Electronic Journal of Qualitative Theory of Differential Equations
2016, No. **44**, 1–13; doi: 10.14232/ejqtde.2016.1.44 http://www.math.u-szeged.hu/ejqtde/

Oscillatory bifurcation for semilinear ordinary differential equations

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Received 24 March 2016, appeared 29 June 2016
Communicated by Michal Fečkan

Abstract. We consider the nonlinear eigenvalue problem

$$u''(t) + \lambda f(u(t)) = 0, \ \ u(t) > 0, \ \ t \in I := (-1, 1), \ \ u(1) = u(-1) = 0,$$

where $f(u) = u + (1/2)\sin^k u$ $(k \geq 2)$ and $\lambda > 0$ is a bifurcation parameter. It is known that λ is parameterized by the maximum norm $\alpha = \|u_\lambda\|_\infty$ of the solution u_λ associated with λ and is written as $\lambda = \lambda(k, \alpha)$. When we focus on the asymptotic behavior of $\lambda(k, \alpha)$ as $\alpha \to \infty$, it is natural to expect that $\lambda(k, \alpha) \to \pi^2/4$, and its convergence rate is common to k. Contrary to this expectation, we show that $\lambda(2n_1 + 1, \alpha)$ tends to $\pi^2/4$ faster than $\lambda(2n_2, \alpha)$ as $\alpha \to \infty$, where $n_1 \geq 1$, $n_2 \geq 1$ are arbitrary given integers.

Keywords: oscillatory bifurcation, asymptotic behavior, asymptotic length of bifurcation curves.

2010 Mathematics Subject Classification: 34C23, 34F10.

1 Introduction

This paper is concerned with the following nonlinear eigenvalue problems

$$-u''(t) = \lambda \left(u(t) + \frac{1}{2}\sin^k u(t) \right), \qquad t \in I := (-1, 1), \tag{1.1}$$

$$u(t) > 0, \qquad t \in I, \tag{1.2}$$

$$u(1) = u(-1) = 0, \tag{1.3}$$

where $k \geq 1$ is a given integer and $\lambda > 0$ is a parameter. If k is fixed, then it is well known that, for any given $\alpha > 0$, there exists a unique solution pair (λ, u_α) of (1.1)–(1.3) with $\alpha = \|u_\alpha\|_\infty$. Besides, λ is parameterized by α and a C^1-function of α (cf. [1], [12, Theorem 2.1]). So we write $\lambda = \lambda(k, \alpha)$ in what follows. Then the solution set of (1.1)–(1.3) consists of the set $\Lambda := \{(\lambda(k, \alpha), u_\alpha) | \text{ sol. of (1.1)–(1.3) with } \|u_\alpha\|_\infty = \alpha\} \subset \mathbb{R}_+ \times C^2(\bar{I})$.

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The qualitative properties of oscillating bifurcation curves have been studied intensively. We refer to [7, 9–11, 13] and the references therein. Recently, when $k = 1$, the asymptotic properties of the bifurcation curve $\lambda = \lambda(1, \alpha)$ have been studied in [14], in which there are two main stress. Firstly, since $\lambda(1, \alpha)$ tends to $\pi^2/4$ as $\alpha \to \infty$, and is oscillating infinitely many times, it is interesting to study the global behavior of $\lambda(1, \alpha)$ and $\lambda'(1, \alpha) := d\lambda(1, \alpha)/d\alpha$, and the precise asymptotic formulas for $\lambda(1, \alpha)$ and $\lambda'(1, \alpha)$ as $\alpha \to \infty$ were established. Secondly, for $\alpha \gg 1$, the asymptotic length

$$L_k(\alpha) := \int_{\alpha}^{2\alpha} \sqrt{1 + (\lambda'(k, x))^2} dx \tag{1.4}$$

for $k = 1$, which seems to be a new concept, has been proposed for the application to inverse bifurcation problems, and precise asymptotic formula for $L_1(\alpha)$ was obtained.

Theorem 1.0 ([14]). *Let $k = 1$ in (1.1). Then as $\alpha \to \infty$*

$$\lambda(1, \alpha) = \frac{\pi^2}{4} - \frac{\pi}{2\alpha}\sqrt{\frac{\pi}{2\alpha}} \sin\left(\alpha - \frac{1}{4}\pi\right) + o(\alpha^{-3/2}), \tag{1.5}$$

$$\lambda'(1, \alpha) = -\frac{\pi}{2\alpha}\sqrt{\frac{\pi}{2\alpha}} \cos\left(\alpha - \frac{\pi}{4}\right) + o(\alpha^{-3/2}), \tag{1.6}$$

$$L_1(\alpha) = \alpha + \frac{3\pi^3}{256}\alpha^{-2} + o(\alpha^{-2}). \tag{1.7}$$

In this paper, we expand the argument in [14] and characterize the meaningful properties of $\lambda(k, \alpha)$ for the case $k \geq 2$.

As for asymptotic behavior of $\lambda(k, \alpha)$ as $\alpha \to \infty$, it is clear that as $\alpha \to \infty$,

$$\lambda(k, \alpha) \to \frac{\pi^2}{4} \tag{1.8}$$

(see Fig. 1 and Fig. 2 below)

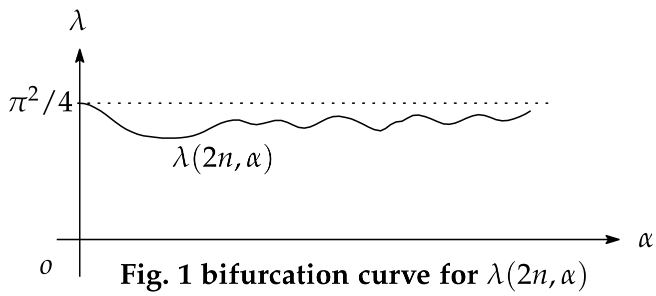

Fig. 1 bifurcation curve for $\lambda(2n, \alpha)$

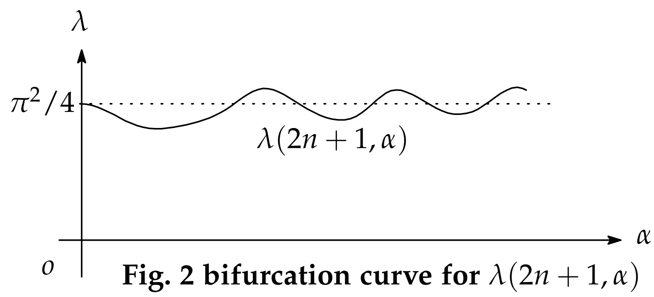

Fig. 2 bifurcation curve for $\lambda(2n + 1, \alpha)$

and it is quite natural to expect that the rate of convergence of $\lambda(2n, \alpha)$ to $\pi^2/4$ as $\alpha \to \infty$ is the same as that of $\lambda(2n + 1, \alpha)$. However, contrary to our expectation, it will turn out that the following inequality holds.

$$|\lambda(2n_1 + 1, \alpha) - \pi^2/4| \ll |\lambda(2n_2, \alpha) - \pi^2/4| \to 0, \tag{1.9}$$

where $n_1 \geq 1$ and $n_2 \geq 1$ are arbitrary given integers. To show (1.9), we calculate the asymptotic behavior of $\sqrt{\lambda(k,\alpha)}$ precisely.

Secondly, we show that as $\alpha \to \infty$,

$$|\lambda'(2n_1+1,\alpha)| \sim |\lambda'(2n_2,\alpha)|, \qquad L_{2n_1+1}(\alpha) - \alpha \sim L_{2n_2}(\alpha) - \alpha. \tag{1.10}$$

The reason why (1.10) holds is explained as follows. $L_k(\alpha)$ ($k \geq 1$) depends on $|\lambda'(k,\alpha)|$, and its rate of convergence to 0 is common to k.

Now we state our main results.

Theorem 1.1. *Let $k = 2n$ ($n \geq 1$). Then as $\alpha \to \infty$*

$$\lambda(2n,\alpha) = \frac{\pi^2}{4} - \frac{\pi^{3/2}}{2^{2n+1}\alpha}\binom{2n}{n} - \frac{\pi}{2^{2n+1}\alpha^{3/2}}\sum_{r=0}^{n-1}(-1)^{n-r}\binom{2n}{r} \times \frac{1}{\sqrt{n-r}}\sin\left((2n-2r)\alpha + \frac{\pi}{4}\right) + O(\alpha^{-2}), \tag{1.11}$$

$$\lambda'(2n,\alpha) = -\frac{\pi^{3/2}}{2^{2n}}\alpha^{-3/2}\sum_{r=0}^{n-1}(-1)^{n-r}\binom{2n}{r}\sqrt{n-r}\cos\left((2n-2r)\alpha + \frac{\pi}{4}\right) + O(\alpha^{-2}), \tag{1.12}$$

$$L_{2n}(\alpha) = \alpha + \frac{3\pi^3}{2^{4n+5}\alpha^2}\sum_{r=0}^{n-1}\binom{2n}{r}^2(n-r) + O(\alpha^{-5/2}). \tag{1.13}$$

Theorem 1.2. *Let $k = 2n+1$ ($n \geq 0$). Then as $\alpha \to \infty$*

$$\lambda(2n+1,\alpha) = \frac{\pi^2}{4} - \frac{\pi^{3/2}}{2^{2n+1}\alpha^{3/2}}\sum_{r=0}^{n}(-1)^{n+r}\binom{2n+1}{r} \times \sqrt{\frac{1}{2(2n-2r+1)}}\sin\left((2n-2r+1)\alpha - \frac{1}{4}\pi\right) + O(\alpha^{-2}), \tag{1.14}$$

$$\lambda'(2n+1,\alpha) = -\frac{\pi^{3/2}}{2^{2n+1}\alpha^{3/2}}\sum_{r=0}^{n}(-1)^{n+r}\binom{2n+1}{r}\sqrt{\frac{(2n-2r+1)}{2}} \times \cos\left((2n-2r+1)\alpha - \frac{\pi}{4}\right) + O(\alpha^{-2}), \tag{1.15}$$

$$L_{2n+1}(\alpha) = \alpha + \frac{3\pi^3}{2^{4n+8}\alpha^2}\sum_{r=0}^{n}\binom{2n+1}{r}^2(2n-2r+1) + O(\alpha^{-5/2}). \tag{1.16}$$

Remarks. (i) It seems that (1.12) (resp. (1.15)) can be obtained easily by (1.11) (resp. (1.14)). However, since it is not clear whether the formal derivative of (1.11) (resp. (1.14)) is correct or not, we should be careful to prove (1.12) and (1.15).

(ii) Let $n \geq 0$ be fixed. Then we see from (1.14) that $\lambda(2n+1,\alpha)$ crosses the line $\lambda = \pi^2/4$ infinitely many times. Indeed, let q be an arbitrary large even integer, $\alpha_1 = (q+(1/2))\pi$ and $\alpha_2 = (q+(3/2))\pi$. Then it is easy to see that for $0 \leq r \leq n$,

$$\sin\left((2n-2r+1)\alpha_1 - \frac{1}{4}\pi\right) = \sin\left((n-r)\pi + \frac{1}{4}\pi\right),$$

$$\sin\left((2n-2(r+1)+1)\alpha_1 - \frac{1}{4}\pi\right) = -\sin\left((n-r)\pi + \frac{1}{4}\pi\right),$$

$$\sin\left((2n-2r+1)\alpha_2 - \frac{1}{4}\pi\right) = -\sin\left((n-r)\pi + \frac{1}{4}\pi\right),$$

$$\sin\left((2n-2(r+1)+1)\alpha_2 - \frac{1}{4}\pi\right) = \sin\left((n-r)\pi + \frac{1}{4}\pi\right).$$

This implies that for $0 \le r \le n$,

$$(-1)^{n+r}\sin\left((2n-2r+1)\alpha_1 - \frac{1}{4}\pi\right) = (-1)^n \sin\left(\left(n+\frac{1}{4}\right)\pi\right)$$
$$(-1)^{n+r}\sin\left((2n-2r+1)\alpha_2 - \frac{1}{4}\pi\right) = -(-1)^n \sin\left(\left(n+\frac{1}{4}\right)\pi\right).$$

By this and (1.12), for the constant

$$C_n := \frac{(-1)^n \pi^{3/2}}{2^{2n+1}} \sin\left(\left(n+\frac{1}{4}\right)\pi\right) \sum_{r=0}^{n} \binom{2n+1}{r} \sqrt{\frac{1}{2(2n-2r+1)}},$$

we obtain

$$\lambda(2n+1,\alpha_1) = \frac{\pi^2}{4} - C_n \alpha_1^{-3/2} + O(\alpha_1^{-2}), \tag{1.17}$$
$$\lambda(2n+1,\alpha_2) = \frac{\pi^2}{4} + C_n \alpha_2^{-3/2} + O(\alpha_2^{-2}). \tag{1.18}$$

This implies our assertion.
(iii) The study of bifurcation problems has a long history and there are so many topics. For the readers who are interested in this field, we refer to [2–6].

The proofs of Theorems 1.1 and 1.2 depend on the time map arguments used in [14]. However, we understand easily from Theorems 1.0–1.2 that all the terms in Theorems 1.1 and 1.2 are extremely more complicated than those in Theorem 1.0. Therefore, we proceed all the steps of the calculation very carefully.

2 Proof of Theorem 1.1

In this section, let $k = 2n$ ($n \ge 1$) be fixed. In what follows, let $\alpha \gg 1$. Furthermore, C denotes various positive constants independent of $\alpha \gg 1$. For simplicity, we write $\lambda = \lambda(\alpha) = \lambda(2n, \alpha)$. We have (cf. [8, p. 31])

$$g(\theta) := \sin^{2n}\theta = \frac{1}{2^{2n}}\left[\sum_{r=0}^{n-1}(-1)^{n-r}2\binom{2n}{r}\cos(2n-2r)\theta + \binom{2n}{n}\right], \tag{2.1}$$

$$G(u) := \int_0^u g(\theta)d\theta \tag{2.2}$$
$$= \frac{1}{2^{2n}}\left[\sum_{r=0}^{n-1}(-1)^{n-r}\binom{2n}{r}\frac{1}{n-r}\sin(2n-2r)u + \binom{2n}{n}u\right].$$

By (1.1), we have

$$\left(u_\alpha''(t) + \lambda\left(u_\alpha(t) + \frac{1}{2}\sin^{2n}u_\alpha(t)\right)\right)u_\alpha'(t) = 0.$$

By this, we obtain

$$\frac{1}{2}u_\alpha'(t)^2 + \frac{1}{2}\lambda(u_\alpha(t)^2 + G(u_\alpha(t))) = \text{constant} = \frac{1}{2}\lambda(\alpha^2 + G(\alpha)).$$

This implies that for $-1 \le t \le 0$,

$$u_\alpha'(t) = \sqrt{\lambda}\sqrt{\alpha^2 - u_\alpha(t)^2 + G(\alpha) - G(u_\alpha(t))}.$$

By this, putting $s := u_\alpha(t)/\alpha$, we obtain the time map

$$\begin{aligned} \sqrt{\lambda} &= \int_{-1}^{0} \frac{u_\alpha'(t)}{\sqrt{\alpha^2 - u_\alpha(t)^2 + G(\alpha) - G(u_\alpha(t))}} dt \\ &= \int_0^1 \frac{1}{\sqrt{1 - s^2 + (G(\alpha) - G(\alpha s))/\alpha^2}} ds. \end{aligned} \tag{2.3}$$

Proof of (1.11). By (2.2), for $\alpha \gg 1$ and $0 \le s \le 1$, we have

$$\begin{aligned} G(\alpha) - G(\alpha s) &= \frac{1}{2^{2n}} \binom{2n}{n} \alpha(1 - s) \\ &\quad + \frac{1}{2^{2n}} \sum_{r=0}^{n-1} (-1)^{n-r} \binom{2n}{r} \frac{1}{n - r} \{\sin((2n - 2r)\alpha) - \sin((2n - 2r)\alpha s)\}. \end{aligned} \tag{2.4}$$

By this, for $\alpha \gg 1$ and $0 \le s \le 1$

$$\begin{aligned} \left| \frac{G(\alpha) - G(\alpha s)}{\alpha^2(1 - s^2)} \right| &\le C \frac{\alpha(1 - s)}{\alpha^2(1 - s^2)} + C \sum_{r=0}^{n-1} \left| \frac{\sin((2n - 2r)\alpha) - \sin((2n - 2r)\alpha s)}{\alpha^2(1 - s^2)} \right| \\ &\le C\alpha^{-1} + \frac{C}{\alpha^2(1 - s^2)} \sum_{r=0}^{n-1} \left| \int_{(2n-2r)\alpha s}^{(2n-2r)\alpha} \cos\theta d\theta \right| \\ &\le C\alpha^{-1}. \end{aligned} \tag{2.5}$$

By this, (2.3), (2.4), Taylor expansion and Lebesgue's convergence theorem, we obtain

$$\begin{aligned} \sqrt{\lambda} &= \int_0^1 \frac{1}{\sqrt{1 - s^2}} \left(1 + \frac{G(\alpha) - G(\alpha s)}{\alpha^2(1 - s^2)} \right)^{-1/2} ds \\ &= \int_0^1 \frac{1}{\sqrt{1 - s^2}} \left(1 - \frac{G(\alpha) - G(\alpha s)}{2\alpha^2(1 - s^2)} + O(\alpha^{-2}) \right) ds \\ &= \frac{\pi}{2} - \frac{1}{2^{2n+1}\alpha} \binom{2n}{n} \int_0^1 \frac{1}{\sqrt{1 - s^2}(1 + s)} ds \\ &\quad - \frac{1}{2^{2n+1}\alpha^2} \sum_{r=0}^{n-1} (-1)^{n-r} \binom{2n}{r} \frac{1}{n - r} I_r + O(\alpha^{-2}), \end{aligned} \tag{2.6}$$

where

$$I_r := \int_0^1 \frac{\sin((2n - 2r)\alpha) - \sin((2n - 2r)\alpha s)}{(1 - s^2)^{3/2}} ds. \tag{2.7}$$

Let $Y_\nu(\alpha)$ and $\mathbf{E}_\nu(\alpha)$ be the Neumann functions and Weber functions. For $\alpha \gg 1$, we have (cf. [8, p. 929, p. 958])

$$Y_\nu(\alpha) = \sqrt{\frac{2}{\pi\alpha}} \sin\left(\alpha - \frac{\pi}{2}\nu - \frac{\pi}{4} \right) + O(\alpha^{-3/2}), \tag{2.8}$$

$$\mathbf{E}_\nu(\alpha) = -Y_\nu(\alpha) + O(\alpha^{-1}). \tag{2.9}$$

By putting $s = \sin\theta$, $\theta = \pi/2 - t$, integration by parts and l'Hôpital's rule, (2.7)–(2.9) and

[8, p. 425], we have

$$\begin{aligned} I_r &= \int_0^{\pi/2} \frac{1}{\cos^2\theta}\{\sin(2n-2r)\alpha - \sin((2n-2r)\alpha\sin\theta)\}d\theta \\ &= \int_0^{\pi/2} (\tan\theta)'\{\sin(2n-2r)\alpha - \sin((2n-2r)\alpha\sin\theta)\}d\theta \\ &= [\tan\theta\{\sin(2n-2r)\alpha - \sin((2n-2r)\alpha\sin\theta)\}]_0^{\pi/2} \\ &\quad + (2n-2r)\alpha\int_0^{\pi/2}\sin\theta\cos((2n-2r)\alpha\sin\theta)d\theta \\ &= (2n-2r)\alpha\int_0^{\pi/2}\sin\theta\cos((2n-2r)\alpha\sin\theta)d\theta \\ &= (2n-2r)\alpha\int_0^{\pi/2}\cos t\cos((2n-2r)\alpha\cos t)dt \\ &= \frac{(n-r)\pi\alpha}{2}(\mathbf{E}_1((2n-2r)\alpha) - \mathbf{E}_{-1}((2n-2r)\alpha)) \\ &= \frac{(n-r)\pi\alpha}{2}(-Y_1((2n-2r)\alpha) + Y_{-1}((2n-2r)\alpha) + O(\alpha^{-1})) \\ &= \sqrt{(n-r)\pi\alpha}\sin\left((2n-2r)\alpha + \frac{\pi}{4}\right) + O(1). \end{aligned} \tag{2.10}$$

By this and (2.6), we obtain

$$\begin{aligned} \sqrt{\lambda(2n,\alpha)} &= \frac{\pi}{2} - \frac{1}{2^{2n+1}\alpha}\binom{2n}{n} - \frac{1}{2^{2n+1}\alpha^{3/2}}\sum_{r=0}^{n-1}(-1)^{n-r}\binom{2n}{r} \\ &\quad \times\sqrt{\frac{\pi}{n-r}}\sin\left((2n-2r)\alpha + \frac{\pi}{4}\right) + O(\alpha^{-2}). \end{aligned} \tag{2.11}$$

By this, we directly obtain (1.11). Thus the proof is complete. □

We next prove (1.12). To do this, we prepare the following lemma.

Lemma 2.1. *As $\alpha \to \infty$,*

$$(\sqrt{\lambda})' = -\frac{1}{2^{2n}}\alpha^{-3/2}\sum_{r=0}^{n-1}(-1)^{n-r}\binom{2n}{r}\sqrt{(n-r)\pi}\cos\left((2n-2r)\alpha + \frac{\pi}{4}\right) + O(\alpha^{-2}). \tag{2.12}$$

Proof. By (2.1), we have

$$g(\alpha) - sg(\alpha s) = \frac{1}{2^{2n-1}}\sum_{r=0}^{n-1}(-1)^{n-r}\binom{2n}{r}\{\cos((2n-2r)\alpha) - s\cos((2n-2r)\alpha s)\}. \tag{2.13}$$

By (2.5), (2.6) and the argument in [1, (2.7)], we have

$$\begin{aligned} (\sqrt{\lambda})' &= \frac{1}{2}\int_0^1 \frac{1}{\sqrt{1-s^2}}\left(1 + \frac{G(\alpha)-G(\alpha s)}{\alpha^2(1-s^2)}\right)^{-3/2} \\ &\quad \times\left\{2\alpha^{-3}\frac{G(\alpha)-G(\alpha s)}{1-s^2} - \frac{g(\alpha)-sg(\alpha s)}{\alpha^2(1-s^2)}\right\}ds \\ &= -\frac{1}{2}(1+O(\alpha^{-1}))\int_0^1\frac{1}{\sqrt{1-s^2}}\frac{g(\alpha)-sg(\alpha s)}{\alpha^2(1-s^2)}ds + O(\alpha^{-2}) \\ &= -\frac{1}{2^{2n}}\alpha^{-2}(1+O(\alpha^{-1}))\sum_{r=0}^{n-1}(-1)^{n-r}\binom{2n}{r}M_r + O(\alpha^{-2}). \end{aligned} \tag{2.14}$$

Here, by (2.13),

$$M_r := \int_0^1 \frac{1}{(1-s^2)^{3/2}} \{\cos((2n-2r)\alpha) - s\cos((2n-2r)\alpha s)\}\, ds. \tag{2.15}$$

By this and putting $s = \sin\theta$, integration by parts and using l'Hôpital's rule, we obtain

$$\begin{aligned} M_r &= \int_0^{\pi/2} \frac{1}{\cos^2\theta} \{\cos((2n-2r)\alpha) - \sin\theta\cos((2n-2r)\alpha\sin\theta)\}\, d\theta \\ &= \int_0^{\pi/2} (\tan\theta)' \{\cos((2n-2r)\alpha) - \sin\theta\cos((2n-2r)\alpha\sin\theta)\}\, d\theta \\ &= [\tan\theta \{\cos((2n-2r)\alpha) - \sin\theta\cos((2n-2r)\alpha\sin\theta)\}]_0^{\pi/2} \\ &\quad - \int_0^{\pi/2} \tan\theta \big\{-\cos\theta\cos((2n-2r)\alpha\sin\theta) \\ &\qquad\qquad + (2n-2r)\alpha\sin\theta\cos\theta\sin((2n-2r)\alpha\sin\theta)\big\} d\theta \\ &= \int_0^{\pi/2} \sin\theta\cos((2n-2r)\alpha\sin\theta) d\theta \\ &\quad - (2n-2r)\alpha \int_0^{\pi/2} \sin^2\theta\sin((2n-2r)\alpha\sin\theta) d\theta \\ &:= O(1) - (2n-2r)\alpha M_{r,1}. \end{aligned} \tag{2.16}$$

Let $\mathbf{H}_\nu(\alpha)$ be Struve functions. For $\alpha \gg 1$, we have (cf. [8, p. 952])

$$\mathbf{H}_\nu(\alpha) = Y_\nu(\alpha) + \frac{1}{\pi} \sum_{m=0}^{p-1} \frac{\Gamma\left(m+\frac{1}{2}\right)\left(\frac{\alpha}{2}\right)^{-2m+\nu-1}}{\Gamma\left(\nu+\frac{1}{2}-m\right)} + O(\alpha^{\nu-2p-1}). \tag{2.17}$$

By putting $\theta = \pi/2 - t$, (2.8) and (2.17), we obtain (cf. [8, p. 425])

$$\begin{aligned} M_{r,1} &= \int_0^{\pi/2} \cos^2 t \sin((2n-2r)\alpha\cos t) dt \\ &= \int_0^{\pi/2} \sin((2n-2r)\alpha\cos t) dt - \int_0^{\pi/2} \sin^2 t\sin((2n-2r)\alpha\cos t) dt \\ &= \frac{\sqrt{\pi}}{2}\Gamma\left(\frac{1}{2}\right)\mathbf{H}_0((2n-2r)\alpha) - \frac{\sqrt{\pi}}{2}\left(\frac{2}{(2n-2r)\alpha}\right)\Gamma\left(\frac{3}{2}\right)\mathbf{H}_1((2n-2r)\alpha) \\ &= \frac{\pi}{2} Y_0((2n-2r)\alpha) - \frac{\sqrt{\pi}}{2}\left(\frac{2}{(2n-2r)\alpha}\right)\Gamma\left(\frac{3}{2}\right) Y_1((2n-2r)\alpha) + O(\alpha^{-1}) \\ &= \frac{1}{2}\sqrt{\frac{\pi}{(n-r)\alpha}} \sin\left((2n-2r)\alpha - \frac{\pi}{4}\right) + O\left(\alpha^{-1}\right) \\ &= -\frac{1}{2}\sqrt{\frac{\pi}{(n-r)\alpha}} \cos\left((2n-2r)\alpha + \frac{\pi}{4}\right) + O\left(\alpha^{-1}\right). \end{aligned} \tag{2.18}$$

By (2.14), (2.15), (2.16) and (2.18), we obtain (2.12). Thus the proof is complete. □

Proof of (1.12). By the argument in [1], we have

$$\left(\sqrt{\lambda(\alpha)}\right)' = \frac{1}{2}\lambda(\alpha)^{-1/2}\lambda'(\alpha).$$

By this, (1.11) and Lemma 2.1, we have

$$\lambda'(\alpha) = 2\sqrt{\lambda(\alpha)}\frac{d}{d\alpha}\left(\sqrt{\lambda(\alpha)}\right) = 2\frac{\pi}{2}(1 + O(\alpha^{-1}))(\sqrt{\lambda})' \tag{2.19}$$
$$= -\frac{\pi}{2^{2n}}\alpha^{-3/2}\sum_{r=0}^{n-1}(-1)^{n-r}\binom{2n}{r}\sqrt{(n-r)\pi}\cos\left((2n-2r)\alpha + \frac{\pi}{4}\right) + O(\alpha^{-2}).$$

Thus we obtain (1.12). □

Proof of (1.13). By (2.19) and Taylor expansion, we obtain

$$L_{2n}(\alpha) = \int_\alpha^{2\alpha}\sqrt{1 + \lambda'(x)^2}\,dx \tag{2.20}$$
$$= \int_\alpha^{2\alpha}\left\{1 + \frac{\pi^3}{2^{4n+1}}x^{-3}\sum_{r=0,m=0}^{n-1}(-1)^{2n-r-m}\binom{2n}{r}\binom{2n}{m}\sqrt{(n-r)(n-m)}\right.$$
$$\left.\times\cos\left((2n-2r)x + \frac{\pi}{4}\right)\cos\left((2n-2m)x + \frac{\pi}{4}\right) + O(x^{-7/2})\right\}dx.$$

Then we have two cases to consider.
(i) Let $r \ne m$. Then

$$S_{r,m} := \int_\alpha^{2\alpha}\frac{\cos\left((2n-2r)x + \frac{\pi}{4}\right)\cos\left((2n-2m)x + \frac{\pi}{4}\right)}{x^3}dx \tag{2.21}$$
$$= \frac{1}{2}\int_\alpha^{2\alpha}\frac{\cos\left((4n-2r-2m)x + \frac{\pi}{2}\right)}{x^3}dx + \frac{1}{2}\int_\alpha^{2\alpha}\frac{\cos((2r-2m)x)}{x^3}dx$$
$$= -\frac{1}{2}\int_\alpha^{2\alpha}\frac{\sin\left((4n-2r-2m)x\right)}{x^3}dx + \frac{1}{2}\int_\alpha^{2\alpha}\frac{\cos((2r-2m)x)}{x^3}dx$$
$$:= S_{m,r,1} + S_{m,r,2}.$$

Then by integration by parts, we obtain

$$S_{r,m,1} = \frac{1}{2(4n-2r-2m)}\left[\frac{1}{x^3}\cos((4n-2r-2m)x)\right]_\alpha^{2\alpha}$$
$$+ \frac{3}{2(4n-2r-2m)}\int_\alpha^{2\alpha}\frac{\cos((4n-2r-2m)x)}{x^4}dx \tag{2.22}$$
$$= O(\alpha^{-3}).$$

By the same calculation as that just above, we obtain

$$S_{r,m,2} = O(\alpha^{-3}). \tag{2.23}$$

(ii) Let $r = m$. Then by (2.21) and (2.22), we obtain

$$S_{r,r} := -\frac{1}{2}\int_\alpha^{2\alpha}\frac{\sin\left((4n-4r)x\right)}{x^3}dx + \frac{1}{2}\int_\alpha^{2\alpha}\frac{1}{x^3}dx \tag{2.24}$$
$$= O(\alpha^{-3}) + \frac{3}{16\alpha^2}.$$

Therefore, by (2.20)–(2.24), we obtain

$$L_{2n}(\alpha) = \alpha + \frac{3\pi^3}{2^{4n+5}\alpha^2}\sum_{r=0}^{n-1}\binom{2n}{r}^2(n-r) + O(\alpha^{-5/2}). \tag{2.25}$$

Thus we obtain (1.13). □

3 Proof of Theorem 1.2

In this section, let $k = 2n+1$ $(n \geq 1)$. For simplicity, we write $\lambda = \lambda(\alpha) = \lambda(2n+1, \alpha)$. We use the same notations as those in Section 2. By [8, p. 31], we have

$$g(\theta) := \sin^{2n+1}\theta = \frac{1}{2^{2n}} \sum_{r=0}^{n} (-1)^{n+r} \binom{2n+1}{r} \sin(2n-2r+1)\theta, \tag{3.1}$$

$$\begin{aligned} G(u) &:= \int_0^u g(\theta)d\theta \\ &= \frac{1}{2^{2n}} \sum_{r=0}^{n} (-1)^{n+r} \binom{2n+1}{r} \frac{1}{2n-2r+1}(1-\cos((2n-2r+1)u)). \end{aligned} \tag{3.2}$$

By (3.2), we have

$$\begin{aligned} G(\alpha) - G(\alpha s) = \frac{1}{2^{2n}} &\sum_{r=0}^{n} (-1)^{n+r} \binom{2n+1}{r} \frac{1}{2n-2r+1} \\ &\times \{-\cos((2n-2r+1)\alpha) + \cos((2n-2r+1)\alpha s)\}. \end{aligned} \tag{3.3}$$

We first prove (1.14). By the same argument as that to obtain (2.6) and (3.3), we obtain

$$\begin{aligned} \sqrt{\lambda} &= \frac{\pi}{2} - \frac{1}{2\alpha^2} \int_0^1 \frac{G(\alpha) - G(\alpha s)}{(1-s^2)^{3/2}} ds + O(\alpha^{-2}) \\ &= \frac{\pi}{2} - \frac{1}{2^{2n+1}\alpha^2} \sum_{r=0}^{n} (-1)^{n+r} \binom{2n+1}{r} \frac{1}{2n-2r+1} K_r + O(\alpha^{-2}), \end{aligned} \tag{3.4}$$

where

$$K_r := \int_0^1 \frac{-\cos((2n-2r+1)\alpha) + \cos((2n-2r+1)\alpha s)}{(1-s^2)^{3/2}} ds. \tag{3.5}$$

Lemma 3.1. *As $\alpha \to \infty$,*

$$K_r = \sqrt{\frac{(2n-2r+1)\pi\alpha}{2}} \sin\left((2n-2r+1)\alpha - \frac{1}{4}\pi\right) + O(\alpha^{-1/2}). \tag{3.6}$$

Proof. Let $J_{\pm\nu}(z)$ be the Bessel function. For $\alpha \gg 1$, we have (cf. [8, p. 929])

$$J_{\pm\nu}(\alpha) = \sqrt{\frac{2}{\pi\alpha}} \cos\left(\alpha \mp \frac{\pi}{2}\nu - \frac{\pi}{4}\right) + O(\alpha^{-3/2}). \tag{3.7}$$

By putting $s = \sin\theta$, $t = \pi/2 - \theta$ and using the same integration by parts as that in (2.10) and (3.7), we obtain (cf. [8, p. 425])

$$\begin{aligned} K_r &= \int_0^{\pi/2} \frac{1}{\cos^2\theta} \{-\cos((2n-2r+1)\alpha) + \cos((2n-2r+1)\alpha\sin\theta)\} d\theta \\ &= \alpha(2n-2r+1) \int_0^{\pi/2} \sin\theta \sin((2n-2r+1)\alpha\sin\theta) d\theta \\ &= \alpha(2n-2r+1) \int_0^{\pi/2} \cos t \sin((2n-2r+1)\alpha\cos t) dt \\ &= \alpha(2n-2r+1)\frac{\pi}{2} J_1((2n-2r+1)\alpha) \\ &= \alpha(2n-2r+1)\frac{\pi}{2} \sqrt{\frac{2}{\pi(2n-2r+1)\alpha}} \cos\left((2n-2r+1)\alpha - \frac{3}{4}\pi\right) + O(\alpha^{-1/2}) \\ &= \sqrt{\frac{\pi(2n-2r+1)\alpha}{2}} \sin\left((2n-2r+1)\alpha - \frac{1}{4}\pi\right) + O(\alpha^{-1/2}). \end{aligned} \tag{3.8}$$

This implies (3.6). Thus the proof is complete. □

Proof of (1.14). By (3.4) and Lemma 3.1, we obtain

$$\sqrt{\lambda(2n+1,\alpha)} = \frac{\pi}{2} - \frac{1}{2^{2n+1}\alpha^{3/2}} \sum_{r=0}^{n} (-1)^{n+r} \binom{2n+1}{r} \times \sqrt{\frac{\pi}{2(2n-2r+1)}} \sin\left((2n-2r+1)\alpha - \frac{1}{4}\pi\right) + O(\alpha^{-2}). \tag{3.9}$$

This implies (1.14) in Theorem 1.2. □

Now we prove (1.15) and (1.16).

Lemma 3.2. *For $\alpha \gg 1$,*

$$(\sqrt{\lambda})' = -\frac{\sqrt{\pi}}{2^{2n+1}}\alpha^{-3/2} \sum_{r=0}^{n} (-1)^{n+r} \binom{2n+1}{r} \sqrt{\frac{(2n-2r+1)}{2}} \cos\left((2n-2r+1)\alpha - \frac{\pi}{4}\right) + O(\alpha^{-2}). \tag{3.10}$$

Proof. Following the argument in Lemma 2.1, we calculate (2.14). By (3.1), we have

$$g(\alpha) - sg(\alpha s) = \frac{1}{2^{2n}} \sum_{r=0}^{n} (-1)^{n+r} \binom{2n+1}{r} \times \{\sin((2n-2r+1)\alpha) - s\sin((2n-2r+1)\alpha s)\}. \tag{3.11}$$

By this and (2.14), we have

$$\begin{aligned}(\sqrt{\lambda})' &= -\frac{1}{2}(1+O(\alpha^{-1})) \int_0^1 \frac{1}{\sqrt{1-s^2}} \frac{g(\alpha) - sg(\alpha s)}{\alpha^2(1-s^2)} ds + O(\alpha^{-2}) \\ &= -\frac{1}{2^{2n+1}}\alpha^{-2}(1+O(\alpha^{-1})) \sum_{r=0}^{n-1} (-1)^{n+r} \binom{2n+1}{r} R_r + O(\alpha^{-2}),\end{aligned} \tag{3.12}$$

where

$$R_r := \int_0^1 \frac{1}{(1-s^2)^{3/2}} \{\sin((2n-2r+1)\alpha) - s\sin((2n-2r+1)\alpha s)\} ds. \tag{3.13}$$

By this and putting $s = \sin\theta$, we obtain

$$\begin{aligned} R_r &= \int_0^{\pi/2} \frac{1}{\cos^2\theta} \{\sin((2n-2r+1)\alpha) - \sin\theta \sin((2n-2r+1)\alpha\sin\theta)\} d\theta \\ &= [\tan\theta\{\sin((2n-2r+1)\alpha) - \sin\theta\sin((2n-2r+1)\alpha\sin\theta)\}]_0^{\pi/2} \\ &\quad - \int_0^{\pi/2} \tan\theta \big\{-\cos\theta\sin((2n-2r+1)\alpha\sin\theta) \\ &\qquad - \alpha(2n-2r+1)\sin\theta\cos\theta\cos((2n-2r+1)\alpha\sin\theta)\big\} d\theta \\ &= R_{r,1} + \alpha(2n-2r+1)R_{r,2} \\ &:= \int_0^{\pi/2} \sin\theta\sin((2n-2r-1)\alpha\sin\theta) d\theta \\ &\quad + \alpha(2n-2r+1)\int_0^{\pi/2} \sin^2\theta\cos((2n-2r+1)\alpha\sin\theta) d\theta. \end{aligned} \tag{3.14}$$

By this, (3.7), [8, p. 425] and putting $t = \pi/2 - \theta$, we obtain

$$\begin{aligned} R_{r,1} &= \int_0^{\pi/2} \cos t \sin((2n-2r+1)\alpha \cos t) dt \\ &= \frac{\pi}{2} J_1(2n-2r+1)\alpha) \\ &= \sqrt{\frac{\pi}{2(2n-2r+1)\alpha}} \cos\left((2n-2r+1)\alpha - \frac{3}{4}\pi\right) + O(\alpha^{-3/2}). \end{aligned} \tag{3.15}$$

By (3.7), [8, p. 425] and putting $t = \pi/2 - \theta$, we obtain

$$\begin{aligned} R_{r,2} &= \int_0^{\pi/2} \cos^2\theta \cos((2n-2r+1)\alpha\cos\theta) d\theta \\ &= \frac{1}{2}\int_0^{\pi/2} (1+\cos 2\theta)\cos((2n-2r+1)\alpha\cos\theta)d\theta \\ &= \frac{1}{2}\int_0^{\pi/2} \cos((2n-2r+1)\alpha\cos\theta)d\theta + \frac{1}{2}\int_0^{\pi/2}\cos 2\theta\cos((2n-2r+1)\alpha\cos\theta)d\theta \\ &= \frac{\pi}{4}J_0((2n-2r+1)\alpha) - \frac{\pi}{4}J_2((2n-2r+1)\alpha) \\ &= \frac{\pi}{4}\sqrt{\frac{2}{\pi(2n-2r+1)\alpha}}\left(\cos\left((2n-2r+1)\alpha - \frac{\pi}{4}\right) - \cos\left((2n-2r+1)\alpha - \frac{5\pi}{4}\right)\right) \\ &\quad + O(\alpha^{-3/2}) \\ &= \sqrt{\frac{\pi}{2(2n-2r+1)\alpha}}\cos\left((2n-2r+1)\alpha - \frac{\pi}{4}\right) + O(\alpha^{-3/2}). \end{aligned} \tag{3.16}$$

By (3.14)–(3.16), we obtain

$$R_r = \sqrt{\frac{(2n-2r+1)\pi\alpha}{2}}\cos\left((2n-2r+1)\alpha - \frac{\pi}{4}\right) + O(\alpha^{-1/2}). \tag{3.17}$$

By (3.12) and (3.17), we obtain

$$\begin{aligned} (\sqrt{\lambda})' &= -\frac{\sqrt{\pi}}{2^{2n+1}}\alpha^{-3/2}\sum_{r=0}^{n}(-1)^{n+r}\binom{2n+1}{r}\sqrt{\frac{(2n-2r+1)}{2}}\cos\left((2n-2r+1)\alpha - \frac{\pi}{4}\right) \\ &\quad + O(\alpha^{-2}). \end{aligned} \tag{3.18}$$

This implies (3.10). Thus the proof is complete. □

Proof of (1.15). By (2.19), Lemmas 3.1 and 3.2, we obtain

$$\begin{aligned} \lambda'(\alpha) &= 2\sqrt{\lambda(\alpha)}\frac{d}{d\alpha}\left(\sqrt{\lambda(\alpha)}\right) \\ &= -\frac{\pi^{3/2}}{2^{2n+1}\alpha^{3/2}}\sum_{r=0}^{n}(-1)^{n+r}\binom{2n+1}{r}\sqrt{\frac{(2n-2r+1)}{2}}\cos\left((2n-2r+1)\alpha - \frac{\pi}{4}\right) \\ &\quad + O(\alpha^{-2}). \end{aligned} \tag{3.19}$$

Thus we obtain (1.15). □

Proof of (1.16). By (3.19) and Taylor expansion, we obtain

$$\begin{aligned} L_{2n+1}(\alpha) &= \int_\alpha^{2\alpha} \sqrt{1+\lambda'(x)^2}dx \qquad (3.20)\\ &= \int_\alpha^{2\alpha} \left\{ 1 + \frac{\pi^3}{2^{4n+3}x^3}\left[\sum_{r=0}^{n}(-1)^{n+r}\binom{2n+1}{r}\sqrt{\frac{(2n-2r+1)}{2}} \right.\right. \\ &\qquad \left.\left. \times \cos\left((2n-2r+1)x - \frac{\pi}{4}\right)\right]^2 + O(\alpha^{-7/2})\right\} dx. \end{aligned}$$

By direct calculation, we obtain

$$\begin{aligned} Q_{m,r} &= \int_\alpha^{2\alpha} \frac{\cos\left((2n-2r+1)x-\frac{\pi}{4}\right)\cos\left((2n-2m+1)x-\frac{\pi}{4}\right)}{x^3}dx \qquad (3.21)\\ &= \frac{1}{2}\int_\alpha^{2\alpha}\frac{\sin(4n-2r-2m+2)x}{x^3}dx + \frac{1}{2}\int_\alpha^{2\alpha}\frac{\cos(2r-2m)x}{x^3}dx \\ &:= \frac{1}{2}Q_{m,r,1} + \frac{1}{2}Q_{m,r,2}. \end{aligned}$$

By integration by parts and direct calculation as that to obtain (2.22), we obtain

$$Q_{m,r,1} = O(\alpha^{-3}) \qquad (1 \le r,\ m \le n), \qquad (3.22)$$

$$Q_{m,r,2} = O(\alpha^{-3}) \qquad (m \ne r). \qquad (3.23)$$

Finally, for $0 \le m \le n$, we obtain

$$Q_{m,m,2} = \int_\alpha^{2\alpha} \frac{1}{x^3}dx = \frac{3}{8\alpha^2}. \qquad (3.24)$$

By (3.20)–(3.24), we obtain (1.16). Thus the proof is complete. □

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