Electronic Journal of Qualitative Theory of Differential Equations
2016, No. **111**, 1–25; doi: 10.14232/ejqtde.2016.1.111 http://www.math.u-szeged.hu/ejqtde/

Sharp Gronwall–Bellman type integral inequalities with delay

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Received 25 March 2016, appeared 29 November 2016
Communicated by Leonid Berezansky

Abstract. Various attempts have been made to give an upper bound for the solutions of the delayed version of the Gronwall–Bellman integral inequality, but the obtained estimations are not sharp. In this paper a new approach is presented to get sharp estimations for the nonnegative solutions of the considered delayed inequalities. The results are based on the idea of the generalized characteristic inequality. Our method gives sharp estimation, and therefore the results are more exact than the earlier ones.

Keywords: delayed Gronwall–Bellman integral inequality, sharp solution, generalized characteristic inequality.

2010 Mathematics Subject Classification: 26D15, 35A23.

1 Introduction

The Gronwall–Bellman integral inequality (see [5] and [8]) plays an important role in the qualitative theory of the solutions of differential and integral equations with and without delay.

It states that if a and x are nonnegative and continuous functions on the interval $[t_0, T[$ $(t_0 < T \leq \infty)$ satisfying

$$x(t) \leq c + \int_{t_0}^{t} a(s)x(s)ds, \qquad t_0 \leq t < T \tag{1.1}$$

for some $c \geq 0$, then

$$x(t) \leq c \exp\left(\int_{t_0}^{t} a(s)ds \right), \qquad t_0 \leq t < T. \tag{1.2}$$

This estimation is precise, since the function

$$t \to c \exp\left(\int_{t_0}^{t} a(s)ds \right), \qquad t_0 \leq t < T$$

satisfies (1.1) with equality.

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A number of generalizations of this inequality have been developed and studied, we refer to the classical books [3,7,16,17] and the literature cited these.

Various attempts have been made to give a sharp upper bound for the solutions of the following delayed version of (1.1)

$$x(t) \leq c + \int_{t_0}^{\alpha(t)} a(s)x(s)ds, \qquad t_0 \leq t < T, \tag{1.3}$$

where a, $x : [t_0, T[\to \mathbb{R}_+$ are continuous, and $\alpha : [t_0, T[\to [t_0, T[$ is a continuously differentiable and increasing function with $\alpha(t) \leq t$ $(t_0 \leq t < T)$ (see [1,2,14,15,18,19]). The obtained estimation is

$$x(t) \leq c \exp\left(\int_{t_0}^{\alpha(t)} a(s)ds\right), \qquad t_0 \leq t < T,$$

which is not sharp in contrast with (1.2).

Essentially, there are two different methods to give upper bounds for the solutions of either (1.1) or (1.3): the first one is to obtain a differential inequality from the considered integral inequality (see [2, 14, 18, 19]), while the second one is based on iterative techniques (see [4, 11, 12]). In case of applying iterative techniques, some standard integral inequalities are used, which can be found in a very general form in [13].

In this paper a new approach is presented to get sharp estimation for the nonnegative solutions of the delayed inequality

$$x(t) \leq c + \int_{t_0}^{t} a(u)x(u - \tau(u))du, \qquad t_0 \leq t < T, \tag{1.4}$$

where $a : [t_0, T[\to \mathbb{R}_+$ is locally integrable, $\tau : [t_0, T[\to \mathbb{R}_+$ is a measurable function such that

$$t_0 - r \leq t - \tau(t), \qquad t_0 \leq t < T$$

with some $r \geq 0$, and $x : [t_0 - r, T[\to \mathbb{R}_+$ is Borel measurable and locally bounded. By making a substitution, it can be shown that inequality (1.3) is a special case of (1.4).

Our treatment of the inequality (1.4) uses the following observation. Under suitable conditions, by introducing the function $y : [t_0 - r, T[\to \mathbb{R}_+$

$$y(t) = \begin{cases} c + \int_{t_0}^{t} a(u)x(u - \tau(u))du, & t_0 \leq t < T, \\ x(t), & t_0 - r \leq t < t_0, \end{cases}$$

the integral inequality (1.4) can be transformed to the delayed differential inequality

$$y'(t) \leq a(t)y(u - \tau(u)), \qquad t_0 \leq t < T. \tag{1.5}$$

Thus the nonnegative solutions of (1.4) can be estimated by the nonnegative solutions of the differential inequality (1.5) or the nonnegative solutions of the nonautonomous linear delay differential equation

$$y'(t) = a(t)y(u - \tau(u)), \qquad t_0 \leq t < T. \tag{1.6}$$

The results in this paper are based on the idea of the generalized characteristic inequality

$$a(t) \exp\left(-\int_{t-\tau(t)}^{t} \gamma(s)ds\right) \leq \gamma(t), \qquad t_0 \leq t < T, \tag{1.7}$$

and the generalized characteristic equation

$$a(t)\exp\left(-\int_{t-\tau(t)}^{t}\gamma(s)ds\right)=\gamma(t),\qquad t_0\le t<T,$$

which is obtained by looking for solutions of (1.6) in the form

$$y(t)=\exp\left(\int_{t_0}^{t}\gamma(s)ds\right),\qquad t_0\le t<T.$$

The generalized characteristic equation has been introduced for nonautonomous linear delay differential equations to obtain some powerful comparison results (see [10]). For a recent application we refer to [6].

We shall use the solutions $\gamma:[t_0-r,T[\,\to\mathbb{R}_+$ of (1.7) to estimate the solutions of (1.4). Our method gives sharp estimation for the solutions of (1.4), and therefore much better upper bounds can be obtained for the solutions of (1.3) than the earlier ones.

2 A sharp Gronwall–Belmann type estimation for delay dependent linear integral inequalities

The set of nonnegative numbers and the set of nonnegative integers will be denoted by $\mathbb{R}_+$ and $\mathbb{N}$ respectively.

Throughout this paper measurable means Lebesgue measurable, while Borel measurability is always indicated.

Definition 2.1. Let $p\in\mathbb{R}$, $p<T\le\infty$, and $f:[p,T[\,\to\mathbb{R}_+$.

(a) We say that f is locally integrable if it is integrable over $[p,t]$ for every $t\in[p,T[$.

(b) We say that f is locally bounded if it is bounded on $[p,t]$ for every $t\in[p,T[$.

We come now to one of the principal results of this paper.

Theorem 2.2. *Let $t_0\in\mathbb{R}$, $t_0<T\le\infty$, $c\ge0$, and $a:[t_0,T[\,\to\mathbb{R}_+$ be locally integrable. Assume $r\ge0$, and $\tau:[t_0,T[\,\to\mathbb{R}_+$ is a measurable function such that*

$$t_0-r\le t-\tau(t),\qquad t_0\le t<T.$$

If $x:[t_0-r,T[\,\to\mathbb{R}_+$ is Borel measurable and locally bounded such that

$$x(t)\le c+\int_{t_0}^{t}a(u)x(u-\tau(u))du,\qquad t_0\le t<T,\tag{2.1}$$

then

$$x(t)\le K\exp\left(\int_{t_0}^{t}\gamma(s)ds\right),\qquad t_0\le t<T,\tag{2.2}$$

where the function $\gamma:[t_0-r,T[\,\to\mathbb{R}_+$ is locally integrable, and satisfies the characteristic inequality

$$a(t)\exp\left(-\int_{t-\tau(t)}^{t}\gamma(s)ds\right)\le\gamma(t),\qquad t_0\le t<T,\tag{2.3}$$

and

$$K:=\max\left(c\exp\left(\int_{t_0-r}^{t_0}\gamma(u)du\right),\ \sup_{t_0-r\le s\le t_0}x(s)\exp\left(\int_{s}^{t_0}\gamma(u)du\right)\right).$$

Remark 2.3. Let $t_0 \in \mathbb{R}$, $t_0 < T \leq \infty$, $c \geq 0$, and $a : [t_0, T[\to \mathbb{R}_+$ be measurable. Assume $r \geq 0$, and $\tau : [t_0, T[\to \mathbb{R}_+$ is a measurable function such that

$$t_0 - r \leq t - \tau(t), \qquad t_0 \leq t < T.$$

If $x : [t_0 - r, T[\to \mathbb{R}_+$ is Borel measurable, the function

$$t \to a(t)x(t - \tau(t)), \qquad t_0 \leq t < T$$

is locally integrable, and (2.1) holds, then x is locally bounded on $[t_0, T[$, since the function defined by the right hand side of (2.1) is continuous. This shows that the assumption of local boundedness on x is natural.

When a and τ are constant functions, we get the following corollary.

Corollary 2.4. *Let $t_0 \in \mathbb{R}$, $t_0 < T \leq \infty$, and $c, a, \tau \geq 0$. If $x : [t_0 - \tau, T[\to \mathbb{R}_+$ is Borel measurable and locally bounded such that*

$$x(t) \leq c + \int_{t_0}^{t} a x(u - \tau) du, \qquad t_0 \leq t < T,$$

then

$$x(t) \leq K e^{\gamma(t - t_0)}, \qquad t_0 \leq t < T,$$

where the nonnegative number γ satisfies the inequality

$$a \leq \gamma e^{\gamma \tau},$$

and

$$K := \max\left(c e^{\gamma \tau}, \sup_{t_0 - r \leq s \leq t_0} x(s) e^{\gamma(t_0 - s)} \right).$$

Since a is locally integrable in Theorem 2.2, it is clear that the function

$$\gamma : [t_0 - r, T[\to \mathbb{R}_+, \qquad \gamma(t) = \begin{cases} a(t), & t \geq t_0 \\ 0, & t_0 - r \leq t < t_0 \end{cases}$$

is locally integrable and satisfies the inequality (2.3). Thus we get the next Gronwall–Bellman type estimation.

Corollary 2.5. *Let $t_0 \in \mathbb{R}$, $t_0 < T \leq \infty$, $c \geq 0$, and $a : [t_0, T[\to \mathbb{R}_+$ be locally integrable. Assume $r \geq 0$, and $\tau : [t_0, T[\to \mathbb{R}_+$ is a measurable function such that*

$$t_0 - r \leq t - \tau(t), \qquad t_0 \leq t < T.$$

If $x : [t_0 - r, T[\to \mathbb{R}_+$ is Borel measurable, locally bounded and satisfies the integral inequality (2.1), then

$$x(t) \leq K \exp\left(\int_{t_0}^{t} a(s) ds \right), \qquad t_0 \leq t < T, \tag{2.4}$$

where

$$K := \max\left(c, \sup_{t_0 - r \leq s \leq t_0} x(s) \right).$$

It is worth to note that in the non-delay case inequality (2.1) reduces to

$$x(t) \le c + \int_{t_0}^{t} a(u)x(u)du, \qquad t_0 \le t < T,$$

and the function $x_0 : [t_0, T[\to \mathbb{R}_+$ defined by

$$x_0(t) = c\exp\left(\int_{t_0}^{t} a(s)ds\right), \qquad t_0 \le t < T$$

satisfies

$$x_0(t) = c + \int_{t_0}^{t} a(u)x_0(u)du, \qquad t_0 \le t < T.$$

This yields that the estimation (2.4) is precise if $\tau(t) = 0$, $t \ge t_0$.

Definition 2.6. We say that the function $\gamma : [t_0 - r, T[\to \mathbb{R}_+$ satisfying (2.3) provides a sharp estimation with respect to the nonnegative solutions of the inequality (2.1) if

$$x(t) \le K\exp\left(\int_{t_0}^{t} \gamma(s)ds\right), \qquad t_0 \le t < T$$

holds for any nonnegative solution $x : [t_0 - r, T[\to \mathbb{R}_+$ of (2.1), and there exists a nonnegative solution $x_0 : [t_0 - r, T[\to \mathbb{R}_+$ of (2.1) such that

$$0 < \liminf_{t\to T-} x_0(t)\exp\left(-\int_{t_0}^{t} \gamma(s)ds\right) \le \limsup_{t\to T-} x_0(t)\exp\left(-\int_{t_0}^{t} \gamma(s)ds\right) < \infty. \tag{2.5}$$

In the delayed case there exists $\gamma : [t_0 - r, T[\to \mathbb{R}_+$ satisfying (2.3) which provides sharp estimation as we shall see from the following result. Further, we give the smallest γ which satisfies (2.3).

Theorem 2.7. *Let $t_0 \in \mathbb{R}$, $t_0 < T \le \infty$, and $a : [t_0, T[\to \mathbb{R}_+$ be locally integrable. Assume $r \ge 0$, and $\tau : [t_0, T[\to \mathbb{R}_+$ is a measurable function such that*

$$t_0 - r \le t - \tau(t), \qquad t_0 \le t < T.$$

(a) There exists a unique function $\hat{\gamma} : [t_0 - r, T[\to \mathbb{R}_+$ such that $\hat{\gamma}$ is locally integrable and satisfies the integral equation

$$a(t)\exp\left(-\int_{t-\tau(t)}^{t} \gamma(s)ds\right) = \gamma(t), \qquad t_0 \le t < T \tag{2.6}$$

with the initial condition

$$\gamma(t) = 0, \qquad t_0 - r \le t < t_0. \tag{2.7}$$

(b) If the function $\gamma : [t_0 - r, T[\to \mathbb{R}_+$ is locally integrable, and satisfies the inequality

$$a(t)\exp\left(-\int_{t-\tau(t)}^{t} \gamma(s)ds\right) \le \gamma(t), \qquad t_0 \le t < T,$$

then

$$\hat{\gamma}(t) \le \gamma(t), \qquad t_0 - r \le t < T.$$

(c) Assume $c \geq 0$. The function $\hat{x} : [t_0 - r, T[\to \mathbb{R}_+$ defined by

$$\hat{x}(t) = c \exp\left(\int_{t_0}^{t} \hat{\gamma}(s) ds \right), \qquad t_0 - r \leq t < T$$

is the unique solution of the integral equation

$$x(t) = c + \int_{t_0}^{t} a(u) x(u - \tau(u)) du, \qquad t_0 \leq t < T, \tag{2.8}$$

with the initial condition

$$x(t) = c, \qquad t_0 - r \leq t \leq t_0. \tag{2.9}$$

Remark 2.8. It comes from Theorem 2.2 and from Theorem 2.7 (c) that $\hat{\gamma}$ provides a sharp estimation with respect to the nonnegative solutions of the inequality (2.1).

Supplementing Theorem 2.2 the next assertion is presented.

Theorem 2.9. *Let $t_0 \in \mathbb{R}$, $t_0 < T \leq \infty$, and $c \geq 0$. Let $a, b : [t_0, T[\to \mathbb{R}_+$ be locally integrable. Assume $r \geq 0$, and $\tau : [t_0, T[\to \mathbb{R}_+$ is a measurable function such that*

$$t_0 - r \leq t - \tau(t), \qquad t_0 \leq t < T.$$

Extend the function b to $[t_0 - r, \infty[$ such that $b(t) = 0$ for $t_0 - r \leq t < t_0$.

If $x : [t_0 - r, T[\to \mathbb{R}_+$ is Borel measurable and locally bounded such that

$$x(t) \leq c + \int_{t_0}^{t} b(u) x(u) du + \int_{t_0}^{t} a(u) x(u - \tau(u)) du, \qquad t_0 \leq t < T, \tag{2.10}$$

then

$$x(t) \leq K \exp\left(\int_{t_0}^{t} (b(s) + \gamma(s)) ds \right), \qquad t_0 \leq t < T,$$

where the function $\gamma : [t_0 - r, T[\to \mathbb{R}_+$ is locally integrable and satisfies the inequality

$$a(t) \exp\left(- \int_{t-\tau(t)}^{t} (b(s) + \gamma(s) ds) \right) \leq \gamma(t), \qquad t_0 \leq t < T, \tag{2.11}$$

and

$$K := \max\left(c \exp\left(\int_{t_0 - r}^{t_0} \gamma(u) du \right), \sup_{t_0 - r \leq s \leq t_0} x(s) \exp\left(\int_{s}^{t_0} \gamma(u) du \right) \right).$$

Considerations similar to those involved in Corollary 2.5 give: under the conditions of Theorem 2.9 the function

$$\gamma : [t_0 - r, T[\to \mathbb{R}_+, \qquad \gamma(t) = \begin{cases} a(t) \exp\left(- \int_{t-\tau(t)}^{t} b(s) ds \right), & t \geq t_0 \\ 0, & t_0 - r \leq t < t_0 \end{cases}$$

is locally integrable and satisfies the inequality (2.11), and thus we get the following Gronwall–Bellman type estimation for x.

Corollary 2.10. *Let $t_0 \in \mathbb{R}$, $t_0 < T \leq \infty$, and $c \geq 0$. Let $a, b : [t_0, T[\to \mathbb{R}_+$ be locally integrable. Assume $r \geq 0$, and $\tau : [t_0, T[\to \mathbb{R}_+$ is a measurable function such that*

$$t_0 - r \leq t - \tau(t), \qquad t_0 \leq t < T.$$

Extend the function b to $[t_0 - r, \infty[$ such that $b(t) = 0$ for $t_0 - r \leq t < t_0$.

If $x : [t_0 - r, T[\to \mathbb{R}_+$ is Borel measurable and locally bounded such that (2.10) holds, then

$$x(t) \leq K \exp\left(\int_{t_0}^{t} b(s)ds + \int_{t_0}^{t} a(s) \exp\left(-\int_{s-\tau(s)}^{s} b(u)du \right) ds \right), \qquad t_0 \leq t < T,$$

where

$$K := \max\left(c, \sup_{t_0 - r \leq s \leq t_0} x(s) \right), \qquad t_0 \leq t < T,$$

The next result gives a γ which provides a sharp estimation with respect to the nonnegative solutions of the inequality (2.10) (we need to slightly re-formulate Definition 2.6).

Theorem 2.11. *Let $t_0 \in \mathbb{R}$, $t_0 < T \leq \infty$, and $c \geq 0$. Let $a, b : [t_0, T[\to \mathbb{R}_+$ be locally integrable. Assume $r \geq 0$, and $\tau : [t_0, T[\to \mathbb{R}_+$ is a measurable function such that*

$$t_0 - r \leq t - \tau(t), \qquad t_0 \leq t < T.$$

Extend the function b to $[t_0 - r, \infty[$ such that $b(t) = 0$ for $t_0 - r \leq t < t_0$.

(a) There exists a unique locally integrable function $\hat{\gamma} : [t_0 - r, \infty[\to \mathbb{R}_+$ which satisfies the integral equation

$$a(t) \exp\left(-\int_{t-\tau(t)}^{t} (b(s) + \gamma(s)ds) \right) = \gamma(t), \qquad t_0 \leq t < T. \tag{2.12}$$

with the initial condition

$$\gamma(t) = 0, \qquad t_0 - r \leq t < t_0. \tag{2.13}$$

(b) If $\gamma : [t_0 - r, \infty[\to \mathbb{R}_+$ is a locally integrable function such that

$$a(t) \exp\left(-\int_{t-\tau(t)}^{t} (b(s) + \gamma(s)ds) \right) \leq \gamma(t), \qquad t_0 \leq t < T,$$

holds, then

$$\gamma(t) \leq \hat{\gamma}(t), \qquad t_0 \leq t < T.$$

(c) The function $\hat{x} : [t_0 - r, \infty[\to \mathbb{R}_+$ defined by

$$\hat{x}(t) = c \exp\left(\int_{t_0}^{t} (b(s) + \gamma(s))ds \right)$$

is the unique solution of the integral equation

$$x(t) = c + \int_{t_0}^{t} b(u)x(u)du + \int_{t_0}^{t} a(u)x(u - \tau(u))du, \qquad t_0 \leq t < T$$

with the initial condition

$$x(t) = c, \qquad t_0 - r \leq t \leq t_0.$$

Proof.

(a) By Theorem 2.7 (a), the initial value problem (2.12) and (2.13) has a unique solution.

(b) It follows from (a) by using Theorem 2.7 (b).

(c) We can follow the proof of Theorem 2.7 (c).

The proof is complete. □

Another result, analogous to Theorem 2.9, emerges.

Theorem 2.12. *Let $t_0 \in \mathbb{R}$, and $t_0 < T \leq \infty$. Let a, $b : [t_0, T[\to \mathbb{R}_+$ be locally integrable, and let $c : [t_0, T[\to \mathbb{R}_+$ be a positive, measurable and increasing function. Assume $r \geq 0$, and $\tau : [t_0, T[\to \mathbb{R}_+$ is a measurable function such that*

$$t_0 - r \leq t - \tau(t), \qquad t_0 \leq t < T.$$

Extend the function b to $[t_0 - r, \infty[$ such that $b(t) = 0$ for $t_0 - r \leq t < t_0$.

If $x : [t_0 - r, T[\to \mathbb{R}_+$ is Borel measurable and locally bounded such that

$$x(t) \leq c(t) + \int_{t_0}^{t} b(u)x(u)du + \int_{t_0}^{t} a(u)x(u - \tau(u))du, \qquad t_0 \leq t < T, \tag{2.14}$$

then

$$x(t) \leq Kc(t)\exp\left(\int_{t_0}^{t} b(s)ds + \int_{t_0}^{t} \gamma(s)ds\right), \qquad t_0 \leq t < T,$$

where the function $\gamma : [t_0 - r, T[\to \mathbb{R}_+$ is locally integrable, and satisfies the inequality (2.11), and

$$K := \max\left(\exp\left(\int_{t_0 - r}^{t_0} \gamma(u)du\right), \max_{t_0 - r \leq s \leq t_0} \frac{x(s)}{c(s)} \exp\left(\int_{s}^{t_0} \gamma(u)du\right)\right).$$

3 Applicability of the main results

First, we compare Theorem 2.2 to a frequently used result from [18]. Another remarkable result in [14] is its special case.

We need some notations.

Definition 3.1. Let $p \in \mathbb{R}$ and $p < T \leq \infty$.

(c) The set of all continuous and nonnegative functions on $[p, T[$ will be denoted by $C([p, T[, \mathbb{R}_+)$.

(d) The set of all continuously differentiable functions from $[p, T[$ into $[p, T[$ will be denoted by $C^1([p, T[, [p, T[)$

Theorem A (see [18])**.** *Let $t_0 \in \mathbb{R}$, $t_0 < T \leq \infty$, $c \geq 0$, and $f, g \in C([t_0, T[, \mathbb{R}_+)$. Assume $\alpha \in C^1([t_0, T[, [t_0, T[)$ is an increasing function with $\alpha(t) \leq t$ $(t_0 \leq t < T)$. If $x \in C([t_0, T[, \mathbb{R}_+)$ satisfies the inequality*

$$x(t) \leq c + \int_{t_0}^{t} f(s)x(s)ds + \int_{t_0}^{\alpha(t)} g(u)x(u)du, \qquad t_0 \leq t < T, \tag{3.1}$$

then

$$x(t) \leq c\exp\left(\int_{t_0}^{t} f(s)ds + \int_{t_0}^{\alpha(t)} g(s)ds\right), \qquad t_0 \leq t < T. \tag{3.2}$$

As we shall see in Remark 3.3, the above result is a consequence of the next theorem which is a far-reaching generalization of it, and which comes from Theorem 2.2.

Theorem 3.2. *Let $t_0 \in \mathbb{R}$, $t_0 < T \le \infty$, and $c \ge 0$. Let $f, g \in C([t_0, T[, \mathbb{R}_+)$. Assume $\alpha \in C^1([t_0, T[, [t_0, T[)$ is an increasing function with $\alpha(t) \le t$ $(t_0 \le t < T)$. If $x \in C([t_0, T[, \mathbb{R}_+)$ satisfies the inequality (3.1), then*

(a)

$$x(t) \le c \exp\left(\int_{t_0}^{t} (f(s) + \gamma(s)) ds \right), \qquad t_0 \le t < T, \tag{3.3}$$

where the function $\gamma : [t_0, T[\to \mathbb{R}_+$ is locally integrable, and satisfies the inequality

$$\alpha'(t) g(\alpha(t)) \exp\left(-\int_{\alpha(t)}^{t} (f(s) + \gamma(s) ds) \right) \le \gamma(t), \qquad t_0 \le t < T. \tag{3.4}$$

(b) For every $t_0 \le t < T$

$$x(t) \le c \exp\left(\int_{t_0}^{t} f(s) ds + \int_{t_0}^{t} \alpha'(s) g(\alpha(s)) \exp\left(-\int_{\alpha(s)}^{s} f(u) du \right) ds \right). \tag{3.5}$$

Proof.

(a) By using a substitution, we get

$$\int_{t_0}^{\alpha(t)} g(u) x(u) du = \int_{t_0}^{t} \alpha'(s) g(\alpha(s)) x(\alpha(s)) ds, \qquad t_0 \le t < T,$$

and hence (3.1) holds if and only if

$$x(t) \le c + \int_{t_0}^{t} f(s) x(s) ds + \int_{t_0}^{t} a(u) x(\alpha(u)) du, \qquad t_0 \le t < T \tag{3.6}$$

with

$$a(t) = \alpha'(t) g(\alpha(t)), \qquad t_0 \le t < T. \tag{3.7}$$

Now (3.3) follows from Theorem 2.9 with the function γ given in (3.4).

(b) It follows immediately from Corollary 2.10 by using (3.6) and (3.7).

The proof is complete. □

Remark 3.3. (a) Under the conditions of Theorem 3.2 the function $\gamma \in C([t_0, T[, \mathbb{R}_+)$ defined by $\gamma(t) = \alpha'(t) g(\alpha(t))$ $(t_0 \le t < T)$ satisfies (3.4), and therefore (3.2) can be obtained from (3.3) by applying this γ. We can see that Theorem A follows from Theorem 3.2.

Moreover, the explicit upper bound in (3.5) is also better than the upper bound in (3.2), since

$$\int_{t_0}^{t} \alpha'(s) g(\alpha(s)) \exp\left(-\int_{\alpha(s)}^{s} f(u) du \right) ds \le \int_{t_0}^{\alpha(t)} g(s) ds, \qquad t_0 \le t < T.$$

(b) It is worth to note that the proof of Theorem 3.2 (a) shows that inequality (3.1) can be transformed to an equivalent inequality having the form (2.10), but the converse is not true in general.

Remark 3.4. The extension

$$x(t) \le c(t) + \int_{t_0}^{t} f(s)x(s)ds + \int_{t_0}^{\alpha(t)} g(u)x(u)du, \qquad t_0 \le t < T$$

of (3.1) have been studied in [19] under the conditions of Theorem A, and where $c \in C([t_0, T[, \mathbb{R}_+)$ is positive and increasing. Like Theorem 3.2, it can be obtained an essential generalization of the main result of [19] from Theorem 2.12.

We illustrate by two examples that (3.3) can give much better explicit upper bound for the solutions of (3.1) than (3.2).

Example 3.5. (a) Consider the inequality

$$x(t) \le c + \int_{1}^{\sqrt{t}} \frac{1}{\sqrt{u}} x(u)du, \qquad t \ge 1, \tag{3.8}$$

where $c \ge 0$ and $x \in C([1, \infty[, \mathbb{R}_+)$.

Then by Theorem A,

$$x(t) \le c \exp\left(\int_{1}^{\sqrt{t}} \frac{1}{\sqrt{u}} du\right) = c \exp\left(2\left(\sqrt[4]{t} - 1\right)\right), \qquad t \ge 1. \tag{3.9}$$

Theorem 3.2 (a) gives by choosing

$$\gamma : [1, \infty[\to \mathbb{R}_+, \qquad \gamma(t) = \frac{1}{2t} \tag{3.10}$$

that

$$x(t) \le c\sqrt{t}, \qquad t \ge 1, \tag{3.11}$$

which is not exponential estimation in contrast with (3.9).

It can be checked easily that if (3.10) holds, then the inequality (3.4) is satisfied with equality, and hence Theorem 2.7 (b) and (c) show that (3.11) is the best upper bound for the solutions of (3.8).

(b) Consider the inequality

$$x(t) \le c + \int_{0}^{t/2} x(u)du, \qquad t \ge 0, \tag{3.12}$$

where $c \ge 0$ and $x \in C(\mathbb{R}_+, \mathbb{R}_+)$.

From Theorem A, we have that

$$x(t) \le c \exp\left(\int_{0}^{t/2} 1 ds\right) = c \exp\left(\frac{t}{2}\right), \qquad t \ge 0. \tag{3.13}$$

Some easy calculations give that the function

$$\gamma :]0, \infty[\to \mathbb{R}_+, \qquad \gamma(t) = \frac{1}{\sqrt{t}}$$

satisfies the inequality (3.4) which now has the form

$$\frac{1}{2} \exp\left(-\int_{t/2}^{t} \gamma(s)ds\right) \le \gamma(t), \qquad t \ge 0.$$

Therefore Theorem 3.2 (a) implies

$$x(t) \le c \exp\left(2\sqrt{t}\right), \qquad t \ge 0, \tag{3.14}$$

which is much better than (3.13).

Remark 3.6. We mention that Gronwall–Bellman type integral inequalities have been extended and studied in measure spaces in [12]. Estimation for the solutions of (3.12) can be obtained from the results of [12] too:

$$x(t) \le c\left(\frac{1}{2} + \frac{t}{4} + \frac{1}{2}\exp\left(\frac{t}{2}\right)\right), \qquad t \ge 0. \tag{3.15}$$

In spite of the very general settings in [12], the upper bound (3.15) is also sharper than the upper bound (3.13) coming from Theorem A for every $t \ge 0$.

Next, we demonstrate the scope of the different estimations by applying them to the delay differential equation

$$y'(t) = -\frac{2}{t}y(t) + \frac{q}{t}y(qt), \qquad t \ge 1, \tag{3.16}$$

with the parameter $q \in \,]0,1]$. We note that (3.16) is not a delay equation if $q > 1$.

We say that $y \in C([q,\infty[,\mathbb{R})$ is a solution of (3.16) if y is differentiable on $[1,\infty[$ and satisfies (3.16) for every $t \ge 1$.

By using our basic result Theorem 2.2, we have the following.

Proposition 3.7. *For every solution y of* (3.16)

$$\lim_{t\to\infty} y(t) = 0,$$

for any $q \in \,]0,1]$.

Proof. By applying the variation of constants formula to (3.16) we obtain for all $t \ge 1$ that

$$\begin{aligned} y(t) &= y(1)\exp\left(\int_1^t -\frac{2}{s}ds\right) + \int_1^t \left(\frac{q}{s}\exp\left(\int_s^t -\frac{2}{u}du\right)y(qs)\right)ds \\ &= y(1)\frac{1}{t^2} + \frac{q}{t^2}\int_1^t sy(qs)ds, \qquad t \ge 1. \end{aligned} \tag{3.17}$$

By introducing

$$x(t) := t^2 y(t), \qquad t \ge q$$

we have from (3.17)

$$x(t) = x(1) + \int_1^t \frac{1}{qs}x(qs)ds, \qquad t \ge 1.$$

The solution $x \in C([q,\infty],\mathbb{R})$ of this problem satisfies the integral inequality

$$|x(t)| \le |x(1)| + \int_1^t \frac{1}{qs}|x(qs)|ds, \qquad t \ge 1. \tag{3.18}$$

It is an easy task to calculate that the function

$$\gamma_1 : [q,\infty[\, \to \mathbb{R}_+, \qquad \gamma_1(t) = \frac{1}{t}$$

satisfies the equation

$$\frac{1}{qt}\exp\left(-\int_{qt}^t \gamma_1(s)ds\right) = \gamma_1(t), \qquad t \ge 1,$$

for any $q \in]0,1]$. Thus it follows from Theorem 2.2 that

$$\begin{aligned} |y(t)| &= \frac{1}{t^2}|x(t)| \le \frac{1}{q}\max_{q\le s\le 1}|y(s)|\frac{1}{t^2}\exp\left(\int_1^t \frac{1}{s}ds\right) \\ &\le \frac{1}{q}\max_{q\le s\le 1}|y(s)|\frac{1}{t}, \qquad t \ge 1, \end{aligned} \tag{3.19}$$

which implies the result.

The proof is complete. □

The previous result can be proved only partially by using other estimates.

Remark 3.8. (a) By applying the classical Gronwall–Bellman type estimation to (3.18) (see Corollary 2.5), we have that y obeys the inequality:

$$|y(t)| = \frac{1}{t^2}|x(t)| \le |y(1)|\frac{1}{t^2}\exp\left(\int_1^t \frac{1}{qs}ds\right) = |y(1)|t^{\frac{1}{q}-2}, \qquad t \ge 1. \tag{3.20}$$

It follows from this that every solution of (3.16) tends to zero at infinity if $\frac{1}{2} < q \le 1$. At the same time estimation (3.20) is useless if $0 < q \le \frac{1}{2}$.

(b) Theorem A cannot be applied for (3.18) directly: the inequality (3.18) is equivalent to

$$|x(t)| \le |x(1)| + \int_q^{qt} \frac{1}{qu}|x(u)|du, \qquad t \ge 1,$$

where the range of the function $\alpha : [1,\infty[\to \mathbb{R}$, $\alpha(t) = qt$ is not a subset of $[1,\infty[$ for any $0 < q < 1$.

(c) It is not hard to check that the function

$$\gamma_2 : [q,\infty[\to \mathbb{R}_+, \qquad \gamma_2(t) = \frac{1}{\sqrt{qt}}$$

satisfies the inequality

$$\frac{1}{qt}\exp\left(-\int_{qt}^t \gamma_2(s)ds\right) \le \gamma_2(t), \qquad t \ge 1,$$

and therefore Theorem 2.2 implies

$$\begin{aligned} |y(t)| &= \frac{1}{t^2}|x(t)| \le q^{-\frac{1}{\sqrt{q}}}\max_{q\le s\le 1}|y(s)|\frac{1}{t^2}\exp\left(\int_1^t \frac{1}{\sqrt{qs}}ds\right) \\ &\le q^{-\frac{1}{\sqrt{q}}}\max_{q\le s\le 1}|y(s)|t^{\frac{1}{\sqrt{q}}-2}, \qquad t \ge 1, \end{aligned}$$

which shows that every solution of (3.16) tends to zero at infinity only if $\frac{1}{4} < q \le 1$.

(d) Since $y : [q,\infty[\to \mathbb{R}$, $y(t) = \frac{1}{t}$ is a solution of (3.16), (3.19) is the best estimation in the sense that it gives the best convergence rate for the solutions.

Finally, we consider the integral inequality

$$x(t) \le c + \int_0^t a(s)x(s)ds, \qquad t \ge 0 \tag{3.21}$$

together with its approximating inequality

$$u(t) \le c + \int_0^t a\Big(\Big[\frac{s}{h}\Big]h\Big)u\Big(\Big[\frac{s}{h}\Big]h\Big)ds, \qquad t \ge 0, \tag{3.22}$$

where $h \in \,]0,1]$ is the step size, and $[\cdot]$ denotes the greatest integer function.

Clearly, (3.22) is an integral inequality with the delay function

$$\tau(t) = t - \Big[\frac{t}{h}\Big]h, \qquad t \ge 0,$$

therefore Theorem 2.2 is applicable to get the next statement.

Theorem 3.9. *Assume $c \ge 0$, $a \in C([0,\infty[, \mathbb{R}_+)$, and $h \in \,]0,1]$. Then the following statements hold.*

(a) Any measurable and locally bounded solution $x : [0,\infty[\, \to \mathbb{R}_+$ of (3.21) obeys

$$x(t) \le c\exp\Big(\int_0^t a(s)ds\Big), \qquad t \ge 0.$$

(b) Any Borel measurable and locally bounded solution $u : [0,\infty[\, \to \mathbb{R}_+$ of (3.22) obeys

$$u(t) \le c\exp\Big(\int_0^t \gamma_h(s)ds\Big), \qquad t \ge 0, \tag{3.23}$$

where

$$\gamma_h(t) = \frac{a\big([\frac{t}{h}]h\big)}{1 + a\big([\frac{t}{h}]h\big)\big(t - [\frac{t}{h}]h\big)}, \qquad t \ge 0.$$

(c) We have

$$\gamma_h(t) \le a(t), \qquad t \ge 0,$$

and

$$\lim_{h \to 0} \gamma_h(t) = a(t), \qquad t \ge 0.$$

Proof. (a) and (c) are obvious.

(b) This follows from Theorem 2.2 and the identity

$$a\Big(\Big[\frac{t}{h}\Big]h\Big) = \gamma_h(t)\exp\Big(\int_{[\frac{t}{h}]h}^t \gamma_h(s)ds\Big), \qquad t \ge 0. \tag{3.24}$$

(3.24) can be written in the equivalent form

$$a\Big(\Big[\frac{t}{h}\Big]h\Big) = a(nh) = \frac{a(nh)}{1 + a(nh)(t - nh)}\exp\Big(\int_{nh}^t \frac{a(nh)}{1 + a(nh)(s - nh)}ds\Big),$$

where $nh \le t < (n+1)h$ for some nonnegative integer n (of course, n depends on h).

The proof is complete. □

It is worth to note that (3.23) gives for every nonnegative integer n

$$u((n+1)h) \le c\prod_{i=0}^n \exp\Big(\int_{ih}^{(i+1)h} \gamma_h(s)ds\Big) = c\prod_{i=0}^n (1 + ha(ih)). \tag{3.25}$$

On the other hand we have from (3.22)

$$u((n+1)h) \leq c + \int_0^{(n+1)h} a\Big(\Big[\frac{s}{h}\Big]h\Big)u\Big(\Big[\frac{s}{h}\Big]h\Big)ds, \qquad n \geq 0,$$

or equivalently

$$u((n+1)h) \leq c + \sum_{i=0}^{n} ha(ih)u(ih), \qquad n \geq 0.$$

Suppose that the latest inequality is an equality, that is

$$u((n+1)h) = c + \sum_{i=0}^{n} ha(ih)u(ih), \qquad n \geq 0$$

with

$$u(0) = c.$$

The previous initial value problem can be solved easily:

$$u((n+1)h) - u(nh) = ha(nh)u(nh), \qquad n \geq 0,$$

that is

$$u((n+1)h) = (1 + ha(nh))u(nh), \qquad n \geq 0,$$

and hence

$$u((n+1)h) = c\prod_{i=0}^{n}(1 + ha(ih)), \qquad n \geq 0.$$

This verifies again that estimation (3.25) is sharp.

4 On sharpness of the classical estimation in the delayed case

Consider the inequality

$$x(t) \leq c + \int_{t_0}^{t} a(u)x(u - \tau(u))du, \qquad t_0 \leq t < T \tag{4.1}$$

under the conditions in Theorem 2.2. It follows from Corollary 2.5 that the estimation

$$x(t) \leq K\exp\Big(\int_{t_0}^{t} a(s)ds\Big), \qquad t_0 \leq t < T \tag{4.2}$$

holds for every nonnegative solution x of (4.1) ($x : [t_0 - r, T[\to \mathbb{R}_+$ is Borel measurable and locally bounded), but Theorem 2.7 ensures that (2.2) is more exact than (4.2). This means (see Remark 2.8) that there exist a function $\gamma : [t_0 - r, T[\to \mathbb{R}_+$ satisfying (2.3) which provides a sharp estimation with respect to the nonnegative solutions of the inequality (4.1) in the sense of Definition 2.6, but a does not provide a sharp estimation with respect to the nonnegative solutions of the inequality (4.1) in general. Thus a natural question is: for which classes of inequalities (4.1) is the estimation (4.2) sharp?

The next proposition shows that the estimation (4.2) is sharp under some integral condition.

Proposition 4.1. *Let $t_0 \in \mathbb{R}$, $t_0 < T \leq \infty$, and $a : [t_0, T[\to \mathbb{R}_+$ be locally integrable. Assume $r \geq 0$, and $\tau : [t_0, T[\to \mathbb{R}_+$ is a measurable function such that*

$$t_0 - r \leq t - \tau(t), \qquad t_0 \leq t < T.$$

Extend the function a to $[t_0 - r, T[$ such that $a(t) = 0$ if $t_0 - r \leq t < t_0$.

(a) The function a provides a sharp estimation with respect to the nonnegative solutions of the inequality (4.1) if and only if

$$\int_{t_0}^{T} (a(s) - \hat{\gamma}(s))ds < \infty. \tag{4.3}$$

(b) If

$$\int_{t_0}^{T} a(s)\left(1 - \exp\left(-\int_{s-\tau(s)}^{s} a(u)du\right)\right)ds < \infty, \tag{4.4}$$

then (4.3) is satisfied.

(c) If

$$\int_{t_0}^{T} \left(a(s)\int_{s-\tau(s)}^{s} a(u)du\right)ds < \infty,$$

then (4.4) is satisfied.

Proof. (a) Assume (4.3) holds. By Theorem 2.7 (c),

$$x_0(t) = \exp\left(\int_{t_0}^{t} \hat{\gamma}(s)ds\right), \qquad t_0 - r \leq t < T$$

is a solution of (2.1), and hence (2.5) is true with this solution.

Conversely, assume the existence of a solution $x_0 : [t_0 - r, T[\to \mathbb{R}_+$ of (4.1) such that (2.5) holds. By Theorem 2.2,

$$\begin{aligned} 0 &< \liminf_{t \to T-} x_0(t) \exp\left(-\int_{t_0}^{t} a(s)ds\right) \\ &\leq K \liminf_{t \to T-} \exp\left(-\int_{t_0}^{t} (a(s) - \hat{\gamma}(s))ds\right), \qquad t_0 \leq t < T \end{aligned}$$

which implies (4.3).

(b) Since $0 \leq \hat{\gamma}(t) \leq a(t)$ $(t_0 - r \leq t < T)$, it follows from Theorem 2.7 (a) that

$$\begin{aligned} \int_{t_0}^{T} (a(s) - \hat{\gamma}(s))ds &\leq \int_{t_0}^{T} a(s)\left(1 - \exp\left(-\int_{s-\tau(s)}^{s} \hat{\gamma}(u)du\right)\right)ds \\ &\leq \int_{t_0}^{T} a(s)\left(1 - \exp\left(-\int_{s-\tau(s)}^{s} a(u)du\right)\right)ds. \end{aligned}$$

(c) According to $1 - e^{-x} \leq x$ $(x \geq 0)$,

$$\int_{t_0}^{T} a(s)\left(1 - \exp\left(-\int_{s-\tau(s)}^{s} a(u)du\right)\right)ds \leq \int_{t_0}^{T} \left(a(s)\int_{s-\tau(s)}^{s} a(u)du\right)ds.$$

The proof is complete. □

5 Monotone dependence of the estimation with respect to the delay

As an illustration, consider the delay differential equation

$$x'(t) = ax(t-\tau), \qquad t \geq 0, \tag{5.1}$$

where $a > 0$ is fixed and $\tau \geq 0$ is a parameter. The characteristic equation of (5.1) is

$$\gamma = ae^{-\gamma\tau}. \tag{5.2}$$

The unique solution of (5.2) is denoted by $\gamma(\tau)$. It is easy to check that $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is decreasing that is $0 \leq \tau_1 < \tau_2$ implies $0 < \gamma(\tau_2) < \gamma(\tau_1)$. It follows from this that for $0 \leq \tau_1 < \tau_2$ the solutions

$$x_i(t) = e^{\gamma(\tau_i)t}, \qquad t \geq 0, \quad i = 1,2$$

of the initial value problems

$$\left.\begin{array}{l} x'(t) = ax(t-\tau_i), \quad t \geq 0 \\ x(t) = 1, \quad -\tau_i \leq t \leq 0 \end{array}\right\}, \qquad i = 1,2,$$

satisfy

$$x_2(t) \leq x_1(t), \qquad t \geq 0.$$

These simple observations are generalized in this subsection.

Let $t_0 \in \mathbb{R}$, $t_0 < T \leq \infty$, $c \geq 0$, and $a : [t_0, T[\to \mathbb{R}$ be continuous. Assume $r \geq 0$, $\tau : [t_0, T[\to \mathbb{R}_+$ is a continuous function such that

$$t_0 - r \leq t - \tau(t), \qquad t_0 \leq t < T$$

and $\varphi : [t_0 - r, t_0] \to \mathbb{R}$ is continuous. It is well known that the initial value problem

$$\left.\begin{array}{l} x'(t) = a(t)x(t-\tau(t)), \quad t_0 \leq t < T \\ x(t) = \varphi(t), \quad -r \leq t \leq t_0 \end{array}\right\} \tag{5.3}$$

is equivalent to the integral equation

$$x(t) = x(t_0) + \int_{t_0}^{t} a(u)x(u-\tau(u))du, \qquad t_0 \leq t < T \tag{5.4}$$

with the same initial condition.

Consequently, we consider integral equations first.

Proposition 5.1. *Let $t_0 \in \mathbb{R}$, $t_0 < T \leq \infty$, $c \geq 0$, and $a : [t_0, T[\to \mathbb{R}_+$ be locally integrable. Assume $r \geq 0$, and τ_1, $\tau_2 : [t_0, T[\to \mathbb{R}_+$ are measurable functions such that*

$$t_0 - r \leq t - \tau_i(t), \qquad t_0 \leq t < T, \quad i = 1,2,$$

and

$$\tau_1(t) \leq \tau_2(t), \qquad t_0 \leq t < T. \tag{5.5}$$

(a) *If $\hat{\gamma}_i : [t_0 - r, T[\to \mathbb{R}_+$ is the unique solution of the integral equation (see Theorem 2.7 (a))*

$$a(t) \exp\left(- \int_{t-\tau_i(t)}^{t} \gamma(s) ds \right) = \gamma(t), \qquad t_0 \le t < T, \quad i = 1, 2$$

with the initial condition

$$\gamma(t) = 0, \qquad t_0 - r \le t < t_0,$$

then

$$\hat{\gamma}_2(t) \le \hat{\gamma}_1(t), \qquad t_0 \le t < T.$$

(b) *If the function $\hat{x}_i : [t_0 - r, T[\to \mathbb{R}_+$ defined by*

$$\hat{x}_i(t) = c \exp\left(\int_{t_0}^{t} \hat{\gamma}_i(s) ds \right), \qquad t_0 - r \le t < T, \quad i = 1, 2$$

is the unique solution (see Theorem 2.7 (c)) of the integral equation

$$x(t) = c + \int_{t_0}^{t} a(u) x(u - \tau_i(u)) du, \qquad t_0 \le t < T,$$

with the initial condition

$$x(t) = c, \qquad t_0 - r \le t \le t_0,$$

then

$$x_2(t) \le x_1(t), \qquad t \ge 0.$$

Proof.

(a) Since $\hat{\gamma}_1$ is nonnegative, (5.5) yields that

$$\int_{t-\tau_2(t)}^{t} \hat{\gamma}_1(s) ds \ge \int_{t-\tau_1(t)}^{t} \hat{\gamma}_1(s) ds, \qquad t_0 \le t < T,$$

and therefore

$$a(t) \exp\left(- \int_{t-\tau_2(t)}^{t} \hat{\gamma}_1(s) ds \right) \le \hat{\gamma}_1(t), \qquad t_0 \le t < T.$$

Now the result follows from Theorem 2.7 (b).

(b) It is an immediate consequence of (a).

The proof is complete. □

We arrive now at an application of the foregoing results to differential equations.

Proposition 5.2. *Let $t_0 \in \mathbb{R}$, $t_0 < T \le \infty$, $c \ge 0$, and $a : [t_0, T[\to \mathbb{R}$ be continuous. Assume $r \ge 0$, $\tau_1, \tau_2 : [t_0, T[\to \mathbb{R}_+$ are continuous functions such that*

$$t_0 - r \le t - \tau_i(t), \qquad t_0 \le t < T, \quad i = 1, 2, \tag{5.6}$$

and

$$\tau_1(t) \le \tau_2(t), \qquad t_0 \le t < T,$$

and assume $\varphi : [t_0 - r, t_0] \to \mathbb{R}$ is continuous. If $x : [t_0 - r, T[\to \mathbb{R}$ is a solution of the initial value problem

$$\left.\begin{aligned} x'(t) &= a(t)x(t-\tau_2(t)), \quad t_0 \le t < T \\ x(t) &= \varphi(t), \quad -r \le t \le t_0 \end{aligned}\right\},$$

and the function $\gamma_1 : [t_0 - r, T[\to \mathbb{R}_+$ is locally integrable and satisfies the inequality

$$|a(t)| \exp\left(-\int_{t-\tau_1(t)}^{t} \gamma_1(s)ds\right) \le \gamma_1(t), \qquad t_0 \le t < T, \tag{5.7}$$

then

$$|x(t)| \le \max_{t_0 - r \le s \le t_0} |\varphi(s)| \exp\left(\int_{t_0}^{t} \gamma_1(s)ds\right), \qquad t_0 \le t < T.$$

Proof. By Proposition 5.1 (a), if $\hat{\gamma}_i : [t_0 - r, T[\to \mathbb{R}_+$ is the unique solution of the integral equation (see Theorem 2.7 (a))

$$|a(t)| \exp\left(-\int_{t-\tau_i(t)}^{t} \gamma(s)ds\right) = \gamma(t), \qquad t_0 \le t < T, \quad i = 1, 2$$

with the initial condition

$$\gamma(t) = 0, \qquad t_0 - r \le t < t_0,$$

then

$$\hat{\gamma}_2(t) \le \hat{\gamma}_1(t), \qquad t_0 \le t < T. \tag{5.8}$$

Since γ_1 satisfies the inequality (5.7), Theorem 2.7 (b) and (5.8) yield

$$\hat{\gamma}_2(t) \le \hat{\gamma}_1(t) \le \gamma_1(t), \qquad t_0 \le t < T. \tag{5.9}$$

According to the equivalence of (5.3) and (5.4) we have

$$|x(t)| \le |\varphi(t_0)| + \int_{t_0}^{t} |a(u)||x(u - \tau_2(u))|du, \qquad t_0 \le t < T,$$

where

$$|x(t)| = |\varphi(t)|, \qquad -r \le t \le t_0,$$

and therefore by Theorem 2.2 and (5.9),

$$|x(t)| \le K \exp\left(\int_{t_0}^{t} \hat{\gamma}_2(t)ds\right) \le K \exp\left(\int_{t_0}^{t} \gamma_1(s)ds\right), \qquad t_0 \le t < T$$

with

$$K = \max_{t_0 - r \le s \le t_0} |\varphi(s)|.$$ □

Remark 5.3. Let $t_0 \in \mathbb{R}$, $t_0 < T \le \infty$, $c \ge 0$, and $a : [t_0, T[\to \mathbb{R}_+$ be positive and continuous. Assume $r \ge 0$, $\tau : [t_0, T[\to \mathbb{R}_+$ is a continuous function such that

$$t_0 - r \le t - \tau(t), \qquad t_0 \le t < T.$$

Consider the unstable type delay differential equation

$$x'(t) = a(t)x(t - \tau(t)), \qquad t_0 \le t < T. \tag{5.10}$$

The interesting meaning of the above theorem is that the positive solutions of (5.10) growth faster at infinity if τ is replaced by a smaller delay.

6 Proofs of the main results

Proof of Theorem 2.2. (i) First we prove the result when the restriction of x to $[t_0, T[$ is continuous.

Let $y : [t_0 - r, T[\to \mathbb{R}_+$ be defined by

$$y(t) = x(t)\exp\Big(-\int_{t_0-r}^{t}\gamma(s)ds\Big).$$

Then y is Borel measurable and locally bounded, continuous on $[t_0, T[$, and (2.1) implies that

$$y(t) \le c\exp\Big(-\int_{t_0-r}^{t}\gamma(s)ds\Big) + \exp\Big(-\int_{t_0-r}^{t}\gamma(s)ds\Big) \times \int_{t_0}^{t} a(u)\exp\Big(\int_{t_0-r}^{u-\tau(u)}\gamma(s)ds\Big)y(u-\tau(u))du, \qquad t_0 \le t < T.$$

Hence for every $t_0 \le t < T$

$$y(t) \le c\exp\Big(-\int_{t_0-r}^{t}\gamma(s)ds\Big) + \exp\Big(-\int_{t_0-r}^{t}\gamma(s)ds\Big) \times \int_{t_0}^{t} a(u)\exp\Big(-\int_{u-\tau(u)}^{u}\gamma(s)ds\Big)\exp\Big(\int_{t_0-r}^{u}\gamma(s)ds\Big)y(u-\tau(u))du.$$

By applying (2.3), we have

$$y(t) \le c\exp\Big(-\int_{t_0-r}^{t}\gamma(s)ds\Big) + \exp\Big(-\int_{t_0-r}^{t}\gamma(s)ds\Big) \times \int_{t_0}^{t} \gamma(u)\exp\Big(\int_{t_0-r}^{u}\gamma(s)ds\Big)y(u-\tau(u))du, \quad t_0 \le t < T. \tag{6.1}$$

Let

$$L := \max\Big(c, \sup_{t_0-r\le s\le t_0} x(s)\exp\Big(-\int_{t_0-r}^{s}\gamma(u)du\Big)\Big),$$

and let $L_1 > L$. The definition of L yields

$$y(t) \le L < L_1, \qquad t_0 - r \le t \le t_0,$$

and therefore the continuity of y on $[t_0, T[$ implies that there exists $q > 0$ for which $t_0 + q < T$ and

$$y(t) < L_1, \qquad t_0 \le t \le t_0 + q.$$

Assume there is a t_1 from $]t_0 + q, T[$ such that $y(t_1) = L_1$. Since y is continuous on $[t_0, T[$, it can be supposed that $y(t) < L_1$ for every $t \in [t_0, t_1[$.

It follows from (6.1) and $L_1 > L \ge c$ that

$$\begin{aligned} y(t_1) &\le c\exp\Big(-\int_{t_0-r}^{t_1}\gamma(s)ds\Big) \\ &\quad + \exp\Big(-\int_{t_0-r}^{t_1}\gamma(s)ds\Big)\int_{t_0}^{t_1}\gamma(u)\exp\Big(\int_{t_0-r}^{u}\gamma(s)ds\Big)L_1du \\ &= c\exp\Big(-\int_{t_0-r}^{t_1}\gamma(s)ds\Big) + L_1\Big(1-\exp\Big(-\int_{t_0}^{t_1}\gamma(s)ds\Big)\Big) \\ &= L_1 + \exp\Big(-\int_{t_0-r}^{t_1}\gamma(s)ds\Big)\Big(c - L_1\exp\Big(\int_{t_0-r}^{t_0}\gamma(s)ds\Big)\Big) < L_1, \end{aligned}$$

which contradicts $y(t_1) = L_1$. Hence

$$y(t) < L_1, \qquad t_0 \le t < T. \tag{6.2}$$

Since $L_1 > L$ is arbitrary, (6.2) shows that

$$y(t) \le L, \qquad t_0 \le t < T.$$

From what we have proved already follows that

$$\begin{aligned} x(t) &= y(t)\exp\left(\int_{t_0-r}^{t}\gamma(s)ds\right) \\ &\le L\exp\left(\int_{t_0-r}^{t}\gamma(s)ds\right) = K\exp\left(\int_{t_0}^{t}\gamma(s)ds\right), \qquad t_0 \le t < T. \end{aligned}$$

(ii) Now assume x is Borel measurable and locally bounded.
Introduce the function $z : [t_0 - r, T[\to \mathbb{R}_+$,

$$z(t) = \begin{cases} x(t), & t_0 - r \le t < t_0 \\ c + \int_{t_0}^{t} a(u)x(u-\tau(u))du, & t_0 \le t < T \end{cases}.$$

Then (2.1) and the definition of z imply that $x(t) \le z(t)$ $(t_0 - r \le t < T)$, and therefore

$$\begin{aligned} z(t) &= c + \int_{t_0}^{t} a(u)x(u-\tau(u))du \\ &\le c + \int_{t_0}^{t} a(u)z(u-\tau(u))du, \qquad t_0 \le t < T. \end{aligned}$$

Since z is continuous on $[t_0, T[$, it follows from the first part of the proof that

$$x(t) \le z(t) \le K\exp\left(\int_{t_0-r}^{t}\gamma(s)ds\right), \qquad t_0 \le t < T,$$

where

$$\begin{aligned} K &:= \max\left(c, \sup_{t_0-r\le s\le t_0} z(s)\exp\left(-\int_{t_0-r}^{s}\gamma(u)du\right)\right) \\ &= \max\left(c, \sup_{t_0-r\le s\le t_0} x(s)\exp\left(-\int_{t_0-r}^{s}\gamma(u)du\right)\right). \end{aligned}$$

The proof is complete. □

Proof of Theorem 2.7. (a) Extend the function a to $[t_0 - r, T[$ such that $a(t) = 0$ if $t_0 - r \le t < t_0$.
The functions $\gamma_n : [t_0 - r, T[\to \mathbb{R}_+$ $(n \in \mathbb{N})$ are defined inductively by the formulae

$$\gamma_0(t) = a(t), \qquad t_0 - r \le t < T \tag{6.3}$$

and

$$\gamma_{n+1}(t) = a(t)\exp\left(-\int_{t-\tau(t)}^{t}\gamma_n(s)ds\right), \qquad t_0 - r \le t < T. \tag{6.4}$$

It can be proved by induction easily that $0 \le \gamma_n(t) \le a(t)$ $(t_0 - r \le t < T)$ and γ_n is locally integrable for all $n \in \mathbb{N}$.

We show by another induction argument on k that for every $k \in \mathbb{N}$

$$\begin{aligned} 0 &\le \gamma_1(t) \le \gamma_3(t) \le \cdots \le \gamma_{2k+1}(t) \le \cdots \\ &\le \gamma_{2k}(t) \le \cdots \le \gamma_2(t) \le \gamma_0(t) = a(t), \qquad t_0 - r \le t < T. \end{aligned} \tag{6.5}$$

By using (6.3) and (6.4), it is easy to check that

$$0 \le \gamma_1(t) \le \gamma_3(t) \le \gamma_2(t) \le \gamma_0(t) = a(t), \qquad t_0 - r \le t < T.$$

Now, assume

$$0 \le \gamma_{2k-1}(t) \le \gamma_{2k+1}(t) \le \gamma_{2k+2}(t) \le \gamma_{2k}(t) \le a(t), \qquad t_0 - r \le t < T. \tag{6.6}$$

Then

$$\begin{aligned} \gamma_{2k+3}(t) &= a(t) \exp\left(-\int_{t-\tau(t)}^{t} \gamma_{2k+2}(s) ds \right) \\ &\ge a(t) \exp\left(-\int_{t-\tau(t)}^{t} \gamma_{2k}(s) ds \right) = \gamma_{2k+1}(t), \qquad t_0 - r \le t < T, \end{aligned}$$

and thus

$$\begin{aligned} \gamma_{2k+4}(t) &= a(t) \exp\left(-\int_{t-\tau(t)}^{t} \gamma_{2k+3}(s) ds \right) \\ &\le a(t) \exp\left(-\int_{t-\tau(t)}^{t} \gamma_{2k+1}(s) ds \right) = \gamma_{2k+2}(t), \qquad t_0 - r \le t < T, \end{aligned}$$

Similarly,

$$\begin{aligned} \gamma_{2k+3}(t) &= a(t) \exp\left(-\int_{t-\tau(t)}^{t} \gamma_{2k+2}(s) ds \right) \\ &\le a(t) \exp\left(-\int_{t-\tau(t)}^{t} \gamma_{2k+1}(s) ds \right) = \gamma_{2k+2}(t), \qquad t_0 - r \le t < T, \end{aligned}$$

and therefore

$$\begin{aligned} \gamma_{2k+4}(t) &= a(t) \exp\left(-\int_{t-\tau(t)}^{t} \gamma_{2k+3}(s) ds \right) \\ &\ge a(t) \exp\left(-\int_{t-\tau(t)}^{t} \gamma_{2k+2}(s) ds \right) = \gamma_{2k+3}(t), \qquad t_0 - r \le t < T, \end{aligned}$$

We have proved that (6.6) implies

$$0 \le \gamma_{2k+1}(t) \le \gamma_{2k+3}(t) \le \gamma_{2k+4}(t) \le \gamma_{2k+2}(t) \le a(t), \qquad t_0 - r \le t < T.$$

In the next step we show that the sequence $(\gamma_n)_{n=0}^{\infty}$ is convergent and the limit is a solution of the integral equation (2.6) with the initial condition (2.7).

Due to (6.5), the sequences $(\gamma_{2k})_{k=0}^{\infty}$ and $(\gamma_{2k+1})_{k=0}^{\infty}$ are convergent on $[t_0 - r, T[$, and the functions

$$\gamma_{\text{up}} : [t_0 - r, T[\to \mathbb{R}_+, \qquad \gamma_{\text{up}}(t) = \lim_{n\to\infty} \gamma_{2k}(t)$$

and

$$\gamma_{\text{low}} : [t_0 - r, T[\to \mathbb{R}_+, \qquad \gamma_{\text{low}}(t) = \lim_{n\to\infty} \gamma_{2k+1}(t)$$

are locally integrable and $\gamma_{\text{low}} \leq \gamma_{\text{up}}$.

Now, (6.4) and Lebesgue's convergence theorem (it can be applied by (6.5)) insure that

$$\gamma_{\text{up}}(t) = a(t)\exp\left(-\int_{t-\tau(t)}^{t} \gamma_{\text{low}}(s)ds\right), \qquad t_0 - r \leq t < T \tag{6.7}$$

and

$$\gamma_{\text{low}}(t) = a(t)\exp\left(-\int_{t-\tau(t)}^{t} \gamma_{\text{up}}(s)ds\right), \qquad t_0 - r \leq t < T. \tag{6.8}$$

Since $|e^{-x} - e^{-y}| \leq |x - y|$ if $x, y \geq 0$, it follows from (6.7) and (6.8) that

$$\begin{aligned} 0 &\leq \gamma_{\text{up}}(t) - \gamma_{\text{low}}(t) \\ &\leq a(t)\int_{t_0-r}^{t} (\gamma_{\text{up}}(s) - \gamma_{\text{low}}(s))ds, \qquad t_0 - r \leq t < T, \end{aligned} \tag{6.9}$$

and therefore the Gronwall–Bellman inequality gives that $\hat{\gamma} := \gamma_{\text{up}} = \gamma_{\text{low}}$. By (6.7) $\hat{\gamma}$ is a solution of the integral equation (2.6) with the initial condition (2.7).

Only the task of confirming the uniqueness of $\hat{\gamma}$ remains. Assume γ_1 and γ_2 are solutions of the integral equation (2.6) such that the initial condition (2.7) holds. Then

$$a(t)\left|\exp\left(-\int_{t-\tau(t)}^{t} \gamma_1(s)ds\right) - \exp\left(-\int_{t-\tau(t)}^{t} \gamma_2(s)ds\right)\right| = |\gamma_1(t) - \gamma_2(t)|, \qquad t_0 - r \leq t < T.$$

Following the argument of (6.9), we have from this

$$|\gamma_1(t) - \gamma_2(t)| \leq a(t)\int_{t_0-r}^{t} |\gamma_1(s) - \gamma_2(s)|ds, \qquad t_0 - r \leq t < T,$$

and hence the Gronwall–Bellman inequality can be applied again.

(b) We define the functions $\lambda_n : [t_0 - r, T[\to \mathbb{R}_+$ $(n \in \mathbb{N})$ by recursion similarly to (6.3) and (6.4):

$$\lambda_0(t) = \gamma(t), \qquad t_0 - r \leq t < T$$

and

$$\lambda_{n+1}(t) = a(t)\exp\left(-\int_{t-\tau(t)}^{t} \lambda_n(s)ds\right), \qquad t_0 - r \leq t < T.$$

It can be proved by induction that $0 \leq \lambda_n(t) \leq a(t)$ $(t_0 - r \leq t < T)$ for all $n \geq 1$ and λ_n is locally integrable for all $n \in \mathbb{N}$.

As in (a), another induction argument on k gives that for every $k \geq 1$

$$\gamma_{2k-1} \leq \lambda_{2k} \leq \gamma_{2k-2}$$

and

$$\gamma_{2k-1} \leq \lambda_{2k+1} \leq \gamma_{2k},$$

where (γ_n) is the sequence defined by (6.3) and (6.4).

These inequalities, $\lambda_1 \leq \lambda_0 = \gamma$, and the properties of (γ_n) give that

$$\hat{\gamma} = \lim_{n\to\infty} \lambda_n \leq \gamma.$$

(c) Since $\hat{\gamma}(t) = 0$ for all $t_0 - r \le t < t_0$, $\hat{x}(t) = c$ for all $t_0 - r \le t \le t_0$. By the definition of $\hat{x}$ and the meaning of $\hat{\gamma}$,

$$\begin{aligned} c + \int_{t_0}^{t} a(u)\hat{x}(u-\tau(u))du &= c + \int_{t_0}^{t} a(u)c\exp\left(\int_{t_0}^{u-\tau(u)} \hat{\gamma}(v)dv\right)du \\ &= c\left(1 + \int_{t_0}^{t} a(u)\exp\left(-\int_{u-\tau(u)}^{u} \hat{\gamma}(v)dv\right)\exp\left(\int_{t_0}^{u}\hat{\gamma}(v)dv\right)du\right) \\ &= c\left(1 + \int_{t_0}^{t} \hat{\gamma}(u)\exp\left(\int_{t_0}^{u}\hat{\gamma}(v)dv\right)du\right) \\ &= c\left(1 + \exp\left(\int_{t_0}^{t}\hat{\gamma}(v)dv\right) - 1\right) = c\exp\left(\int_{t_0}^{t}\hat{\gamma}(v)dv\right) \\ &= \hat{x}(t), \qquad t_0 \le t < T. \end{aligned}$$

We have proved that $\hat{x}$ is a solution of (2.8) with the initial value (2.9).

Suppose x_1 and x_2 are solutions of (2.8) with the initial value (2.9). Then they are continuous,

$$|x_1(t) - x_2(t)| \le \int_{t_0}^{t} a(u)|x_1(u-\tau(u)) - x_2(u-\tau(u))|du, \qquad t_0 \le t < T,$$

and

$$|x_1(t) - x_2(t)| = 0, \qquad t_0 - r \le t \le t_0.$$

By Theorem 2.2, with $\gamma = \hat{\gamma}$, we have

$$|x_1(t) - x_2(t)| = 0, \qquad t_0 - r \le t < T,$$

and thus the uniqueness of the solutions is also proved.

The proof is now complete. □

Proof of Theorem 2.9. By applying the variation of constants formula to (2.10) we get

$$x(t) \le c\exp\left(\int_{t_0}^{t} b(v)dv\right) + \int_{t_0}^{t} a(u)x(u-\tau(u))\exp\left(\int_{u}^{t} b(v)dv\right)du, \qquad t_0 \le t < T. \tag{6.10}$$

Define $y \in C([t_0 - r, \infty[, \mathbb{R}_+)$ by

$$y(t) = x(t)\exp\left(\int_{t_0}^{t} -b(v)dv\right), \qquad t_0 - r \le t < T. \tag{6.11}$$

It follows from (6.10) that

$$\begin{aligned} y(t) &\le c + \exp\left(\int_{t_0}^{t} -b(v)dv\right) \\ &\quad\times \int_{t_0}^{t} a(u)y(u-\tau(u))\exp\left(\int_{t_0}^{u-\tau(u)} b(v)dv + \int_{u}^{t} b(v)dv\right)du \\ &= c + \int_{t_0}^{t} a(u)\exp\left(-\int_{u-\tau(u)}^{u} b(v)dv\right)y(u-\tau(u))du, \qquad t_0 \le t < T. \end{aligned} \tag{6.12}$$

The function

$$t \to a(t)\exp\left(-\int_{t-\tau(t)}^{t} b(v)dv\right), \qquad t_0 \le t < T$$

is nonnegative, and the measurability of τ yields that it is also locally integrable. This and (2.11) show that Theorem 2.2 can be applied to (6.12), and therefore

$$y(t) \le K \exp\left(\int_{t_0}^{t} \gamma(s)ds\right), \qquad t_0 \le t < T, \tag{6.13}$$

where

$$K := \max\left(c \exp\left(\int_{t_0-r}^{t_0} \gamma(u)du\right), \sup_{t_0-r\le s\le t_0} x(s) \exp\left(\int_{s}^{t_0} \gamma(u)du\right)\right). \tag{6.14}$$

The result follows from (6.13) and (6.14).

The proof is complete. □

Proof of Theorem 2.12. Extend the function c to $[t_0 - r, T[$ such that $c(t) = c(0)$ if $t_0 - r \le t < t_0$.

Since c is positive and increasing, and the other functions are nonnegative, we have from (2.14) that

$$\frac{x(t)}{c(t)} \le 1 + \int_{t_0}^{t} b(u)\frac{x(u)}{c(u)}du + \int_{t_0}^{t} a(u)\frac{x(u-\tau(u))}{c(u-\tau(u))}du, \qquad t_0 \le t < T. \tag{6.15}$$

Introducing the function

$$y : [t_0 - r, T[\to \mathbb{R}_+, \qquad y(t) := \frac{x(t)}{c(t)},$$

(6.15) yields

$$y(t) \le 1 + \int_{t_0}^{t} b(u)y(u)du + \int_{t_0}^{t} a(u)y(u-\tau(u))du, \qquad t_0 \le t < T,$$

and therefore Theorem 2.9 can be applied. □

Acknowledgement

The research of the authors has been supported by Hungarian National Foundations for Scientific Research Grant No. K120186.

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