note that condition (A_2), which is weaker than the coercivity assumption: $V(x) \to \infty$ as $|x| \to \infty$, was originally discussed by *Bartsch* and *Wang* in [7] to overcome the lack of compactness.

Without further mentioning, we always assume $a, b \in \mathcal{V}(\mathbb{R}^N)$. Now we introduce some notation. Assume $1 < m < \infty$. Let $D^{s,m}(\mathbb{R}^N)$ denote the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the Gagliardo seminorm

$$[u]_{s,m} := \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^m}{|x-y|^{N+ms}} dxdy \right)^{1/m}.$$

Let $\omega \in \mathcal{V}(\mathbb{R}^N)$ and define

$$E_{m,\omega} := \left\{ u \in D^{s,m}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \omega(x)|u(x)|^m dx < \infty \right\},$$

endowed with the norm

$$\|u\|_{E_{m,\omega}} := \left([u]_{s,m}^m + \|u\|_{m,\omega}^m \right)^{1/m},$$

where $\|u\|_{m,\omega} = \left(\int_{\mathbb{R}^N} \omega |u|^m dx \right)^{1/m}$. Define

$$\mathbf{W} := E_{p,a} \times E_{q,b},$$

endowed with the norm

$$\|(u,v)\| := \|u\|_{E_{p,a}} + \|v\|_{E_{q,b}}.$$

By the embedding $E_{p,a} \hookrightarrow L^p(\mathbb{R}^N)$ (see [26]), we can define

$$\lambda^* = \inf \left\{ \|u\|_{E_{p,a}}^p + \|v\|_{E_{q,b}}^p : \int_{\mathbb{R}^N} |(u,v)|^p dx = 1 \right\},$$

and deduce that $\lambda^* > 0$.

Finally, the conditions we impose on the nonlinearity are:

(H_1) $H \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$ such that $H(x,u,v) > 0$ if $(u,v) \neq (0,0)$, $H(x,0,0) = 0$, and $H_u(x,u,v) = 0$ if $u \leq 0$; $H_v(x,u,v) = 0$ if $v \leq 0$;

(H_2) there exist $r \in (p, q_s^*)$ and $C > 0$ such that

$$|H_z(x,z)| \leq C(1 + |z|^{r-1}), \quad \text{for all } (x,z) \in \mathbb{R}^N \times \mathbb{R}^2,$$

where $H_z(x,z) = (H_u(x,z), H_v(x,z))$;

(H_3) $\lim\limits_{|z| \to \infty} \dfrac{H_z(x,z) \cdot z}{|z|^p} = \infty$, uniformly in $x \in \mathbb{R}^N$;

(H_4) there exists $g \in L^1(\mathbb{R}^N)_+$ such that

$$F(x, t_1 z) \leq F(x, t_2 z) + g(x), \quad \text{for all } (x,z) \in \mathbb{R}^N \times \mathbb{R}^2 \text{ and } 0 < t_1 \leq t_2 \text{ or } 0 < t_2 \leq t_1,$$

where $F(x,z) := H_z(x,z) \cdot z - pH(x,z)$;

(H_5) There exists $0 \leq \lambda(x) \leq \|\lambda\|_\infty < \lambda^*$ for all $x \in \mathbb{R}^N$ such that

$$\limsup_{|z| \to 0} \frac{|H_z(x,z)|}{|z|^{p-1}} \leq \lambda(x), \quad \text{uniformly in } x \in \mathbb{R}^N.$$

Remark 1.1. Obviously, the functions $f_1(t)$ and $f_2(t)$ mentioned before satisfy (H_1)–(H_5). For the single equation, it is easy to see that (1.4) implies the more weaker condition (H_5). Condition (H_4) together with (H_5), introduced by *Jeanjean* in [16], was often used to study the existence of nontrivial solutions for superlinear problems without (AR) condition in recent years, see for example [20,22].

Now, we give the definition of weak solutions of problem (1.1).

Definition 1.2. We say that $(u, v) \in \mathbf{W}$ is a (weak) *solution* of problem (1.1), if

$$\begin{aligned}\langle u, \varphi\rangle_{s,p} + \langle v, \psi\rangle_{s,q} &+ \int_{\mathbb{R}^N} a(x)|u(x)|^{p-2}u(x)\varphi(x)dx + \int_{\mathbb{R}^N} b(x)|v(x)|^{q-2}v(x)\psi(x)dx \\ &= \int_{\mathbb{R}^N} H_u(x,u,v)\varphi(x)dx + \int_{\mathbb{R}^N} H_v(x,u,v)\psi(x)dx,\end{aligned}$$

$$\langle u, \varphi\rangle_{s,m} := \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{m-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ms}} dxdy$$

for any $(\varphi, \psi) \in \mathbf{W}$.

The main result of this paper is the following.

Theorem 1.3. *Let $0 < s < 1 < q \leq p < Nq/(N-sq), sq < N$, and let (H_1)–(H_5) hold. Assume $a, b \in \mathcal{V}(\mathbb{R}^N)$. Then system (1.1) has at least one pair of nontrivial and nonnegative weak solution $(u, v) \in \mathbf{W}$.*

This article is organized as follows. In Section 2, we give some necessary definitions and properties of the fractional Sobolev space $\mathbf{W}$. In Section 3, using the mountain pass theorem, we obtain the existence of solutions for system (1.1).

2 Preliminaries

In this section, we give some basic results of fractional Sobolev space $\mathbf{W}$ that will be used in the next section.

By Lemma 10 of [26], one has $E_{p,a} = (E_{p,a}, \|\cdot\|_{E_{p,a}})$ and $E_{q,b} = (E_{q,b}, \|\cdot\|_{E_{q,b}})$ are two separable, reflexive Banach spaces. Hence, by Theorem 1.12 of [1], we have the following.

Lemma 2.1. $\mathbf{W} = (\mathbf{W}, \|\cdot\|_{\mathbf{W}})$ *is a separable and reflexive Banach space.*

Lemma 2.2. *The embedding $\mathbf{W} \hookrightarrow L^\nu(\mathbb{R}^N) \times L^\nu(\mathbb{R}^N)$ is continuous if $\nu \in [p, p_s^*]$, and*

$$\|(u, v)\|_\nu \leq C_\nu \|(u, v)\| \text{ for all } (u, v) \in \mathbf{W}. \tag{2.1}$$

Proof. By Lemma 1 of [26], there exists $C_\nu > 0$ such that

$$\|u\|_\nu \leq C_\nu \|u\|_{E_{p,a}} \text{ and } \|v\|_\nu \leq C_\nu \|v\|_{E_{q,b}} \text{ for all } (u, v) \in \mathbf{W}.$$

Hence,

$$\begin{aligned}\|(u, v)\|_\nu = \left\|\sqrt{u^2+v^2}\right\|_{L^\nu(\mathbb{R}^N)} &\leq \|u+v\|_{L^\nu(\mathbb{R}^N)} \\ &\leq \|u\|_{L^\nu(\mathbb{R}^N)} + \|v\|_{L^\nu(\mathbb{R}^N)} \\ &\leq C_\nu(\|u\|_{E_{p,a}} + \|v\|_{E_{q,b}}) \\ &= C_\nu\|(u, v)\|.\end{aligned}$$

Hence the lemma is proved. □

Similar to Proposition A.10 of [4], we have the following lemma.

Lemma 2.3. *Let $\{(u_n, v_n)\}_n \subset \mathbf{W}$ be such that $(u_n, v_n) \rightharpoonup (u, v)$ weakly in $\mathbf{W}$ as $n \to \infty$. Then, up to a subsequence, $(u_n, v_n) \to (u, v)$ a.e. in $\mathbb{R}^N$ as $n \to \infty$.*

By using the same discussion as in [26, Theorem 2.1], we have the following compact embedding.

Lemma 2.4. *Suppose that $a, b \in \mathcal{V}(\mathbb{R}^N)$ and $1 < q \le p < q_s^*$. Let $\nu \in [p, q_s^*)$ be a fixed exponent. Then the embeddings $E_{p,a} \hookrightarrow L^\nu(\mathbb{R}^N)$ and $E_{q,b} \hookrightarrow L^\nu(\mathbb{R}^N)$ are compact. Moreover, the embedding $\mathbf{W} \hookrightarrow L^\nu(\mathbb{R}) \times L^\nu(\mathbb{R})$ is compact.*

3 Proof of Theorem 1.3

To prove Theorem 1.3, we need the following mountain pass theorem under condition (C).

Theorem 3.1 (see [17, Theorem 6]). *Let E be a real Banach space with its dual space E^*, and suppose that $J \in C^1(E, \mathbb{R})$ satisfies*

$$\max\{J(0), J(e)\} \le \alpha < \beta \le \inf_{\|u\|=\rho} J(u),$$

for some $\alpha < \beta$, $\rho > 0$ and $e \in E$ with $\|e\| > \rho$. Let $c \ge \beta$ be characterized by

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t))$$

where $\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = e\}$ is the set of continuous paths joining 0 and e, then there exists a sequence $\{u_n\}_n \subset E$ such that

$$J(u_n) \to c \ge \beta \quad \text{and} \quad \|J'(u_n)\|_{E^*}(1 + \|u_n\|) \to 0, \quad \text{as } n \to \infty.$$

This kind of sequence is usually called a Cerami sequence.

Definition 3.2. We say that a functional $J : E \to \mathbb{R}$ of class C^1 satisfies the Cerami condition ((C) in short) if any Cerami sequence associated with J has a strongerly convergent subsequence in E.

The Euler–Lagrange functional associated with system (1.1) is

$$\begin{aligned} I(u, v) = \frac{1}{p} \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dxdy + \frac{1}{p} \int_{\mathbb{R}^N} a(x)|u|^p dx \\ + \frac{1}{q} \iint_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^q}{|x - y|^{N+qs}} dxdy + \frac{1}{q} \int_{\mathbb{R}^N} b(x)|v|^q dx - \int_{\mathbb{R}^N} H(x, u, v) dx. \end{aligned}$$

Clearly, the functional I is well-defined in $\mathbf{W}$. Under conditions (H_1)–(H_5), it is easy to see that the functional I is of class C^1, and for $(u, v) \in \mathbf{W}$

$$\begin{aligned} \langle I'(u, v), (\varphi, \psi) \rangle = \langle u, \varphi \rangle_{s,p} + \int_{\mathbb{R}^N} a|u|^{p-2} u\varphi dx + \langle v, \psi \rangle_{s,q} + \int_{\mathbb{R}^N} b|v|^{q-2} v\psi dx \\ - \int_{\mathbb{R}^N} H_u(x, u, v)\varphi dx - \int_{\mathbb{R}^N} H_v(x, u, v)\psi dx, \end{aligned}$$

for all $(\varphi, \psi) \in \mathbf{W}$.

Lemma 3.3. *Any Cerami sequence associated with the functional I is bounded in* $\mathbf{W}$.

Proof. Let $\{(u_n, v_n)\}_n \subset \mathbf{W}$ be a Cerami sequence associated with I. Then there exists $C > 0$ independent of n such that $|I(u_n, v_n)| \leq C$ and $(1 + \|(u_n, v_n)\|)I'(u_n, v_n) \to 0$ as $n \to \infty$. In the sequel, we will use C to denote various positive constant that does not depend on n. Hence there exists $\varepsilon_n > 0$, with $\varepsilon_n \to 0$, such that

$$|\langle I'(u_n, v_n), (\varphi, \psi)\rangle| \leq \frac{\varepsilon_n \|(\varphi, \psi)\|}{1 + \|(u_n, v_n)\|}, \quad \text{for all } (\varphi, \psi) \in \mathbf{W} \text{ and } n \in \mathbb{N}. \tag{3.1}$$

Choosing $(\varphi, \psi) = (u_n, v_n)$ in (3.1), we deduce

$$\begin{aligned} &\left| \langle u_n, u_n \rangle_{s,p} + \int_{\mathbb{R}^N} a|u_n|^p dx + \langle v_n, v_n \rangle_{s,q} + \int_{\mathbb{R}^N} b|v_n|^q dx \right. \\ &\left. - \int_{\mathbb{R}^N} [H_u(x, u_n, v_n)u_n + H_v(x, u_n, v_n)v_n] dx \right| \\ &\quad = |\langle I'(u_n, v_n), (u_n, v_n)\rangle| \\ &\quad \leq \frac{\varepsilon_n \|(u_n, v_n)\|}{1 + \|(u_n, v_n)\|} \leq \varepsilon_n \leq C. \end{aligned} \tag{3.2}$$

Hence we have

$$-\|u_n\|^p_{E_{p,a}} - \|v_n\|^q_{E_{q,b}} + \int_{\mathbb{R}^N} [H_u(x, u_n, v_n)u_n + H_v(x, u_n, v_n)v_n] dx \leq C \tag{3.3}$$

Next we show that $\{(u_n, v_n)\}_n$ is bounded in $\mathbf{W}$. Arguing by contradiction, we assume that $\|(u_n, v_n)\| \to \infty$. Without loss of generality, we assume that $\|(u_n, v_n)\| \geq 1$ for all $n \in \mathbb{N}$. Set $(X_n, Y_n) := \frac{(u_n, v_n)}{\|(u_n, v_n)\|}$. Clearly, $\|(X_n, Y_n)\| = 1$. Then there exists $(X, Y) \in \mathbf{W}$ such that, up to a subsequence,

$$\begin{aligned} (X_n, Y_n) &\rightharpoonup (X, Y) \quad \text{in } \mathbf{W} \\ (X_n, Y_n) &\to (X, Y) \quad \text{a.e. in } \mathbb{R}^N. \end{aligned}$$

Moreover, by Lemma 2.4, we can assume that, up to a subsequence,

$$(X_n, Y_n) \to (X, Y) \quad \text{in } L^\nu(\mathbb{R}^N) \times L^\nu(\mathbb{R}^N) \quad \text{for any } \nu \in [p, p_s^*).$$

Let $X_n^- = \min\{0, X_n\}$ and $Y_n^- = \min\{0, Y_n\}$. Clearly, $\{(X_n^-, Y_n^-)\}$ is also bounded in $\mathbf{W}$. Choosing $(\varphi, \psi) = (X_n^-, Y_n^-)$ in (3.1), we obtain by $\|(u_n, v_n)\| \to \infty$

$$o(1) = \frac{\langle I'(u_n, v_n), (X_n^-, Y_n^-)\rangle}{\|(u_n, v_n)\|^{p-1}},$$

that is,

$$\begin{aligned}
o(1) &= \frac{1}{\|(u_n,v_n)\|^{p-1}}\left[\langle u_n, X_n^-\rangle_{s,p} + \int_{\mathbb{R}^N} a|u_n|^{p-2}u_nX_n^-dx + \langle v_n, Y_n^-\rangle_{s,q} + \int_{\mathbb{R}^N} b|v_n|^{q-2}v_nY_n^-dx\right] \\
&\quad - \int_{\mathbb{R}^N} \frac{H_u(x,u_n^+,v_n^+)X_n^- + H_v(x,u_n^+,v_n^+)Y_n^-}{\|(u_n,v_n)\|^{p-1}}dx \\
&= \frac{1}{\|(u_n,v_n)\|^{p}}\left[\langle u_n, u_n^-\rangle_{s,p} + \int_{\mathbb{R}^N} a|u_n|^{p-2}u_nu_n^-dx + \langle v_n, v_n^-\rangle_{s,q} + \int_{\mathbb{R}^N} b|v_n|^{q-2}v_nv_n^-dx\right] \\
&\quad - \int_{\mathbb{R}^N} \frac{H_u(x,u_n^+,v_n^+)u_n^- + H_v(x,u_n^+,v_n^+)v_n^-}{\|(u_n,v_n)\|^{p}}dx \\
&= \frac{1}{\|(u_n,v_n)\|^{p}}\left[\langle u_n, u_n^-\rangle_{s,p} + \int_{\mathbb{R}^N} a|u_n|^{p-2}u_nu_n^-dx + \langle v_n, v_n^-\rangle_{s,q} + \int_{\mathbb{R}^N} b|v_n|^{q-2}v_nv_n^-dx\right] \\
&\geq \frac{1}{\|(u_n,v_n)\|^{p}}\left(\|u_n^-\|_{E_{p,a}}^p + \|v_n^-\|_{E_{q,b}}^q\right),
\end{aligned} \tag{3.4}$$

where the last inequality follows from the following elementary inequality

$$|\xi^- - \eta^-|^m \leq |\xi-\eta|^{m-2}(\xi-\eta)(\xi^- - \eta^-), \quad \text{for } \xi,\eta \in \mathbb{R} \text{ and } m > 1.$$

Hence, we deduce from (3.4) that as $n \to \infty$

$$\|X_n^-\|_{E_{p,a}} \to 0.$$

Similarly, by

$$o(1) = \frac{\langle I'(u_n,v_n), (X_n^-, Y_n^-)\rangle}{\|(u_n,v_n)\|^{q-1}},$$

we obtain

$$\|Y_n^-\|_{E_{q,b}} \to 0,$$

as $n \to \infty$. Therefore, we get $(X_n^-, Y_n^-) \to (0,0)$ in $\mathbf{W}$, this implies that $(X^-, Y^-) = (0,0)$ a.e. in $\mathbb{R}^N$. Hence, $X \geq 0$ and $Y \geq 0$ a.e. in $\mathbb{R}^N$.

Set $\Omega^+ = \{x \in \mathbb{R}^N : X > 0 \text{ or } Y > 0\}$ and $\Omega^0 = \{x \in \mathbb{R}^N : (X,Y) = (0,0)\}$. Assume Ω^+ has a positive Lebesgue measure. Recall that $\|(u_n,v_n)\| \to \infty$. Hence we get

$$|(u_n,v_n)| = \|(u_n,v_n)\||(X_n,Y_n)| \to \infty \quad \text{a.e. in } \Omega^+.$$

Thus, by (H_3)

$$\lim_{n\to\infty} \frac{H(x,u_n,v_n)}{\|(u_n,v_n)\|^p} = \lim_{n\to\infty} \frac{H(x,u_n,v_n)|(X_n,Y_n)|^p}{|(u_n,v_n)|^p} = \infty$$

a.e. in Ω^+. Then we deduce from Fatou's lemma that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{H(x,u_n,v_n)}{\|(u_n,v_n)\|^p}dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{H(x,u_n,v_n)|(X_n,Y_n)|^p}{|(u_n,v_n)|^p}dx = \infty. \tag{3.5}$$

On the other hand, by$|I(u_n,v_n)| \leq C$, one has

$$\frac{1}{p}\|u_n\|_{E_{p,a}}^p + \frac{1}{q}\|v_n\|_{E_{q,b}}^q - \int_{\mathbb{R}^N} H(x,u_n,v_n)dx \leq C.$$

Thus

$$\int_{\mathbb{R}^N} \frac{H(x,u_n,v_n)}{\|(u_n,v_n)\|^p}dx \le \frac{1}{p}\frac{\|u_n\|^p_{E_{p,a}}}{\|(u_n,v_n)\|^p} + \frac{1}{q}\frac{\|v_n\|^q_{E_{q,b}}}{\|(u_n,v_n)\|^p} + \frac{C}{\|(u_n,v_n)\|^p},$$

this together with assumption $\|(u_n,v_n)\| \ge 1$ implies that

$$\int_{\mathbb{R}^N} \frac{H(x,u_n,v_n)}{\|(u_n,v_n)\|^p}dx \le \frac{1}{p} + \frac{1}{q} + \frac{C}{\|(u_n,v_n)\|^p}.$$

Therefore, we deduce from $\|(u_n,v_n)\| \to \infty$ that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} \frac{H(x,u_n,v_n)}{\|(u_n,v_n)\|^p}dx \le \frac{1}{p} + \frac{1}{q},$$

which contradicts with (3.5). Hence Ω^+ has zero measure, that is, $(X,Y) = (0,0)$ a.e. in $\mathbb{R}^N$.

By the continuity of the function $t \in [0,1] \to I(tu_n, tv_n)$, there exists a sequence $\{t_n\}_n \subset [0,1]$ such that

$$I(t_nu_n, t_nv_n) = \max_{0\le t\le 1} I(tu_n, tv_n).$$

Set

$$(U_n, V_n) := (2\theta)^{1/q}(X_n, Y_n) = (2\theta)^{1/q}\frac{1}{\|(u_n,v_n)\|}(u_n,v_n) \in \mathbf{W},$$

where $\theta > 1/2$. By (H_5), for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|H_z(x,z)| \le (\lambda(x)+\varepsilon)|z|^{p-1} \quad \text{for all } x \in \mathbb{R}^N \text{ and } |z| \le \delta.$$

For all $x \in \mathbb{R}^N$ and $|z| > \delta$, we have by (H_2)

$$\begin{aligned}|H_z(x,z)| &\le C(1+|z|^{r-1})\\ &\le C\left(|\frac{z}{\delta}| + |z|^{r-1}\right) \le C\left(\frac{1}{\delta^{r-1}}+1\right)|z|^{r-1}.\end{aligned}$$

Hence, we obtain

$$|H_z(x,z)| \le (\lambda(x)+\varepsilon)|z|^{p-1} + C_\varepsilon|z|^{r-1} \quad \text{for all } (x,z) \in \mathbb{R}^N \times \mathbb{R}^2, \tag{3.6}$$

where $C_\varepsilon = C\left(\frac{1}{\delta^{r-1}}+1\right)$. Observing that $H(x,z) = \int_0^1 H_z(x,tz)\cdot z dt$, we get

$$|H(x,z)| \le \frac{1}{p}(\lambda(x)+\varepsilon)|z|^p + C_\varepsilon|z|^r, \quad \text{for all } (x,z) \in \mathbb{R}^N \times \mathbb{R}^2. \tag{3.7}$$

Since $(U_n,V_n) \to (0,0)$ in $L^\nu(\mathbb{R}^N) \times L^\nu(\mathbb{R}^N)$, with $\nu \in [p, q_s^*)$, by using (3.7) with $\varepsilon = 1$, we have

$$\begin{aligned}\int_{\mathbb{R}^N} H(x,U_n,V_n)dx &\le \int_{\mathbb{R}^N} [(\lambda(x)+1)|(U_n,V_n)|^p + C_1|(U_n,V_n)|^r]\,dx\\ &\le (\lambda^*+1)\|(U_n,V_n)\|^p_{L^p(\mathbb{R}^N)} + C_1\|(U_n,V_n)\|^r_{L^r(\mathbb{R}^N)}\\ &\to 0,\end{aligned}$$

as $n\to\infty$, thanks to $r\in(p,q_s^*)$ and $\lambda(x)<\lambda^*$. So we get

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}H(x,U_n,V_n)dx=0. \tag{3.8}$$

Since $\|(u_n,v_n)\|\to\infty$, there exists n_0 large enough such that $(2\theta)^{1/q}/\|(u_n,v_n)\|\in(0,1)$ for all $n\geq n_0$. Hence, for all $n\geq n_0$, we obtain by $q\leq p$, $\theta>\frac{1}{2}$ and $\|Y_n\|_{E_{q,b}}\leq\|X_n\|_{E_{p,a}}+\|Y_n\|_{E_{q,b}}=1$

$$\begin{aligned}
I(t_nu_n,t_nv_n)&\geq I\left((2\theta)^{1/q}u_n/\|(u_n,v_n)\|,\ (2\theta)^{1/q}v_n/\|(u_n,v_n)\|\right)\\
&=\frac{(2\theta)^{p/q}}{p}\|X_n\|_{E_{p,a}}^p+\frac{2\theta}{q}\|Y_n\|_{E_{q,b}}^q-\int_{\mathbb{R}^N}H(x,U_n,V_n)dx\\
&\geq\frac{2\theta}{p}(\|X_n\|_{E_{p,a}}^p+\|Y_n\|_{E_{q,b}}^q)-\int_{\mathbb{R}^N}H(x,U_n,V_n)dx\\
&\geq\frac{2\theta}{p}(\|X_n\|_{E_{p,a}}^p+\|Y_n\|_{E_{q,b}}^p)-\int_{\mathbb{R}^N}H(x,U_n,V_n)dx\\
&\geq\frac{2\theta}{p2^{p-1}}(\|X_n\|_{E_{p,a}}+\|Y_n\|_{E_{q,b}})^p-\int_{\mathbb{R}^N}H(x,U_n,V_n)dx\\
&=\frac{2\theta}{p2^{p-1}}-\int_{\mathbb{R}^N}H(x,U_n,V_n)dx.
\end{aligned}$$

By (3.8), there exists $n_1\geq n_0$ such that

$$\int_{\mathbb{R}^N}H(x,U_n,V_n)dx\leq\frac{\theta}{p2^{p-1}},\quad\text{for all } n\geq n_1.$$

Then

$$I(t_nu_n,t_nv_n)>\frac{\theta}{p2^{p-1}},\quad\text{for all } n\geq n_1,$$

this together with the arbitrariness of $\theta>1/2$ yields

$$\lim_{n\to\infty}I(t_nu_n,t_nv_n)=\infty. \tag{3.9}$$

Following the same discussion as Lemma 7.3 of [6] and using the facts that $I(0,0)=0$ and $I(u_n,v_n)\leq C$, we can assume that $t_n\in(0,1)$ for all $n\geq n_2\geq n_1$. Thus,

$$\begin{aligned}
&\|t_nu_n\|_{E_{p,a}}^p+\|t_nv_n\|_{E_{q,b}}^q-\int_{\mathbb{R}^N}[H_u(x,t_nu_n,t_nv_n)t_nu_n+H_v(x,t_nu_n,t_nv_n)t_nv_n]dx\\
&\quad=\langle I'(t_nu_n,t_nv_n),(t_nu_n,t_nv_n)\rangle=t_n\frac{d}{dt}\langle I'(tu_n,tv_n),(tu_n,tv_n)\rangle\Big|_{t=t_n}=0,
\end{aligned} \tag{3.10}$$

for all $n\geq n_2$.

From $0\leq t_n\leq 1$ and (H_4), one has

$$\int_{\mathbb{R}^N}F(x,t_nu_n,t_nv_n)dx\leq\int_{\mathbb{R}^N}F(x,u_n,v_n)dx+\int_{\mathbb{R}^N}g(x)dx. \tag{3.11}$$

Combining (3.10) with (3.11), we have

$$\begin{aligned}
\|t_nu_n\|_{E_{p,a}}^p+\|t_nv_n\|_{E_{q,b}}^q&=\int_{\mathbb{R}^N}[H_u(x,t_nu_n,t_nv_n)t_nu_n+H_v(x,t_nu_n,t_nv_n)t_nv_n]dx\\
&=\int_{\mathbb{R}^N}pH(x,t_nu_n,t_nv_n)dx+\int_{\mathbb{R}^N}F(x,t_nu_n,t_nv_n)dx\\
&\leq\int_{\mathbb{R}^N}pH(x,t_nu_n,t_nv_n)dx+\int_{\mathbb{R}^N}F(x,u_n,v_n)dx+\int_{\mathbb{R}^N}g(x)dx,
\end{aligned}$$

for all $n \geq n_2$. Then,

$$\begin{aligned} pI(t_nu_n,t_nv_n) &= \|t_nu_n\|^p_{E_{p,a}} + \frac{p}{q}\|t_nv_n\|^q_{E_{q,b}} - \int_{\mathbb{R}^N} pH(x,t_nu_n,t_nv_n)dx + \|t_nv_n\|^q_{E_{q,b}} - \|t_nv_n\|^q_{E_{q,b}} \\ &= \left(\frac{p}{q}-1\right)\|t_nv_n\|^q_{E_{q,b}} + \|t_nu_n\|^p_{E_{p,a}} + \|t_nv_n\|^q_{E_{q,b}} - \int_{\mathbb{R}^N} pH(x,t_nu_n,t_nv_n)dx \\ &\leq \left(\frac{p}{q}-1\right)\|v_n\|^q_{E_{q,b}} + \int_{\mathbb{R}^N} F(x,u_n,v_n)dx + \int_{\mathbb{R}^N} g(x)dx, \end{aligned}$$

this together with (3.9) yields

$$\left(\frac{p}{q}-1\right)\|v_n\|^q_{E_{q,b}} + \int_{\mathbb{R}^N} F(x,u_n,v_n)dx \to \infty. \tag{3.12}$$

On the other hand, by $|I(u_n,v_n)| \leq C$, we have

$$\|u_n\|^p_{E_{p,a}} + \frac{p}{q}\|v_n\|^q_{E_{q,b}} - p\int_{\mathbb{R}^N} H(x,u_n,v_n)dx = pI(u_n,v_n) \leq C.$$

Adding this inequality to (3.3) and using (H_4), we obtain

$$\left(\frac{p}{q}-1\right)\|v_n\|^q_{E_{q,b}} + \int_{\mathbb{R}^N} F(x,u_n,v_n)dx \leq C,$$

which contradicts with (3.12).

Therefore, we conclude that $\{(u_n,v_n)\}_n$ is bounded in $\mathbf{W}$. □

Lemma 3.4. *The functional I satisfies condition (C).*

Proof. Assume $\{(u_n,v_n)\}_n$ is a Cerami sequence. Then there exists $C > 0$ independent of n such that

$$|I(u_n,v_n)| \leq C \quad \text{and} \quad (1+\|(u_n,v_n)\|)I'(u_n,v_n) \to 0.$$

By Lemma 3.3, $\{(u_n,v_n)\}_n$ is bounded in $\mathbf{W}$. Hence, we can assume that

$$(u_n,v_n) \rightharpoonup (u,v) \quad \text{in } \mathbf{W} \quad \text{and} \quad (u_n,v_n) \to (u,v) \quad \text{in } L^\nu(\mathbb{R}^N) \times L^\nu(\mathbb{R}^N),$$

for any $\nu \in [p, q_s^*)$. By using (3.6) corresponding $\varepsilon = 1$ and applying Lemma 2.4, we deduce

$$\begin{aligned} &\int_{\mathbb{R}^N} |H_u(x,u_n,v_n)(u_n-u) + H_v(x,u_n,v_n)(v_n-v)|dx \\ &\quad\leq \int_{\mathbb{R}^N} (|H_u(x,u_n,v_n)| + |H_v(x,u_n,v_n)|)|(u_n-u,v_n-v)|dx \\ &\quad\leq \sqrt{2}\int_{\mathbb{R}^N} \left[(\lambda^*+1)|(u_n,v_n)|^{p-1}|(u_n-u,v_n-v)| + C|(u_n,v_n)|^{r-1}|(u_n-u,v_n-v)|\right]dx \\ &\quad\leq C(\|(u_n-u,v_n-v)\|_{L^p(\mathbb{R}^N)} + \|(u_n-u,v_n-v)\|_{L^r(\mathbb{R}^N)}) \\ &\quad\to 0, \end{aligned}$$

as $n \to \infty$, where C denotes various positive constants. Thus we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |H_u(x,u_n,v_n)(u_n-u) + H_v(x,u_n,v_n)(v_n-v)|dx = 0.$$

Taking into account that $(1+\|(u_n,v_n)\|)I'(u_n,v_n)\to 0$ and the boundedness of $\{(u_n,v_n)\}$, we have

$$\lim_{n\to\infty}\Big[\langle u_n,u_n-u\rangle_{s,p}+\int_{\mathbb{R}^N}a|u_n|^{p-2}u_n(u_n-u)dx \\ +\langle v_n,v_n-v\rangle_{s,q}+\int_{\mathbb{R}^N}b|v_n|^{q-2}v_n(v_n-v)dx\Big]=0. \tag{3.13}$$

Observe that the linear functional $\mathcal{L}(u,v):\mathbf{W}\to\mathbb{R}$ defined by

$$\langle\mathcal{L}(u,v),(\varphi,\psi)\rangle=\langle u,\varphi\rangle_{s,p}+\int_{\mathbb{R}^N}a|u|^{p-2}u\varphi dx+\langle v,\psi\rangle_{s,q}+\int_{\mathbb{R}^N}b|v|^{q-2}v\psi dx$$

for all $(\varphi,\psi)\in\mathbf{W}$, is bounded in $\mathbf{W}$ by using the Hölder inequality, see [26]. Since $(u_n,v_n)\rightharpoonup(u,v)$ in $\mathbf{W}$, it follows that

$$\lim_{n\to\infty}\langle\mathcal{L}(u,v),(u_n-u,v_n-v)\rangle=0. \tag{3.14}$$

Combining (3.13) with (3.14), we arrive at

$$\lim_{n\to\infty}\Big[\langle\mathcal{L}(u_n,v_n),(u_n-u,v_n-v)\rangle-\langle\mathcal{L}(u,v),(u_n-u,v_n-v)\rangle\Big]=0$$

A similar discussion as that in [26] yields that $(u_n,v_n)\to(u,v)$ in $\mathbf{W}$. □

Now we are in position to prove Theorem 1.3.

Proof of Theorem 1.3. We first show that functional I satisfies a mountain pass geometry.

For all $(u,v)\in\mathbf{W}$, with $\|(u,v)\|\le 1$, we have by (3.7) and the definition of λ^*

$$\begin{aligned}I(u,v)&\ge\frac{1}{p}(\|u\|^p_{E_{p,a}}+\|v\|^q_{E_{q,b}})-\frac{1}{p}\int_{\mathbb{R}^N}(\lambda(x)+\varepsilon)|(u,v)|^pdx-C_\varepsilon\int_{\mathbb{R}^N}|(u,v)|^rdx\\ &\ge\frac{1}{p}(\|u\|^p_{E_{p,a}}+\|v\|^q_{E_{q,b}})-\frac{1}{p}(\|\lambda\|_\infty+\varepsilon)\int_{\mathbb{R}^N}|(u,v)|^pdx-C_\varepsilon C_r^r\|(u,v)\|^r\\ &\ge\frac{1}{p}(\|u\|^p_{E_{p,a}}+\|v\|^p_{E_{q,b}})-\frac{1}{p}\frac{(\|\lambda\|_\infty+\varepsilon)}{\lambda^*}(\|u\|^p_{E_{p,a}}+\|v\|^p_{E_{q,b}})-C_\varepsilon C_r^r\|(u,v)\|^r.\end{aligned}$$

Here we use the fact that $\|v\|^q_{E_{q,b}}\ge\|v\|^p_{E_{q,b}}$, since $\|v\|_{E_{q,b}}\le\|(u,v)\|\le 1$. Taking $\varepsilon=\frac{1}{2}(\lambda^*-\|\lambda\|_\infty)$, we obtain

$$\begin{aligned}I(u,v)&\ge\frac{1}{2p}\left(1-\frac{\|\lambda\|_\infty}{\lambda^*}\right)(\|u\|^p_{E_{p,a}}+\|v\|^p_{E_{q,b}})-C\|(u,v)\|^r\\ &\ge\frac{1}{2^pp}\left(1-\frac{\|\lambda\|_\infty}{\lambda^*}\right)(\|u\|_{E_{p,a}}+\|v\|_{E_{q,b}})^p-C\|(u,v)\|^r\\ &=\frac{1}{2^pp}\left(1-\frac{\|\lambda\|_\infty}{\lambda^*}-C\|(u,v)\|^{r-p}\right)\|(u,v)\|^p.\end{aligned}$$

Now we take $\|(u,v)\|=\rho\in(0,1)$ small enough such that $1-\frac{\|\lambda\|_\infty}{\lambda^*}-C\rho^{r-p}>0$, thanks to $\|\lambda\|_\infty<\lambda^*$. Then

$$I(u,v)\ge\frac{1}{2^pp}\rho^p\left(1-\frac{\|\lambda\|_\infty}{\lambda^*}-C\rho^{r-p}\right)=:\alpha>0$$

for all $(u,v) \in \mathbf{W}$, with $\|(u,v)\| = \rho$.

In view of (H_2) and (H_3), for a positive constant $A > 0$, there exists $C_A > 0$ such that

$$H(x,z) \geq A|z|^p - C_A \quad \text{for all } (x,z) \in \mathbb{R}^N \times \mathbb{R}^2.$$

Let B_1 be the unit ball in $\mathbb{R}^N$ and let $u^*, v^* \in C_0^\infty(B_1)$ two positive functions. Denote by u_0, v_0 the extension of u^*, v^* to zero out of B_1. Then, for $t > 1$

$$\begin{aligned} I(tu_0, tv_0) &= \frac{t^p}{p}\|u_0\|_{E_{p,a}}^p + \frac{t^q}{q}\|v_0\|_{E_{q,b}}^q - \int_{B_1} H(x, tu_0, tv_0)dx \\ &\leq \frac{t^p}{p}(\|u_0\|_{E_{p,a}}^p - A\|(u_0,v_0)\|_{L^p(B_1)}^p) + \frac{t^q}{q}\|v_0\|_{E_{q,b}}^q + C_A|B_1|. \end{aligned}$$

Choosing A large enough such that $\|u_0\|_{E_{p,a}}^p < A\|(u_0,v_0)\|_{L^p(B_1)}^p$, one has $I(tu_0,tv_0) \to -\infty$ as $t \to \infty$. Thus, there exists $(e_1,e_2) = (T_0u_0, T_0v_0)$ such that $\|(e_1,e_2)\| > \rho$ and $I(e_1,e_2) < 0$. Therefore, we have proved that I satisfies a mountain pass geometry. Combining this fact with Lemma 3.4, there exists $(0,0) \neq (u,v) \in \mathbf{W}$ satisfying

$$\begin{aligned} \langle u,\varphi\rangle_{s,p} + \langle v,\psi\rangle_{s,q} + \int_{\mathbb{R}^N} a|u|^{p-2}u\varphi dx + \int_{\mathbb{R}^N} b|v|^{q-2}v\psi dx \\ = \int_{\mathbb{R}^N} \left[H_u(x,u,v)u + H_v(x,u,v)v\right] dx, \end{aligned}$$

for all $(\varphi,\psi) \in \mathbf{W}$. Taking $(\varphi,\psi) = (u^-,v^-)$, we have $(u^-,v^-) = (0,0)$ a.e. in $\mathbb{R}^N$, that is, $u \geq 0$ and $v \geq 0$ a.e. in $\mathbb{R}^N$. This ends the proof. □

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