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On a practically solvable product-type system of difference equations of second order

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Abstract. The problem of solvability of the following second order system of difference equations

$$z_{n+1} = \alpha z_n^a w_n^b, \qquad w_{n+1} = \beta w_n^c z_{n-1}^d, \qquad n \in \mathbb{N}_0,$$

where $a, b, c, d \in \mathbb{Z}$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, $z_{-1}, z_0, w_0 \in \mathbb{C} \setminus \{0\}$, is studied in detail.

Keywords: system of difference equations, second order system, product-type system, practically solvable system.

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1 Introduction

Investigation of various concrete nonlinear difference equations and systems has been of a great recent interest (see, e.g., [1–5], [10–35] and the references therein). Here we mention two subareas of interest. First, motivated by some classes of concrete nonlinear difference equations some experts started investigating the corresponding symmetric or cyclic systems of difference equations, as well as some modifications of the symmetric/cyclic systems (see, e.g., [4, 10–12, 14, 15, 19–21, 23–27, 29–33, 35]). Second, one of the basic problems related to difference equations and systems is the problem of their solvability. Books [7–9] contain numerous classical results on the topic. Solvable difference equations and systems have appeared in the literature from time to time for a long time. It has turned out that some of recently studied concrete nonlinear difference equations and systems are solvable. For example, the solvability of systems in [4] and [21] was discovered during the investigation of the long-term behavior of their solutions. On the other hand, some papers which give solutions of quite concrete equations and systems, without any theoretical explanations, have appeared recently. One of the first papers of this type is [5]. These facts motivated some experts to work on the problem

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of solvability more seriously (see, e.g., [4, 13, 16, 17], [19]-[30], [32–35]). Short note [16] can be regarded as a starting point for the serious investigation, and its importance is found in the method which is used for showing the solvability of an extension of the equation in [5]. The method was later used numerous times (see, e.g., [1, 13, 17, 22, 34]) and was essentially extended and developed in many of the other above mentioned papers on solvability.

Another fact of interest is to note that many difference equations and systems are obtained from product-type ones by some modifications of their right-hand sides (e.g., the equations/systems in [18] and [31] are of this type). Product-type equations and systems are solvable for the case of positive coefficients and initial values. However, this is not always the case if some of them are real or complex numbers. Hence, the problem of their solvability in these cases is of some interest. An investigation in this direction was started by S. Stević and some of his collaborators in [27], [29], [30] and [35], where some product-type systems are solved not only theoretically, but essentially also practically, that is, by presenting concrete formulas for solutions depending on the initial values and involving parameters.

This paper is devoted to the study of solvability of the following second order system of difference equations

$$z_{n+1} = \alpha z_n^a w_n^b, \qquad w_{n+1} = \beta w_n^c z_{n-1}^d, \qquad n \in \mathbb{N}_0, \tag{1.1}$$

where $a, b, c, d \in \mathbb{Z}$, $\alpha, \beta \in \mathbb{C}$ and $z_{-1}, z_0, w_0 \in \mathbb{C}$. Motivated by [27] system (1.1), unlike the ones in papers [29], [30] and [35], has some new parameters, namely, the coefficients α and β. Since the cases $\alpha = 0$ or $\beta = 0$ are very simple, so of not much interest, from now on we will assume that $\alpha, \beta \in \mathbb{C} \setminus \{0\}$.

Note that the domain of undefinable solutions [24] to system (1.1) is a subset of the following set

$$\mathcal{U} = \{(z_{-1}, z_0, w_0) \in \mathbb{C}^3 : z_{-1} = 0 \text{ or } z_0 = 0 \text{ or } w_0 = 0\}.$$

Thus we can regard that (z_{-1}, z_0, w_0) belong to $\mathbb{C}^3 \setminus \mathcal{U}$, although in some cases $\mathbb{C}^3 \setminus \mathcal{U}$ can be even equal to $\mathbb{C}^3$ (for example, if a, b, c and d are natural numbers).

For a system of difference equations of the form

$$\begin{aligned} z_n &= f(z_{n-1}, \ldots, z_{n-k}, w_{n-1}, \ldots, w_{n-l}) \\ w_n &= g(z_{n-1}, \ldots, z_{n-s}, w_{n-1}, \ldots, w_{n-t}), \qquad n \in \mathbb{N}_0, \end{aligned}$$

where $k, l, s, t \in \mathbb{N}$, is said that it is *solvable in closed form* if its general solution can be found in terms of initial values z_{-i}, $i = \overline{1, \max\{k, s\}}$, w_{-j}, $j = \overline{1, \max\{l, t\}}$, delays k, l, s, t, and index n only.

Let us also say that, as usual, the sums of the form $\sum_{j=l}^{m} a_j$, for $m < l$, will be regarded to have value equal to zero.

2 Auxiliary results

Several auxiliary results, which will be used in the proofs of the main results are given in this section. The first lemma is well-known (see, e.g., [9]).

Lemma 2.1. *Let* $i \in \mathbb{N}_0$ *and*

$$s_n^{(i)}(z) = 1 + 2^i z + 3^i z^2 + \cdots + n^i z^{n-1}, \qquad n \in \mathbb{N},$$

where $z \in \mathbb{C}$.

Then

$$s_n^{(0)}(z) = \frac{1-z^n}{1-z}, \tag{2.1}$$

$$s_n^{(1)}(z) = \frac{1-(n+1)z^n + nz^{n+1}}{(1-z)^2}, \tag{2.2}$$

for every $z \in \mathbb{C} \setminus \{1\}$ *and* $n \in \mathbb{N}$.

Remark 2.2. Since by the above mentioned convention $s_n^{(i)}(z) = \sum_{j=1}^n j^i z^{j-1} = 0$, for $n = -1, 0$, a simple calculation shows that (2.1) also holds for $n = 0$ and $z \neq 0$, while (2.2) holds for $n = -1$ and $n = 0$ when $z \neq 0$. Note also that $s_n^{(1)}(1) = \frac{n(n+1)}{2}$.

Lemma 2.3. *Let* α_i, β_i, $i = \overline{0,n}$, *be complex numbers. Then*

$$\sum_{i=0}^{n} \alpha_i \sum_{j=0}^{n-i} \beta_j = \sum_{j=0}^{n} \beta_j \sum_{i=0}^{n-j} \alpha_i, \tag{2.3}$$

for every $n \in \mathbb{N}_0$.

Proof. Let

$$\mathcal{S}_n = \{(i,j) : 0 \le i \le n,\ 0 \le j \le n-i\}.$$

From the definition of set $\mathcal{S}_n$ we see that variable j takes all the values from 0 to n too, and that for a fixed j, from the inequality $j \le n-i$ we have that $i \le n-j$, that is, the upper bound for i is equal $n-j$. Hence

$$\mathcal{S}_n = \{(i,j) : 0 \le j \le n,\ 0 \le i \le n-j\}.$$

Using this fact and by the changing order of summation, equality (2.3) easily follows. □

Remark 2.4. Note that Lemma 2.3 is a simple sort of the Fubini theorem with the discrete measure.

Let

$$f_n(u,v) = \sum_{i=0}^{n} u^i \sum_{j=0}^{n-i} v^j, \tag{2.4}$$

where $u, v \in \mathbb{C}$ and $n \in \mathbb{N}_0$, and where we in this case, as usual, regard that $0^0 = 1$.

Lemma 2.5. *Let* $f_n(u,v)$ *be defined in* (2.4). *Then*

$$f_n(u,v) = f_n(v,u), \tag{2.5}$$

for every $u, v \in \mathbb{C}$ *and* $n \in \mathbb{N}_0$.

Moreover, the following formulas hold.

(a) If $u \neq 1 \neq v$ *and* $u \neq v$, *then*

$$f_n(u,v) = \frac{v - u + u^{n+2} - v^{n+2} + uv^{n+2} - vu^{n+2}}{(1-v)(1-u)(v-u)}, \qquad n \in \mathbb{N}_0. \tag{2.6}$$

(b) If $u = v \neq 1$, then

$$f_n(u,u) = \frac{1-(n+2)u^{n+1}+(n+1)u^{n+2}}{(1-u)^2}, \qquad n \in \mathbb{N}_0. \tag{2.7}$$

(c) If $u \neq 1$ and $v = 1$, then

$$f_n(u,1) = \frac{n+1-(n+2)u+u^{n+2}}{(1-u)^2}, \qquad n \in \mathbb{N}_0. \tag{2.8}$$

(d) If $u = 1$ and $v \neq 1$, then

$$f_n(1,v) = \frac{n+1-(n+2)v+v^{n+2}}{(1-v)^2}, \qquad n \in \mathbb{N}_0. \tag{2.9}$$

(e) If $u = v = 1$, then

$$f_n(1,1) = \frac{(n+1)(n+2)}{2}, \qquad n \in \mathbb{N}_0. \tag{2.10}$$

Proof. By using Lemma 2.3 with $\alpha_i = u^i$, $\beta_j = v^j$, $i,j = \overline{0,n}$, equality (2.5) follows.

(a) By using the formula for the sum of a geometric progression three times (see (2.1)), the conditions $u \neq 1 \neq v$ and $u \neq v$, and some simple calculation, for the case $v \neq 0$, we obtain

$$\begin{aligned}
f_n(u,v) &= \sum_{i=0}^{n} u^i \sum_{j=0}^{n-i} v^j = \sum_{i=0}^{n} u^i \left(\frac{1-v^{n-i+1}}{1-v}\right) \\
&= \frac{1}{1-v}\left(\sum_{i=0}^{n} u^i - v^{n+1}\sum_{i=0}^{n}\left(\frac{u}{v}\right)^i\right) \\
&= \frac{1}{1-v}\left(\frac{1-u^{n+1}}{1-u} - v\frac{v^{n+1}-u^{n+1}}{v-u}\right) \\
&= \frac{v-u+u^{n+2}-v^{n+2}+uv^{n+2}-vu^{n+2}}{(1-v)(1-u)(v-u)}.
\end{aligned}$$

If $v = 0$ and $u \neq 1$, then

$$f_n(u,0) = \sum_{i=0}^{n} u^i = \frac{1-u^{n+1}}{1-u} = \frac{-u+u^{n+2}}{(1-u)(-u)}, \tag{2.11}$$

which is nothing but formula (2.6) when $v = 0$.

(b) By using formula (2.1) twice, the conditions $u = v \neq 1$, and some simple calculation, we obtain

$$\begin{aligned}
f_n(u,u) &= \sum_{i=0}^{n} u^i \sum_{j=0}^{n-i} u^j = \sum_{i=0}^{n} u^i \left(\frac{1-u^{n-i+1}}{1-u}\right) \\
&= \frac{1}{1-u}\left(\sum_{i=0}^{n} u^i - u^{n+1}\sum_{i=0}^{n} 1\right) \\
&= \frac{1}{1-u}\left(\frac{1-u^{n+1}}{1-u} - (n+1)u^{n+1}\right) \\
&= \frac{1-(n+2)u^{n+1}+(n+1)u^{n+2}}{(1-u)^2}.
\end{aligned}$$

(c) By using formula (2.1), the conditions $u \neq 1$ and $v = 1$, formula (2.2) with $z = u$, and some simple calculation, we obtain

$$\begin{aligned} f_n(u,1) &= \sum_{i=0}^{n} u^i \sum_{j=0}^{n-i} 1 = \sum_{i=0}^{n} u^i (n-i+1) \\ &= (n+1)\sum_{i=0}^{n} u^i - u \sum_{i=1}^{n} i u^{i-1} \\ &= (n+1)\frac{1-u^{n+1}}{1-u} - u\frac{1-(n+1)u^n + nu^{n+1}}{(1-u)^2} \\ &= \frac{n+1-(n+2)u + u^{n+2}}{(1-u)^2}, \end{aligned}$$

as desired.

(d) By using equality (2.5) with $u = 1$, we get $f_n(1,v) = f_n(v,1)$. From this and by (2.8) with $u \to v$, formula (2.9) follows.

(e) We have

$$f_n(1,1) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} 1 = \sum_{i=0}^{n}(n-i+1) = \sum_{j=1}^{n+1} j = \frac{(n+1)(n+2)}{2}.$$

□

Remark 2.6. If we note that

$$f_n(u,u) = \sum_{i=0}^{n} u^i \sum_{j=0}^{n-i} u^j = \sum_{i=0}^{n}\sum_{j=0}^{n-i} u^{i+j} = \sum_{l=0}^{n}(l+1)u^l = s_{n+1}^{(1)}(u),$$

since the equation $i + j = l$ has $l + 1$ nonnegative integer solutions, formula (2.7) also follows from (2.2). It is also easy to see that $f_n(u,1) = u^n s_{n+1}^{(1)}(\frac{1}{u})$, $u \neq 0$, from which along with (2.2), formula (2.8) can be obtained. Note also that by using the above mentioned summing convention and some simple calculation is obtained that formulas (2.6), (2.8)–(2.10) holds for $n = -1$ (formula (2.7) holds for $n = -1$ if $u \neq 0$).

The following result is also known (for example, it is a consequence of the Lagrange interpolation formula).

Lemma 2.7. *If all the zeros z_j, $j = \overline{1,k}$, of the polynomial*

$$P_k(z) = \gamma_k z^k + \gamma_{k-1} z^{k-1} + \cdots + \gamma_1 z + \gamma_0,$$

are such that $z_i \neq z_j$, $i \neq j$, then the following formulas hold:

$$\sum_{j=1}^{k} \frac{z_j^l}{P_k'(z_j)} = 0$$

for every $l \in \{0,1,\ldots,k-2\}$, and

$$\sum_{j=1}^{k} \frac{z_j^{k-1}}{P_k'(z_j)} = \frac{1}{\gamma_k}.$$

3 Main results

The main results in this paper are proved in this section. The first one is devoted to the case $b = 0$.

Theorem 3.1. *Assume that $a, c, d \in \mathbb{Z}$, $b = 0$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ and $z_{-1}, z_0, w_0 \in \mathbb{C} \setminus \{0\}$. Then system (1.1) is solvable in closed form.*

Proof. In this case system (1.1) becomes

$$z_{n+1} = \alpha z_n^a, \qquad w_{n+1} = \beta w_n^c z_{n-1}^d, \qquad n \in \mathbb{N}_0. \tag{3.1}$$

From the first equation in (3.1) it is not difficult to see that

$$z_n = \alpha^{\sum_{i=0}^{n-1} a^i} z_0^{a^n}, \qquad n \in \mathbb{N}. \tag{3.2}$$

From (3.2) we easily obtain that for $a \neq 1$

$$z_n = \alpha^{\frac{1-a^n}{1-a}} z_0^{a^n}, \qquad n \in \mathbb{N}, \tag{3.3}$$

while for $a = 1$, we have

$$z_n = \alpha^n z_0, \qquad n \in \mathbb{N}. \tag{3.4}$$

By using (3.2) into the second equation in (3.1), we get

$$w_n = \beta \alpha^{d \sum_{i=0}^{n-3} a^i} z_0^{d a^{n-2}} w_{n-1}^c, \quad \text{for } n \geq 3. \tag{3.5}$$

Hence, by using (3.5) with $n \to n-1$, we have that

$$\begin{aligned} w_n &= \beta \alpha^{d \sum_{i=0}^{n-3} a^i} z_0^{d a^{n-2}} \left(\beta \alpha^{d \sum_{i=0}^{n-4} a^i} z_0^{d a^{n-3}} w_{n-2}^c \right)^c \\ &= \beta^{1+c} \alpha^{d \sum_{i=0}^{n-3} a^i + dc \sum_{i=0}^{n-4} a^i} z_0^{d a^{n-2} + dc a^{n-3}} w_{n-2}^{c^2}, \end{aligned} \tag{3.6}$$

for $n \geq 4$.

Based on (3.5) and (3.6), assume that for some $k \geq 2$ we have proved

$$w_n = \beta^{\sum_{i=0}^{k-1} c^i} \alpha^{d \sum_{j=0}^{k-1} \left(c^j \sum_{i=0}^{n-j-3} a^i \right)} z_0^{d \sum_{i=0}^{k-1} c^i a^{n-i-2}} w_{n-k}^{c^k}, \tag{3.7}$$

for $n \geq k+2$.

Then by using (3.5) with $n \to n-k$ into (3.7), we get

$$\begin{aligned} w_n &= \beta^{\sum_{i=0}^{k-1} c^i} \alpha^{d \sum_{j=0}^{k-1} \left(c^j \sum_{i=0}^{n-j-3} a^i \right)} z_0^{d \sum_{i=0}^{k-1} c^i a^{n-i-2}} \left(\beta \alpha^{d \sum_{i=0}^{n-k-3} a^i} z_0^{d a^{n-k-2}} w_{n-k-1}^c \right)^{c^k} \\ &= \beta^{\sum_{i=0}^{k} c^i} \alpha^{d \sum_{j=0}^{k} \left(c^j \sum_{i=0}^{n-j-3} a^i \right)} z_0^{d \sum_{i=0}^{k} c^i a^{n-i-2}} w_{n-k-1}^{c^{k+1}} \end{aligned} \tag{3.8}$$

for $n \geq k+3$.

From (3.5), (3.8) and the method of induction we see that (3.7) holds for all natural numbers k and n such that $1 \leq k \leq n-2$.

By taking $k = n-2$ into (3.7) we get

$$w_n = \beta^{\sum_{i=0}^{n-3} c^i} \alpha^{d \sum_{j=0}^{n-3} \left(c^j \sum_{i=0}^{n-j-3} a^i \right)} z_0^{d \sum_{i=0}^{n-3} c^i a^{n-i-2}} w_2^{c^{n-2}}, \quad \text{for } n \geq 3. \tag{3.9}$$

On the other hand, from the second equation in (3.1) with $n = 1$, we have

$$w_2 = \beta w_1^c z_0^d = \beta(\beta w_0^c z_{-1}^d)^c z_0^d = \beta^{1+c} w_0^{c^2} z_{-1}^{cd} z_0^d. \tag{3.10}$$

From (3.9) and (3.10), we get

$$\begin{aligned} w_n &= \beta^{\sum_{i=0}^{n-3} c^i} \alpha^{d \sum_{j=0}^{n-3} \left(c^j \sum_{i=0}^{n-j-3} a^i\right)} z_0^{d \sum_{i=0}^{n-3} c^i a^{n-i-2}} \left(\beta^{1+c} w_0^{c^2} z_{-1}^{cd} z_0^d\right)^{c^{n-2}} \\ &= \beta^{\sum_{i=0}^{n-1} c^i} \alpha^{d \sum_{j=0}^{n-3} \left(c^j \sum_{i=0}^{n-j-3} a^i\right)} w_0^{c^n} z_0^{d \sum_{i=0}^{n-2} c^i a^{n-i-2}} z_{-1}^{dc^{n-1}}, \end{aligned} \tag{3.11}$$

for $n \geq 3$.

Now the subcases $a \neq c$ and $a = c$ will be considered separately.

Subcase $a \neq c$. In this case from (3.11), we get

$$w_n = \beta^{\sum_{i=0}^{n-1} c^i} \alpha^{d \sum_{j=0}^{n-3} \left(c^j \sum_{i=0}^{n-j-3} a^i\right)} w_0^{c^n} z_0^{d \frac{a^{n-1} - c^{n-1}}{a-c}} z_{-1}^{dc^{n-1}}, \qquad n \geq 2. \tag{3.12}$$

If $a \neq 1$ and $c \neq 1$, then by formula (2.6) with $n \to n-3$, $u = c$ and $v = a$, (3.12) becomes

$$w_n = \beta^{\frac{1-c^n}{1-c}} \alpha^{d \frac{a-c+c^{n-1} - a^{n-1} + ca^{n-1} - ac^{n-1}}{(1-a)(1-c)(a-c)}} w_0^{c^n} z_0^{d \frac{a^{n-1} - c^{n-1}}{a-c}} z_{-1}^{dc^{n-1}}, \qquad n \geq 2. \tag{3.13}$$

If $a \neq c$ and $a = 1$, then by using formula (2.8) with $n \to n-3$, $u = c$ and $v = 1$, we get

$$w_n = \beta^{\frac{1-c^n}{1-c}} \alpha^{d \frac{n-2-(n-1)c+c^{n-1}}{(1-c)^2}} w_0^{c^n} z_0^{d \frac{1-c^{n-1}}{1-c}} z_{-1}^{dc^{n-1}}, \qquad n \geq 2, \tag{3.14}$$

while if $a \neq c$ and $c = 1$, then by using formula (2.9) with $n \to n-3$, $u = 1$ and $v = a$, we get

$$w_n = \beta^n \alpha^{d \frac{n-2-(n-1)a+a^{n-1}}{(1-a)^2}} w_0 z_0^{d \frac{a^{n-1}-1}{a-1}} z_{-1}^d, \qquad n \geq 2. \tag{3.15}$$

Subcase $a = c$. In this case from (3.11), we get

$$w_n = \beta^{\sum_{i=0}^{n-1} a^i} \alpha^{d \sum_{j=0}^{n-3} \left(a^j \sum_{i=0}^{n-j-3} a^i\right)} w_0^{a^n} z_0^{d(n-1)a^{n-2}} z_{-1}^{da^{n-1}}, \tag{3.16}$$

for $n \geq 3$.

If $a = c \neq 1$, then by using formula (2.7) with $n \to n-3$ and $u = v = a$, (3.16) becomes

$$w_n = \beta^{\frac{1-a^n}{1-a}} \alpha^{d \frac{1-(n-1)a^{n-2}+(n-2)a^{n-1}}{(1-a)^2}} w_0^{a^n} z_0^{d(n-1)a^{n-2}} z_{-1}^{da^{n-1}}, \quad n \geq 3, \tag{3.17}$$

while if $a = c = 1$, then by using formula (2.10) with $n \to n-3$ and $u = v = 1$, we get

$$w_n = \beta^n \alpha^{d \frac{(n-2)(n-1)}{2}} w_0 z_0^{d(n-1)} z_{-1}^d, \quad n \geq 3, \tag{3.18}$$

finishing the proof of the theorem. □

Corollary 3.2. *Consider system (1.1) with $a, c, d \in \mathbb{Z}$, $b = 0$ and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. Assume that $z_{-1}, z_0, w_0 \in \mathbb{C} \setminus \{0\}$. Then the following statements are true.*

(a) If $a \neq c$, $a \neq 1$ and $c \neq 1$ then the general solution to system (1.1) is given by (3.3) and (3.13).

(b) If $a \neq c$ and $a = 1$ then the general solution to system (1.1) is given by (3.4) and (3.14).

(c) If $a \neq c$ and $c = 1$ then the general solution to system (1.1) is given by (3.3) and (3.15).

(d) If $a = c \neq 1$ then the general solution to system (1.1) is given by (3.3) and (3.17).

(e) If $a = c = 1$ then the general solution to system (1.1) is given by (3.4) and (3.18).

Theorem 3.3. *Assume that $a, b, c \in \mathbb{Z}$, $d = 0$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ and $z_{-1}, z_0, w_0 \in \mathbb{C} \setminus \{0\}$. Then system (1.1) is solvable in closed form.*

Proof. A proof of the theorem was essentially given in [27], but we give a slightly modified for the completeness. In this case system (1.1) becomes

$$z_{n+1} = \alpha z_n^a w_n^b, \qquad w_{n+1} = \beta w_n^c, \qquad n \in \mathbb{N}_0. \tag{3.19}$$

From the second equation in (3.19) we have that

$$w_n = \beta^{\sum_{i=0}^{n-1} c^i} w_0^{c^n}, \qquad n \in \mathbb{N}. \tag{3.20}$$

From (3.20) we easily obtain that for $c \neq 1$

$$w_n = \beta^{\frac{1-c^n}{1-c}} w_0^{c^n}, \qquad n \in \mathbb{N}, \tag{3.21}$$

while for $c = 1$, we have

$$w_n = \beta^n w_0, \qquad n \in \mathbb{N}. \tag{3.22}$$

Employing (3.20) into the first equation in (3.19), we get

$$z_n = \alpha \beta^{b \sum_{i=0}^{n-2} c^i} w_0^{bc^{n-1}} z_{n-1}^a, \tag{3.23}$$

for $n \geq 2$.

Hence, by using (3.23) with $n \to n-1$ into itself, we have

$$\begin{aligned} z_n &= \alpha \beta^{b \sum_{i=0}^{n-2} c^i} w_0^{bc^{n-1}} \left(\alpha \beta^{b \sum_{i=0}^{n-3} c^i} w_0^{bc^{n-2}} z_{n-2}^a\right)^a, \\ &= \alpha^{1+a} \beta^{b \sum_{i=0}^{n-2} c^i + ba \sum_{i=0}^{n-3} c^i} w_0^{bc^{n-1} + bac^{n-2}} z_{n-2}^{a^2}, \end{aligned} \tag{3.24}$$

for $n \geq 3$.

Based on (3.23) and (3.24), assume that for some $k \geq 2$ we have proved

$$z_n = \alpha^{\sum_{i=0}^{k-1} a^i} \beta^{b \sum_{j=0}^{k-1} \left(a^j \sum_{i=0}^{n-j-2} c^i\right)} w_0^{b \sum_{i=0}^{k-1} a^i c^{n-i-1}} z_{n-k}^{a^k}, \tag{3.25}$$

for $n \geq k+1$.

Then by using (3.23) with $n \to n-k$ into (3.25), we get

$$\begin{aligned} z_n &= \alpha^{\sum_{i=0}^{k-1} a^i} \beta^{b \sum_{j=0}^{k-1} \left(a^j \sum_{i=0}^{n-j-2} c^i\right)} w_0^{b \sum_{i=0}^{k-1} a^i c^{n-i-1}} \left(\alpha \beta^{b \sum_{i=0}^{n-k-2} c^i} w_0^{bc^{n-k-1}} z_{n-k-1}^a\right)^{a^k} \\ &= \alpha^{\sum_{i=0}^{k} a^i} \beta^{b \sum_{j=0}^{k} \left(a^j \sum_{i=0}^{n-j-2} c^i\right)} w_0^{b \sum_{i=0}^{k} a^i c^{n-i-1}} z_{n-k-1}^{a^{k+1}}, \end{aligned} \tag{3.26}$$

for $n \geq k+2$.

From (3.23), (3.26) and the method of induction we see that (3.25) holds for all natural numbers k and n such that $1 \leq k \leq n-1$.

By taking $k = n - 1$ into (3.25) we get

$$z_n = \alpha^{\sum_{i=0}^{n-2} a^i} \beta^{b \sum_{j=0}^{n-2} \left(a^j \sum_{i=0}^{n-j-2} c^i \right)} w_0^{b \sum_{i=0}^{n-2} a^i c^{n-i-1}} z_1^{a^{n-1}}, \tag{3.27}$$

for $n \geq 2$.

By using the relation $z_1 = \alpha z_0^a w_0^b$ into (3.27) it follows that

$$\begin{aligned} z_n &= \alpha^{\sum_{i=0}^{n-2} a^i} \beta^{b \sum_{j=0}^{n-2} \left(a^j \sum_{i=0}^{n-j-2} c^i \right)} w_0^{b \sum_{i=0}^{n-2} a^i c^{n-i-1}} (\alpha z_0^a w_0^b)^{a^{n-1}} \\ &= \alpha^{\sum_{i=0}^{n-1} a^i} \beta^{b \sum_{j=0}^{n-2} \left(a^j \sum_{i=0}^{n-j-2} c^i \right)} w_0^{b \sum_{i=0}^{n-1} a^i c^{n-i-1}} z_0^{a^n}, \end{aligned} \tag{3.28}$$

for $n \geq 2$.

Now the subcases $a \neq c$ and $a = c$ will be considered separately.

Subcase $a \neq c$. In this case from (3.28), we get

$$z_n = \alpha^{\sum_{i=0}^{n-1} a^i} \beta^{b \sum_{j=0}^{n-2} \left(a^j \sum_{i=0}^{n-j-2} c^i \right)} z_0^{a^n} w_0^{b \frac{a^n - c^n}{a-c}}, \qquad n \in \mathbb{N}. \tag{3.29}$$

If $a \neq 1$ and $c \neq 1$, then by formula (2.6) with $n \to n-2$, $u = a$ and $v = c$ and (2.5), (3.29) becomes

$$z_n = \alpha^{\frac{1-a^n}{1-a}} \beta^{b \frac{a - c + c^n - a^n + ca^n - ac^n}{(1-a)(1-c)(a-c)}} z_0^{a^n} w_0^{b \frac{a^n - c^n}{a-c}}, \qquad n \in \mathbb{N}. \tag{3.30}$$

If $a \neq c$ and $a = 1$, then by using formula (2.9) with $n \to n-2$, $u = 1$ and $v = c$, we get

$$z_n = \alpha^n \beta^{b \frac{n-1-nc+c^n}{(1-c)^2}} z_0 w_0^{b \frac{c^n - 1}{c-1}}, \qquad n \in \mathbb{N}, \tag{3.31}$$

while if $a \neq c$ and $c = 1$, then by using formula (2.8) with $n \to n-2$, $u = a$ and $v = 1$, we get

$$z_n = \alpha^{\frac{1-a^n}{1-a}} \beta^{b \frac{n-1-na+a^n}{(1-a)^2}} z_0^{a^n} w_0^{b \frac{a^n - 1}{a-1}}, \qquad n \in \mathbb{N}. \tag{3.32}$$

Subcase $a = c$. In this case from (3.28), we get

$$z_n = \alpha^{\sum_{i=0}^{n-1} a^i} \beta^{b \sum_{j=0}^{n-2} \left(a^j \sum_{i=0}^{n-j-2} a^i \right)} z_0^{a^n} w_0^{bna^{n-1}}, \qquad n \geq 2. \tag{3.33}$$

If $a = c \neq 1$, then by using formula (2.7) with $n \to n-2$, $u = v = a$, (3.33) becomes

$$z_n = \alpha^{\frac{1-a^n}{1-a}} \beta^{b \frac{1 - na^{n-1} + (n-1)a^n}{(1-a)^2}} z_0^{a^n} w_0^{bna^{n-1}}, \qquad n \geq 2, \tag{3.34}$$

while if $a = c = 1$, then by using formula (2.10) with $n \to n-2$, $u = v = 1$, we get

$$z_n = \alpha^n \beta^{b \frac{(n-1)n}{2}} z_0 w_0^{bn}, \quad n \in \mathbb{N}, \tag{3.35}$$

completing the proof. □

Corollary 3.4. *Consider system (1.1) with $a, b, c \in \mathbb{Z}$, $d = 0$ and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. Assume that $z_{-1}, z_0, w_0 \in \mathbb{C} \setminus \{0\}$. Then the following statements are true.*

(a) If $a \neq c$, $a \neq 1$ and $c \neq 1$ then the general solution to system (1.1) is given by (3.21) and (3.30).

(b) If $a \neq c$ and $a = 1$ then the general solution to system (1.1) is given by (3.21) and (3.31).

(c) If $a \neq c$ and $c = 1$ then the general solution to system (1.1) is given by (3.22) and (3.32).

(d) If $a = c \neq 1$, then the general solution to system (1.1) is given by (3.21) and (3.34).

(e) If $a = c = 1$, then the general solution to system (1.1) is given by (3.22) and (3.35).

Theorem 3.5. *Assume that $a, b, c, d \in \mathbb{Z}$, $bd \neq 0$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ and $z_{-1}, z_0, w_0 \in \mathbb{C} \setminus \{0\}$. Then system (1.1) is solvable in closed form.*

Proof. A simple inductive argument shows that the conditions $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ and $z_{-1}, z_0, w_0 \in \mathbb{C} \setminus \{0\}$, along with the equations in (1.1) imply $z_n \neq 0$ for $n \geq -1$ and $w_n \neq 0$ for $n \in \mathbb{N}_0$. Further, from the first equation in (1.1), for every well-defined solution, we have that

$$w_n^b = \frac{z_{n+1}}{\alpha z_n^a}, \qquad n \in \mathbb{N}_0, \tag{3.36}$$

while by taking the second equation in (1.1) to the b-th power, is obtained

$$w_{n+1}^b = \beta^b w_n^{bc} z_{n-1}^{bd}, \qquad n \in \mathbb{N}_0. \tag{3.37}$$

Using (3.36) into (3.37) we obtain

$$\frac{z_{n+2}}{\alpha z_{n+1}^a} = \beta^b \frac{z_{n+1}^c}{\alpha^c z_n^{ac}} z_{n-1}^{bd}, \qquad n \in \mathbb{N}_0,$$

that is,

$$z_{n+2} = \alpha^{1-c} \beta^b z_{n+1}^{a+c} z_n^{-ac} z_{n-1}^{bd}, \qquad n \in \mathbb{N}_0. \tag{3.38}$$

Let

$$a_1 = a + c, \qquad b_1 = -ac, \qquad c_1 = bd, \tag{3.39}$$

$$x_1 = 1 - c, \qquad y_1 = b. \tag{3.40}$$

Then equation (3.38) can be written as

$$z_{n+2} = \alpha^{x_1} \beta^{y_1} z_{n+1}^{a_1} z_n^{b_1} z_{n-1}^{c_1}, \qquad n \in \mathbb{N}_0. \tag{3.41}$$

By using recurrent relation (3.41) with $n \to n-1$ into itself, we have

$$\begin{aligned} z_{n+2} &= \alpha^{x_1} \beta^{y_1} (\alpha^{x_1} \beta^{y_1} z_n^{a_1} z_{n-1}^{b_1} z_{n-2}^{c_1})^{a_1} z_n^{b_1} z_{n-1}^{c_1}, \\ &= \alpha^{x_1 a_1 + x_1} \beta^{y_1 a_1 + y_1} z_n^{a_1 a_1 + b_1} z_{n-1}^{b_1 a_1 + c_1} z_{n-2}^{c_1 a_1} \\ &= \alpha^{x_2} \beta^{y_2} z_n^{a_2} z_{n-1}^{b_2} z_{n-2}^{c_2}, \end{aligned} \tag{3.42}$$

for $n \in \mathbb{N}$, where

$$a_2 := a_1 a_1 + b_1, \qquad b_2 := b_1 a_1 + c_1, \qquad c_2 := c_1 a_1, \tag{3.43}$$

$$x_2 := x_1 a_1 + x_1, \qquad y_2 := y_1 a_1 + y_1. \tag{3.44}$$

Assume that for a $k \in \mathbb{N}$ such that $2 \leq k \leq n+1$, we have proved that

$$z_{n+2} = \alpha^{x_k} \beta^{y_k} z_{n+2-k}^{a_k} z_{n+1-k}^{b_k} z_{n-k}^{c_k}, \tag{3.45}$$

for $n \ge k-1$, and that

$$a_k = a_1 a_{k-1} + b_{k-1}, \qquad b_k = b_1 a_{k-1} + c_{k-1}, \qquad c_k = c_1 a_{k-1}, \tag{3.46}$$

$$x_k = x_1 a_{k-1} + x_{k-1}, \qquad y_k = y_1 a_{k-1} + y_{k-1}. \tag{3.47}$$

Then by using the relation

$$z_{n+2-k} = \alpha^{x_1}\beta^{y_1} z_{n+1-k}^{a_1} z_{n-k}^{b_1} z_{n-k-1}^{c_1},$$

for $n \ge k$, into (3.45), we obtain

$$\begin{aligned} z_{n+2} &= \alpha^{x_k}\beta^{y_k}\left(\alpha^{x_1}\beta^{y_1} z_{n+1-k}^{a_1} z_{n-k}^{b_1} z_{n-k-1}^{c_1}\right)^{a_k} z_{n+1-k}^{b_k} z_{n-k}^{c_k} \\ &= \alpha^{x_1 a_k + x_k}\beta^{y_1 a_k + y_k} z_{n+1-k}^{a_1 a_k + b_k} z_{n-k}^{b_1 a_k + c_k} z_{n-k-1}^{c_1 a_k} \\ &= \alpha^{x_{k+1}}\beta^{y_{k+1}} z_{n+1-k}^{a_{k+1}} z_{n-k}^{b_{k+1}} z_{n-k-1}^{c_{k+1}}, \end{aligned} \tag{3.48}$$

for $n \ge k$, where

$$a_{k+1} := a_1 a_k + b_k, \qquad b_{k+1} := b_1 a_k + c_k, \qquad c_{k+1} := c_1 a_k, \tag{3.49}$$

$$x_{k+1} := x_1 a_k + x_k, \qquad y_{k+1} := y_1 a_k + y_k. \tag{3.50}$$

Relations (3.48)–(3.50), along with (3.42)–(3.44) and the method of induction shows that relations (3.45), (3.46) and (3.47) hold for all natural numbers k and n such that $2 \le k \le n+1$.

Hence, choosing $k = n+1$ in (3.45), and using the relation $z_1 = \alpha z_0^a w_0^b$ we have

$$\begin{aligned} z_{n+2} &= \alpha^{x_{n+1}}\beta^{y_{n+1}} z_1^{a_{n+1}} z_0^{b_{n+1}} z_{-1}^{c_{n+1}} \\ &= \alpha^{x_{n+1}}\beta^{y_{n+1}} \left(\alpha z_0^a w_0^b\right)^{a_{n+1}} z_0^{b_{n+1}} z_{-1}^{c_{n+1}} \\ &= \alpha^{a_{n+1}+x_{n+1}}\beta^{y_{n+1}} z_0^{a a_{n+1}+b_{n+1}} z_{-1}^{c_{n+1}} w_0^{b a_{n+1}}, \end{aligned} \tag{3.51}$$

for $n \in \mathbb{N}_0$.

From recurrent relations (3.46) we easily obtain that the sequence $(a_k)_{k\ge 4}$ satisfy the following difference equation

$$a_k = a_1 a_{k-1} + b_1 a_{k-2} + c_1 a_{k-3}. \tag{3.52}$$

From this, the linearity of difference equation (3.52), and by using the first and third relation in (3.46) we see that $(b_k)_{k\in\mathbb{N}}$ and $(c_k)_{k\in\mathbb{N}}$ are also solutions of equation (3.52).

On the other hand, from the recurrent relations in (3.47) we have

$$x_k = x_1 \sum_{i=1}^{k-1} a_i + x_1, \qquad y_k = y_1 \sum_{i=1}^{k-1} a_i + y_1, \tag{3.53}$$

for every $k \in \mathbb{N}$.

Since difference equation (3.52) is solvable, it follows that closed form formulas for $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$ and $(c_k)_{k\in\mathbb{N}}$, can be found. From the forms of general solutions of equation (3.52) (see (3.76), (3.96), (3.117)) and by using known formulas for $s_n^{(i)}$, $n \in \mathbb{N}$, $i = 0, 1, 2$, in (3.53) we can easily get formulas for $(x_k)_{k\in\mathbb{N}}$ and $(y_k)_{k\in\mathbb{N}}$. From these facts and (3.51) we see that equation (3.38) is solvable, too.

From the second equation in (1.1), we have that for every well-defined solution

$$z_{n-1}^{d}=\frac{w_{n+1}}{\beta w_n^{c}},\qquad n\in\mathbb{N}_0,\tag{3.54}$$

while by taking the first equation in (1.1) to the d-th power, is obtained

$$z_{n+1}^{d}=\alpha^{d}z_n^{ad}w_n^{bd},\qquad n\in\mathbb{N}_0.\tag{3.55}$$

Using (3.54) into (3.55) we obtain

$$\frac{w_{n+3}}{\beta w_{n+2}^{c}}=\alpha^{d}\frac{w_{n+2}^{a}}{\beta^{a}w_{n+1}^{ac}}w_n^{bd},\qquad n\in\mathbb{N}_0,$$

which can be written as

$$w_{n+3}=\alpha^{d}\beta^{1-a}w_{n+2}^{a+c}w_{n+1}^{-ac}w_n^{bd},\qquad n\in\mathbb{N}_0,\tag{3.56}$$

which is a related difference equation to (3.38) (only the coefficient is different and the indices are shifted forward for one). Note also that sequence $(w_n)_{n\in\mathbb{N}_0}$ satisfies the following initial conditions

$$w_1=\beta w_0^{c}z_{-1}^{d}\quad\text{and}\quad w_2=\beta^{1+c}w_0^{c^2}z_0^{d}z_{-1}^{cd}.\tag{3.57}$$

Hence, the above presented procedure for sequence $(z_n)_{n\geq -1}$ can be repeated and obtained that for all natural numbers k and n such that $1\leq k\leq n+1$

$$w_{n+3}=\alpha^{\widehat{x}_k}\beta^{\widehat{y}_k}w_{n+3-k}^{a_k}w_{n+2-k}^{b_k}w_{n+1-k}^{c_k},\tag{3.58}$$

where $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$ and $(c_k)_{k\in\mathbb{N}}$ satisfy the recurrent relations in (3.46) with initial conditions (3.39), and $(\widehat{x}_k)_{k\in\mathbb{N}}$ and $(\widehat{y}_k)_{k\in\mathbb{N}}$ satisfy the equations in (3.47) respectively, with the following initial conditions

$$\widehat{x}_1=d,\qquad \widehat{y}_1=1-a.\tag{3.59}$$

From (3.58) with $k=n+1$ and by using (3.57), we get

$$\begin{aligned}w_{n+3}&=\alpha^{\widehat{x}_{n+1}}\beta^{\widehat{y}_{n+1}}w_2^{a_{n+1}}w_1^{b_{n+1}}w_0^{c_{n+1}}\\&=\alpha^{\widehat{x}_{n+1}}\beta^{\widehat{y}_{n+1}}\left(\beta^{1+c}w_0^{c^2}z_0^{d}z_{-1}^{cd}\right)^{a_{n+1}}\left(\beta w_0^{c}z_{-1}^{d}\right)^{b_{n+1}}w_0^{c_{n+1}}\\&=\alpha^{\widehat{x}_{n+1}}\beta^{\widehat{y}_{n+1}+(1+c)a_{n+1}+b_{n+1}}w_0^{c^2a_{n+1}+cb_{n+1}+c_{n+1}}z_0^{da_{n+1}}z_{-1}^{cda_{n+1}+db_{n+1}},\end{aligned}\tag{3.60}$$

for $n\in\mathbb{N}_0$.

As above, the solvability of equation (3.52) shows that closed form formulas for $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$, $(c_k)_{k\in\mathbb{N}}$, and consequently for $(\widehat{x}_k)_{k\in\mathbb{N}}$ and $(\widehat{y}_k)_{k\in\mathbb{N}}$, can be found. This fact along with (3.60) implies that equation (3.56) is solvable, too.

A small problem is that formulas (3.51) and (3.60) hold for $n\in\mathbb{N}_0$, that is, z_k can be found for $k\geq 2$ and w_l for $l\geq 3$. To overcome the problem we show that for the case $bd\neq 0$, $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$, $(c_k)_{k\in\mathbb{N}}$, defined by (3.46) and $(x_k)_{k\in\mathbb{N}}$ and $(y_k)_{k\in\mathbb{N}}$, defined by (3.47), can be naturally prolonged for negative indices (similarly is proved that $(\widehat{x}_k)_{k\in\mathbb{N}}$ and $(\widehat{y}_k)_{k\in\mathbb{N}}$ are prolonged for these indices). Here, we show how they are prolonged for $k=-2,-1,0$, and in this way we get a natural set of "initial conditions" for the recurrent relations.

From (3.49) and (3.50) with $k = 0$ we get

$$a_1 = a_1 a_0 + b_0, \qquad b_1 = b_1 a_0 + c_0, \qquad c_1 = c_1 a_0. \tag{3.61}$$

$$x_1 = x_1 a_0 + x_0, \qquad y_1 = y_1 a_0 + y_0. \tag{3.62}$$

Since $c_1 = bd \neq 0$, from the last equation in (3.61) we get $a_0 = 1$. Using this fact in the first two equalities in (3.61) and in (3.62) is obtained $b_0 = c_0 = 0$, $x_0 = y_0 = 0$.

From this and by (3.49) and (3.50) with $k = -1$ we get

$$1 = a_0 = a_1 a_{-1} + b_{-1}, \qquad 0 = b_0 = b_1 a_{-1} + c_{-1}, \qquad 0 = c_0 = c_1 a_{-1}. \tag{3.63}$$

$$0 = x_0 = x_1 a_{-1} + x_{-1}, \qquad 0 = y_0 = y_1 a_{-1} + y_{-1}. \tag{3.64}$$

Since $c_1 \neq 0$, from the last equation in (3.63) we get $a_{-1} = 0$. Using this fact in the other two equalities in (3.63) and in (3.64) is obtained $b_{-1} = 1$, $c_{-1} = 0$, $x_{-1} = y_{-1} = 0$.

From this and by (3.49) and (3.50) with $k = -2$ it follows that

$$0 = a_{-1} = a_1 a_{-2} + b_{-2}, \qquad 1 = b_{-1} = b_1 a_{-2} + c_{-2}, \qquad 0 = c_{-1} = c_1 a_{-2}. \tag{3.65}$$

$$0 = x_{-1} = x_1 a_{-2} + x_{-2}, \qquad 0 = y_{-1} = y_1 a_{-2} + y_{-2}. \tag{3.66}$$

Since $c_1 \neq 0$, from the last equation in (3.65) we get $a_{-2} = 0$. Using this fact in the other two equalities in (3.65), as well as in (3.66) is obtained $b_{-2} = 0$, $c_{-2} = 1$ and $x_{-2} = y_{-2} = 0$.

Hence, sequences $(a_k)_{k \geq -2}$, $(b_k)_{k \geq -2}$ and $(c_k)_{k \geq -2}$ are solutions to linear difference equation (3.52) satisfying the next initial conditions

$$\begin{aligned} &a_{-2} = 0, \quad &&a_{-1} = 0, \quad &&a_0 = 1; \\ &b_{-2} = 0, \quad &&b_{-1} = 1, \quad &&b_0 = 0; \\ &c_{-2} = 1, \quad &&c_{-1} = 0, \quad &&c_0 = 0, \end{aligned} \tag{3.67}$$

respectively, sequences $(x_k)_{k \geq -2}$ and $(y_k)_{k \geq -2}$ satisfy (3.47) and the next conditions

$$x_{-2} = 0,\ x_{-1} = 0,\ x_0 = 0,\ x_1 = 1 - c,\ y_{-2} = 0,\ y_{-1} = 0,\ y_0 = 0,\ y_1 = b, \tag{3.68}$$

and finally $(\widehat{x}_k)_{k \geq -2}$ and $(\widehat{y}_k)_{k \geq -2}$ satisfy (3.47) and the next conditions

$$\widehat{x}_{-2} = 0,\ \widehat{x}_{-1} = 0,\ \widehat{x}_0 = 0,\ \widehat{x}_1 = d,\ \widehat{y}_{-2} = 0,\ \widehat{y}_{-1} = 0,\ \widehat{y}_0 = 0,\ \widehat{y}_1 = 1 - a. \tag{3.69}$$

Using these facts it is easy to see that $(z_n)_{n \geq -1}$ defined by (3.51) and $(w_n)_{n \in \mathbb{N}_0}$ defined by (3.60) is the general solution to system (1.1) with initial values z_{-1}, z_0, w_0, that is, the system is solvable, as claimed. □

Remark 3.6. Note that the condition $a, b, c, d \in \mathbb{Z}$ is naturally posed to avoid appearance of multi-valued solutions to system (1.1) for complex initial values.

From Theorem 3.5 we obtain the following corollary.

Corollary 3.7. *Consider system (1.1) with $a, b, c, d \in \mathbb{Z}$ and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. Assume that $z_{-1}, z_0, w_0 \in \mathbb{C} \setminus \{0\}$ and $bd \neq 0$. Then the general solution to system (1.1) is given by formulas (3.51) and (3.60) for $n \geq -3$, where the sequences $(a_n)_{n \geq -2}$, $(b_n)_{n \geq -2}$, $(c_n)_{n \geq -2}$, satisfy difference equation (3.52) with initial conditions (3.67), sequences $(x_n)_{n \geq -2}$ and $(y_n)_{n \geq -2}$ satisfy the equations in (3.47) with conditions (3.68), and sequences $(\widehat{x}_n)_{n \geq -2}$ and $(\widehat{y}_n)_{n \geq -2}$ satisfy the equations in (3.47) with conditions (3.69).*

From the theory of linear difference equations with constant coefficients we know that their general solutions are completely determined by the set of zeros of their associated characteristic polynomials. For equation (3.52) the polynomial is

$$P_3(\lambda) = \lambda^3 - a_1\lambda^2 - b_1\lambda - c_1, \tag{3.70}$$

where a_1, b_1 and c_1 are defined in (3.39). It is important to note that for the case $bd \neq 0$ the polynomial is of the third degree, so that the zeros can be explicitly found.

A standard procedure shows that the zeros of polynomial (3.70) are:

$$\lambda_1 = \frac{a+c}{3} + \frac{\sqrt[3]{B - \sqrt{4A^3 + B^2}}}{3\sqrt[3]{2}} + \frac{\sqrt[3]{B + \sqrt{4A^3 + B^2}}}{3\sqrt[3]{2}}, \tag{3.71}$$

$$\lambda_2 = \frac{a+c}{3} - \frac{(1+i\sqrt{3})\sqrt[3]{B - \sqrt{4A^3 + B^2}}}{6\sqrt[3]{2}} - \frac{(1-i\sqrt{3})\sqrt[3]{B + \sqrt{4A^3 + B^2}}}{6\sqrt[3]{2}}, \tag{3.72}$$

$$\lambda_3 = \frac{a+c}{3} - \frac{(1-i\sqrt{3})\sqrt[3]{B - \sqrt{4A^3 + B^2}}}{6\sqrt[3]{2}} - \frac{(1+i\sqrt{3})\sqrt[3]{B + \sqrt{4A^3 + B^2}}}{6\sqrt[3]{2}}, \tag{3.73}$$

where

$$A := -a^2 + ac - c^2, \qquad B := 2a^3 + 27bd - 3a^2c - 3ac^2 + 2c^3. \tag{3.74}$$

Now recall that the nature of these zeros depends on the sign of the discriminant

$$D := 4A^3 + B^2 \tag{3.75}$$

(see, e.g., [6]). Namely, if $D > 0$, then one zero is real and two are complex conjugates. If $D = 0$, all the zeros are real and at least two of them are equal. Finally, if $D < 0$, all the zeros are real and different.

Case $D \neq 0$. Since $D \neq 0$, then all the zeros λ_i, $i = \overline{1,3}$, of polynomial (3.70) are mutually different, and the general solution to equation (3.52) has the following form

$$u_n = \alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \alpha_3\lambda_3^n, \qquad n \in \mathbb{N}, \tag{3.76}$$

where α_i, $i = \overline{1,3}$, are arbitrary constants. Since for the case $c_1 \neq 0$, the solution can be prolonged for nonpositive indices then we may assume that formula (3.76) holds also for $n \geq -3$.

From Lemma 2.7 with $P_3(t) = \prod_{j=1}^{3}(t - \lambda_j)$, we have

$$\sum_{j=1}^{3} \frac{\lambda_j^l}{P_3'(\lambda_j)} = 0, \quad \text{for } l = 0, 1, \quad \text{and} \quad \sum_{j=1}^{3} \frac{\lambda_j^2}{P_3'(\lambda_j)} = 1. \tag{3.77}$$

From this, since from (3.67) we have $a_{-2} = a_{-1} = 0$ and $a_0 = 1$, and general solution of (3.52) has the form in (3.76), we obtain

$$a_n = \frac{\lambda_1^{n+2}}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} + \frac{\lambda_2^{n+2}}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} + \frac{\lambda_3^{n+2}}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}, \tag{3.78}$$

for $n \geq -2$.

On the other hand, from (3.46) we get

$$b_n = a_{n+1} - a_1 a_n, \tag{3.79}$$
$$c_n = c_1 a_{n-1}, \tag{3.80}$$

for $n \geq -2$.

By using (3.78) and (3.39) into (3.79), we obtain

$$b_n = \sum_{j=1}^{3} \frac{\lambda_j - a - c}{P_3'(\lambda_j)} \lambda_j^{n+2} \tag{3.81}$$

for $n \geq -2$, while by (3.78), which also holds for $n = -3$, (3.39) and (3.80), we get

$$c_n = \sum_{j=1}^{3} \frac{bd}{P_3'(\lambda_j)} \lambda_j^{n+1} \tag{3.82}$$

for $n \geq -2$.

From (3.53), (3.40), (3.78) and the fact that $a_0 = 1$, we have

$$\begin{aligned} x_n &= x_1 \sum_{i=0}^{n-1} a_i \\ &= (1-c) \sum_{i=0}^{n-1} \left(\frac{\lambda_1^{i+2}}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} + \frac{\lambda_2^{i+2}}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} + \frac{\lambda_3^{i+2}}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \right), \end{aligned} \tag{3.83}$$

and

$$\begin{aligned} y_n &= y_1 \sum_{i=0}^{n-1} a_i \\ &= b \sum_{i=0}^{n-1} \left(\frac{\lambda_1^{i+2}}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} + \frac{\lambda_2^{i+2}}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} + \frac{\lambda_3^{i+2}}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \right), \end{aligned} \tag{3.84}$$

for every $n \in \mathbb{N}$, while from (3.53), (3.59) and (3.78), we have

$$\begin{aligned} \widehat{x}_n &= \widehat{x}_1 \sum_{i=0}^{n-1} a_i \\ &= d \sum_{i=0}^{n-1} \left(\frac{\lambda_1^{i+2}}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} + \frac{\lambda_2^{i+2}}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} + \frac{\lambda_3^{i+2}}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \right), \end{aligned} \tag{3.85}$$

and

$$\begin{aligned} \widehat{y}_n &= \widehat{y}_1 \sum_{i=0}^{n-1} a_i \\ &= (1-a) \sum_{i=0}^{n-1} \left(\frac{\lambda_1^{i+2}}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} + \frac{\lambda_2^{i+2}}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} + \frac{\lambda_3^{i+2}}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \right), \end{aligned} \tag{3.86}$$

for every $n \in \mathbb{N}$.

Now assume that $\lambda_i \neq 1$, $i = 1,2,3$. Then, from formulas (3.83)–(3.86), we have

$$x_n = (1-c)R_n^{(1)}(\lambda_1,\lambda_2,\lambda_3), \tag{3.87}$$
$$y_n = bR_n^{(1)}(\lambda_1,\lambda_2,\lambda_3), \tag{3.88}$$
$$\widehat{x}_n = dR_n^{(1)}(\lambda_1,\lambda_2,\lambda_3), \tag{3.89}$$
$$\widehat{y}_n = (1-a)R_n^{(1)}(\lambda_1,\lambda_2,\lambda_3), \tag{3.90}$$

for $n \in \mathbb{N}$, where

$$\begin{aligned} R_n^{(1)}(\lambda_1,\lambda_2,\lambda_3) = & \frac{\lambda_1^2(\lambda_1^n-1)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-1)} + \frac{\lambda_2^2(\lambda_2^n-1)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-1)} \\ & + \frac{\lambda_3^2(\lambda_3^n-1)}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-1)}. \end{aligned}$$

Formulas (3.87)–(3.90) hold also for every $n \geq -2$. Indeed, if $n = -2,-1,0$, then by some simple calculation and (3.77), we obtain

$$\begin{aligned} R_{-2}^{(1)}(\lambda_1,\lambda_2,\lambda_3) &= -\frac{\lambda_1+1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} - \frac{\lambda_2+1}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} - \frac{\lambda_3+1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} \\ &= -\sum_{j=1}^{3}\frac{1}{P_3'(\lambda_j)} - \sum_{j=1}^{3}\frac{\lambda_j}{P_3'(\lambda_j)} = 0; \\ R_{-1}^{(1)}(\lambda_1,\lambda_2,\lambda_3) &= -\frac{\lambda_1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} - \frac{\lambda_2}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} - \frac{\lambda_3}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} \\ &= -\sum_{j=1}^{3}\frac{\lambda_j}{P_3'(\lambda_j)} = 0; \\ R_0^{(1)}(\lambda_1,\lambda_2,\lambda_3) &= 0. \end{aligned}$$

From these three relations we see that the sequences defined in (3.87)–(3.90) satisfy the conditions in (3.68) and (3.69), from which the statement follows.

If one of the zeros is equal to one, say λ_3, then $1 \neq \lambda_1 \neq \lambda_2 \neq 1$, and we have

$$x_n = (1-c)R_n^{(2)}(\lambda_1,\lambda_2), \tag{3.91}$$
$$y_n = bR_n^{(2)}(\lambda_1,\lambda_2), \tag{3.92}$$
$$\widehat{x}_n = dR_n^{(2)}(\lambda_1,\lambda_2), \tag{3.93}$$
$$\widehat{y}_n = (1-a)R_n^{(2)}(\lambda_1,\lambda_2), \tag{3.94}$$

for $n \in \mathbb{N}$, where

$$R_n^{(2)}(\lambda_1,\lambda_2) = \frac{\lambda_1^2(\lambda_1^n-1)}{(\lambda_1-\lambda_2)(\lambda_1-1)^2} + \frac{\lambda_2^2(\lambda_2^n-1)}{(\lambda_2-\lambda_1)(\lambda_2-1)^2} + \frac{n}{(\lambda_1-1)(\lambda_2-1)}.$$

Formulas (3.91)–(3.94) hold also for every $n \geq -2$. Indeed, if $n = -2,-1,0$, then by some simple calculation and (3.77), we obtain

$$\begin{aligned}
R^{(2)}_{-2}(\lambda_1,\lambda_2) &= -\frac{\lambda_1+1}{(\lambda_1-\lambda_2)(\lambda_1-1)} - \frac{\lambda_2+1}{(\lambda_2-\lambda_1)(\lambda_2-1)} - \frac{1+1}{(1-\lambda_1)(1-\lambda_2)}\\
&= -\sum_{j=1}^{3}\frac{1}{P_3'(\lambda_j)} - \sum_{j=1}^{3}\frac{\lambda_j}{P_3'(\lambda_j)} = 0;\\
R^{(2)}_{-1}(\lambda_1,\lambda_2) &= -\frac{\lambda_1}{(\lambda_1-\lambda_2)(\lambda_1-1)} - \frac{\lambda_2}{(\lambda_2-\lambda_1)(\lambda_2-1)} - \frac{1}{(1-\lambda_1)(1-\lambda_2)}\\
&= -\sum_{j=1}^{3}\frac{\lambda_j}{P_3'(\lambda_j)} = 0;\\
R^{(2)}_{0}(\lambda_1,\lambda_2) &= 0.
\end{aligned}$$

From these three relations we see that the sequences defined in (3.91)–(3.94) satisfy the conditions in (3.68) and (3.69), from which the statement follows.

From the above consideration and Theorem 3.5 we obtain the following corollary for the case $bd \neq 0$ and $D \neq 0$.

Corollary 3.8. *Consider system (1.1) with $a, b, c, d \in \mathbb{Z}$ and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. Assume $z_{-1}, z_0, w_0 \in \mathbb{C} \setminus \{0\}$, $bd \neq 0$ and $D \neq 0$. Then the following statements are true.*

(a) If none of the zeros of characteristic polynomial (3.70) is equal to one, i.e., if $P_3(1) \neq 0$, then the general solution to system (1.1) is given by formulas (3.51) and (3.60), where sequences $(a_n)_{n\geq -2}$, $(b_n)_{n\geq -2}$ and $(c_n)_{n\geq -2}$ are given by (3.78), (3.81) and (3.82) respectively, sequences $(x_n)_{n\geq -2}$ and $(y_n)_{n\geq -2}$ are given by (3.87) and (3.88), while sequences $(\widehat{x}_n)_{n\geq -2}$ and $(\widehat{y}_n)_{n\geq -2}$ are given by (3.89) and (3.90).

(b) If only one of the zeros of characteristic polynomial (3.70) is equal to one, i.e., if $P_3(1) = 0$ and $P_3'(1) \neq 0$, say λ_3, then the general solution to system (1.1) is given by formulas (3.51) and (3.60), where sequences $(a_n)_{n\geq -2}$, $(b_n)_{n\geq -2}$ and $(c_n)_{n\geq -2}$ are given by (3.78), (3.81) and (3.82) respectively, sequences $(x_n)_{n\geq -2}$ and $(y_n)_{n\geq -2}$ are given by (3.91) and (3.92), while sequences $(\widehat{x}_n)_{n\geq -2}$ and $(\widehat{y}_n)_{n\geq -2}$ are given by (3.93) and (3.94).

Remark 3.9. Equation (3.70) will have a zero equal to one if

$$P_3(1) = 1 - a - c + ac - bd = 0,$$

that is, if

$$(a-1)(c-1) = bd,$$

so that

$$P_3(\lambda) = \lambda^3 - (a+c)\lambda^2 + ac\lambda - (a-1)(c-1). \tag{3.95}$$

If $a = 3$ and $c = 0$, then $bd = -2 \neq 0$, $D = -4 \cdot 3^6 + (2 \cdot 27 - 27 \cdot 2)^3 = -2916 \neq 0$,

$$P_3(\lambda) = \lambda^3 - 3\lambda^2 + 2 = (\lambda - 1)(\lambda^2 - 2\lambda - 2),$$

so that the conditions of Corollary 3.8 *(b)* are satisfied. Hence, there are cases such that the only one zero of polynomial (3.70) is equal to one and all three zeros are mutually different. For the symmetry when $a = 0$ and $c = 3$ is obtained the same polynomial.

Case $D=0$. If $D=0$, then, at least two zeros of characteristic polynomial (3.70), say, λ_2 and λ_3 are equal. It is easy to see that the polynomial would have three equal zeros only if $B=0$, which along with $D=0$ would also imply $A=0$. However, since $A=-(a^2-ac+c^2)$ this is only possible if $a=c=0$. Indeed, using the fact that $t^2-t+1>0$, $t\in\mathbb{R}$, it is easily obtained that $A<0$ in the case $a\neq 0\neq c$, while if $a\neq 0=c$ or $a=0\neq c$, it is immediately obtained that $A<0$. But, in the case $a=c=0$ polynomial (3.70) would have the form λ^3-bd. Using the condition $bd\neq 0$ it would follow that such a polynomial has three different zeros, which is a contradiction. Thus, (3.70) has exactly two equal zeros in this case.

Hence, the general solution of (3.52) has the following form

$$a_n=\hat{\alpha}_1\lambda_1^n+(\hat{\alpha}_2+\hat{\alpha}_3 n)\lambda_2^n,\quad n\in\mathbb{N}, \tag{3.96}$$

where $\hat{\alpha}_1,\hat{\alpha}_2$ and $\hat{\alpha}_3$ are arbitrary constants. Since, in our case the following conditions must be satisfied $a_{-2}=a_{-1}=0$ and $a_0=1$, we will find the solution $(a_n)_{n\geq -2}$ of equation (3.52) by letting $\lambda_3\to\lambda_2$ into the solution in (3.78).

We have

$$\begin{aligned}
a_n &= \lim_{\lambda_3\to\lambda_2}\left(\frac{\lambda_1^{n+2}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}+\frac{\lambda_2^{n+2}}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}+\frac{\lambda_3^{n+2}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\right)\\
&= \lim_{\lambda_3\to\lambda_2}\left(\frac{\lambda_1^{n+2}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}+\frac{\lambda_2^{n+2}(\lambda_3-\lambda_1)-\lambda_3^{n+2}(\lambda_2-\lambda_1)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)}\right)\\
&= \frac{\lambda_1^{n+2}}{(\lambda_1-\lambda_2)^2}+\lim_{\lambda_3\to\lambda_2}\frac{\lambda_2\lambda_3(\lambda_2^{n+1}-\lambda_3^{n+1})-\lambda_1(\lambda_2^{n+2}-\lambda_3^{n+2})}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)}\\
&= \frac{\lambda_1^{n+2}-(n+2)\lambda_1\lambda_2^{n+1}+(n+1)\lambda_2^{n+2}}{(\lambda_2-\lambda_1)^2},
\end{aligned}$$

for $n\geq -2$, that is,

$$a_n=\frac{\lambda_1^{n+2}+\big(\lambda_2-2\lambda_1+n(\lambda_2-\lambda_1)\big)\lambda_2^{n+1}}{(\lambda_2-\lambda_1)^2},\qquad n\geq -2. \tag{3.97}$$

A direct calculation verify that $a_{-2}=a_{-1}=0$ and $a_0=1$. Clearly, (3.97) is of the form in (3.96) with

$$\hat{\alpha}_1=\frac{\lambda_1^2}{(\lambda_2-\lambda_1)^2},\qquad \hat{\alpha}_2=\frac{\lambda_2^2-2\lambda_1\lambda_2}{(\lambda_2-\lambda_1)^2}\quad\text{and}\quad \hat{\alpha}_3=\frac{\lambda_2}{\lambda_2-\lambda_1}.$$

By using relations (3.97) in (3.79) and (3.80) we get

$$\begin{aligned}
b_n &= \frac{(\lambda_1-a-c)\lambda_1^{n+2}}{(\lambda_2-\lambda_1)^2}\\
&\quad+\frac{\big(\lambda_2(2\lambda_2-3\lambda_1)-(a+c)(\lambda_2-2\lambda_1)+n(\lambda_2-\lambda_1)(\lambda_2-a-c)\big)\lambda_2^{n+1}}{(\lambda_2-\lambda_1)^2},
\end{aligned} \tag{3.98}$$

$$c_n=bd\frac{\lambda_1^{n+1}+(-\lambda_1+n(\lambda_2-\lambda_1))\lambda_2^{n}}{(\lambda_2-\lambda_1)^2}, \tag{3.99}$$

for $n\geq -2$.

From (3.53), (3.40), (3.97) and the fact that $a_0=1$, we have

$$x_n=x_1\sum_{j=0}^{n-1}a_j=(1-c)\sum_{j=0}^{n-1}\frac{\lambda_1^{j+2}+\big(\lambda_2-2\lambda_1+j(\lambda_2-\lambda_1)\big)\lambda_2^{j+1}}{(\lambda_2-\lambda_1)^2}, \tag{3.100}$$

and

$$y_n = y_1 \sum_{j=0}^{n-1} a_j = b \sum_{j=0}^{n-1} \frac{\lambda_1^{j+2} + \big(\lambda_2 - 2\lambda_1 + j(\lambda_2 - \lambda_1)\big)\lambda_2^{j+1}}{(\lambda_2 - \lambda_1)^2}, \tag{3.101}$$

for every $n \in \mathbb{N}$, while from (3.53), (3.59) and (3.97), we have

$$\widehat{x}_n = \widehat{x}_1 \sum_{j=0}^{n-1} a_j = d \sum_{j=0}^{n-1} \frac{\lambda_1^{j+2} + \big(\lambda_2 - 2\lambda_1 + j(\lambda_2 - \lambda_1)\big)\lambda_2^{j+1}}{(\lambda_2 - \lambda_1)^2}, \tag{3.102}$$

and

$$\widehat{y}_n = \widehat{y}_1 \sum_{j=0}^{n-1} a_j = (1-a) \sum_{j=0}^{n-1} \frac{\lambda_1^{j+2} + \big(\lambda_2 - 2\lambda_1 + j(\lambda_2 - \lambda_1)\big)\lambda_2^{j+1}}{(\lambda_2 - \lambda_1)^2}, \tag{3.103}$$

for every $n \in \mathbb{N}$.

If we assume that $\lambda_1 \neq 1 \neq \lambda_2 = \lambda_3$, then from (3.100)–(3.103) and Lemma 2.1, it follows that

$$x_n = (1-c)R_n^{(3)}(\lambda_1, \lambda_2), \tag{3.104}$$
$$y_n = bR_n^{(3)}(\lambda_1, \lambda_2), \tag{3.105}$$
$$\widehat{x}_n = dR_n^{(3)}(\lambda_1, \lambda_2), \tag{3.106}$$
$$\widehat{y}_n = (1-a)R_n^{(3)}(\lambda_1, \lambda_2), \tag{3.107}$$

for every $n \in \mathbb{N}$, where

$$R_n^{(3)}(\lambda_1, \lambda_2) = \frac{\lambda_1^2(\lambda_1^n - 1)}{(\lambda_2 - \lambda_1)^2(\lambda_1 - 1)} + \frac{(\lambda_2 - 2\lambda_1)\lambda_2(\lambda_2^n - 1)}{(\lambda_2 - \lambda_1)^2(\lambda_2 - 1)} + \frac{\lambda_2^2(1 - n\lambda_2^{n-1} + (n-1)\lambda_2^n)}{(\lambda_2 - \lambda_1)(\lambda_2 - 1)^2}.$$

Formulas (3.104)–(3.107) hold also for every $n \geq -2$. Indeed, if $n = -2, -1, 0$, then by some simple calculation, we obtain

$$R_{-2}^{(3)}(\lambda_1, \lambda_2) = -\frac{\lambda_1 + 1}{(\lambda_1 - \lambda_2)^2} - \frac{(\lambda_2 - 2\lambda_1)(\lambda_2 + 1)}{(\lambda_2 - \lambda_1)^2 \lambda_2} + \frac{\lambda_2 + 2}{(\lambda_2 - \lambda_1)\lambda_2} = 0;$$
$$R_{-1}^{(3)}(\lambda_1, \lambda_2) = -\frac{\lambda_1}{(\lambda_1 - \lambda_2)^2} - \frac{\lambda_2 - 2\lambda_1}{(\lambda_2 - \lambda_1)^2} + \frac{1}{\lambda_2 - \lambda_1} = 0;$$
$$R_0^{(3)}(\lambda_1, \lambda_2) = 0.$$

From these three relations we see that the sequences defined in (3.104)–(3.107) satisfy the conditions in (3.68) and (3.69), from which the statement follows.

If we assume that $\lambda_1 \neq 1$ and $\lambda_2 = \lambda_3 = 1$, then from (3.100)–(3.103) it follows that

$$x_n = (1-c)R_n^{(4)}(\lambda_1), \tag{3.108}$$
$$y_n = bR_n^{(4)}(\lambda_1), \tag{3.109}$$
$$\widehat{x}_n = dR_n^{(4)}(\lambda_1), \tag{3.110}$$
$$\widehat{y}_n = (1-a)R_n^{(4)}(\lambda_1), \tag{3.111}$$

for every $n \in \mathbb{N}$, where

$$R_n^{(4)}(\lambda_1) = \frac{\lambda_1^2(\lambda_1^n - 1)}{(\lambda_1 - 1)^3} + \frac{(1-2\lambda_1)n}{(\lambda_1 - 1)^2} + \frac{(n-1)n}{2(1-\lambda_1)}.$$

Formulas (3.108)–(3.111) hold also for every $n \geq -2$. Indeed, if $n = -2, -1, 0$, then by some simple calculation, we obtain

$$\begin{aligned} R_{-2}^{(4)}(\lambda_1) &= -\frac{\lambda_1 + 1}{(\lambda_1 - 1)^2} + \frac{4\lambda_1 - 2}{(\lambda_1 - 1)^2} + \frac{3}{1-\lambda_1} = 0; \\ R_{-1}^{(4)}(\lambda_1) &= -\frac{\lambda_1}{(\lambda_1 - 1)^2} + \frac{2\lambda_1 - 1}{(\lambda_1 - 1)^2} + \frac{1}{1-\lambda_1} = 0; \\ R_0^{(4)}(\lambda_1) &= 0. \end{aligned}$$

From these three relations we see that the sequences defined in (3.108)–(3.111) satisfy the conditions in (3.68) and (3.69), from which the statement follows.

If we assume that $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 \neq 1$, then from formulas (3.100)–(3.103) it follows that

$$x_n = (1-c)R_n^{(5)}(\lambda_2), \tag{3.112}$$

$$y_n = bR_n^{(5)}(\lambda_2), \tag{3.113}$$

$$\widehat{x}_n = dR_n^{(5)}(\lambda_2), \tag{3.114}$$

$$\widehat{y}_n = (1-a)R_n^{(5)}(\lambda_2), \tag{3.115}$$

for every $n \in \mathbb{N}$, where

$$R_n^{(5)}(\lambda_2) = \frac{n}{(\lambda_2 - 1)^2} + \frac{(\lambda_2 - 2)\lambda_2(\lambda_2^n - 1)}{(\lambda_2 - 1)^3} + \frac{\lambda_2^2(1 - n\lambda_2^{n-1} + (n-1)\lambda_2^n)}{(\lambda_2 - 1)^3}.$$

Formulas (3.112)–(3.115) hold also for every $n \geq -2$. Indeed, if $n = -2, -1, 0$, then by some simple calculation, we obtain

$$\begin{aligned} R_{-2}^{(5)}(\lambda_2) &= -\frac{2}{(\lambda_2 - 1)^2} - \frac{(\lambda_2 - 2)(\lambda_2 + 1)}{\lambda_2(\lambda_2 - 1)^2} + \frac{\lambda_2 + 2}{(\lambda_2 - 1)\lambda_2} = 0; \\ R_{-1}^{(5)}(\lambda_2) &= -\frac{1}{(\lambda_2 - 1)^2} - \frac{\lambda_2 - 2}{(\lambda_2 - 1)^2} + \frac{1}{\lambda_2 - 1} = 0; \\ R_0^{(5)}(\lambda_2) &= 0. \end{aligned}$$

From these three relations we see that the sequences defined in (3.112)–(3.115) satisfy the conditions in (3.68) and (3.69), from which the statement follows.

From the above consideration and Theorem 3.5 we obtain the following corollary.

Corollary 3.10. *Consider system (1.1) with $a, b, c, d \in \mathbb{Z}$ and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. Assume $z_{-1}, z_0, w_0 \in \mathbb{C} \setminus \{0\}$, $bd \neq 0$ and $D = 0$. Then the following statements are true.*

(a) If none of the zeros of characteristic polynomial (3.70) is equal to one, i.e., if $P_3(1) \neq 0$, then the general solution to system (1.1) is given by formulas (3.51) and (3.60), where sequences $(a_n)_{n\geq -2}$, $(b_n)_{n\geq -2}$ and $(c_n)_{n\geq -2}$ are given by (3.97), (3.98) and (3.99) respectively, sequences $(x_n)_{n\geq -2}$ and $(y_n)_{n\geq -2}$ are given by (3.104) and (3.105), while sequences $(\widehat{x}_n)_{n\geq -2}$ and $(\widehat{y}_n)_{n\geq -2}$ are given by (3.106) and (3.107).

(b) *If exactly two of the zeros of characteristic polynomial* (3.70) *are equal to one (i.e., if* $P_3(1) = P_3'(1) = 0$ *and* $P_3''(1) \neq 0$*), say* λ_2 *and* λ_3*, then the general solution to system* (1.1) *is given by formulas* (3.51) *and* (3.60)*, where sequences* $(a_n)_{n\geq -2}$*,* $(b_n)_{n\geq -2}$ *and* $(c_n)_{n\geq -2}$ *are given by* (3.97)*,* (3.98) *and* (3.99) *respectively, sequences* $(x_n)_{n\geq -2}$ *and* $(y_n)_{n\geq -2}$ *are given by* (3.108) *and* (3.109)*, while sequences* $(\widehat{x}_n)_{n\geq -2}$ *and* $(\widehat{y}_n)_{n\geq -2}$ *are given by* (3.110) *and* (3.111)*.*

(c) *If only one of the zeros of the characteristic polynomial* (3.70) *is equal to one (i.e., if* $P_3(1) = 0$ *and* $P_3'(1) \neq 0$*), say* λ_1*, then the general solution to system* (1.1) *is given by formulas* (3.51) *and* (3.60)*, where sequences* $(a_n)_{n\geq -2}$*,* $(b_n)_{n\geq -2}$ *and* $(c_n)_{n\geq -2}$ *are given by* (3.97)*,* (3.98) *and* (3.99) *respectively, sequences* $(x_n)_{n\geq -2}$ *and* $(y_n)_{n\geq -2}$ *are given by* (3.112) *and* (3.113)*, while sequences* $(\widehat{x}_n)_{n\geq -2}$ *and* $(\widehat{y}_n)_{n\geq -2}$ *are given by* (3.114) *and* (3.115)*.*

Remark 3.11. As we have already mentioned equation (3.70) will have a zero equal to one if $(a-1)(c-1) = bd$. Since $a, b, c, d \in \mathbb{Z}$, this is possible, for example, if $a = b + 1$ and $c = d + 1$, or $a = d + 1$ and $c = b + 1$. If $\lambda = 1$ is a double zero of (3.70), then it must be $P_3'(1) = 3 - 2a - 2c + ac = 0$, which is possible only if $(a-2)(c-2) = 1$, that is, if $a = c = 3$ or $a = c = 1$. For $a = c = 3$, it follows that $bd = 4$, so that the corresponding polynomial is

$$P_3(\lambda) = \lambda^3 - 6\lambda^2 + 9\lambda - 4 = (\lambda - 1)^2(\lambda - 4), \tag{3.116}$$

while for $a = c = 1$ it follows that $bd = 0$, so that the polynomial is equal to $P_3(\lambda) = \lambda(\lambda - 1)^2$. Since in Corollary 3.10 we have the assumption $bd \neq 0$, the only possibility is that $a = c = 3$ and $bd = 4$. A direct computation shows that, in this case $D = 4A^3 + B^2 = 4(-9)^3 + (27 \cdot 4 - 2 \cdot 27)^2 = 0$. Hence, the only polynomial which satisfies conditions of Corollary 3.10 *(b)* is defined in (3.116).

Remark 3.12. If $a = -3$, $c = 0$ and $bd = (a-1)(c-1)$, then $bd = 4 \neq 0$, $D = -4 \cdot 3^6 + (2(-3)^3 + 27 \cdot 4)^2 = 0$ and

$$P_3(\lambda) = \lambda^3 + 3\lambda^2 - 4 = (\lambda - 1)(\lambda + 2)^2.$$

Hence, there are polynomials which satisfy conditions of Corollary 3.10 *(c)*. Because of the symmetry for $a = 0$ and $c = -3$ is obtained the same polynomial.

Remark 3.13. As we have already showed, the case that all the zeros of polynomial (3.70) are equal, i.e., if $\lambda_1 = \lambda_2 = \lambda_3$, is impossible for the system studied in this paper. Recall, also that if all the zeros of the characteristic polynomial are equal, then the general solution of (3.52) has the following form

$$a_n = (\hat{\beta}_1 + \hat{\beta}_2 n + \hat{\beta}_3 n^2)\lambda_1^n, \qquad n \in \mathbb{N}, \tag{3.117}$$

where $\hat{\beta}_1, \hat{\beta}_2$ and $\hat{\beta}_3$ are arbitrary constants.

Remark 3.14. The formulas presented in this paper can be used in the study of the asymptotic behavior of solutions to system (1.1). We leave the problem to the reader.

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