

# Exponential convergence of a non-autonomous Nicholson's blowflies model with an oscillating death rate

Zhiwen Long  <sup>1,2</sup>

<sup>1</sup>College of Mathematics and Econometrics, Hunan University, Changsha, Hunan 410082, PR China

<sup>2</sup>Department of Mathematics and Finance, Hunan University of Humanities, Science and Technology,  
Loudi, Hunan 417000, PR China

Received 15 March 2016, appeared 14 June 2016

Communicated by Ferenc Hartung

**Abstract.** This paper is concerned with a non-autonomous delayed Nicholson's blowflies model with an oscillating death rate. Under proper conditions, we employ a novel argument to establish a criterion on the global exponential convergence of the zero equilibrium point for this model. The obtained result improves and supplements existing ones. We also use numerical simulations to demonstrate our theoretical results.

**Keywords:** Nicholson's blowflies model, global exponential convergence, delay, oscillating death rate.

**2010 Mathematics Subject Classification:** 34C27, 34D23.

## 1 Introduction

The Nicholson's blowflies model

$$x'(t) = -ax(t) + \beta x(t - \tau)e^{-\gamma x(t - \tau)}, \quad (1.1)$$

was used in Gurney et al. [6] to describe the periodic oscillation in Nicholson's classic experiments [11] with the Australian sheep blowfly, *Lucilia cuprina*. Here  $\beta$  is the maximum per capita daily egg production rate,  $\frac{1}{\gamma}$  is the size at which the blowfly population reproduces at its maximum rate,  $a$  is the per capita daily adult death rate, and  $\tau$  is the generation time. As a classical model of biological systems, model (1.1) and its modifications have also been later used to describe population growth of other species, and thus, have been extensively and intensively studied by many researchers (see, e.g., [3] and the references therein).

When the model is used to describe the population dynamics with periodically varying environment, the coefficients and delays in the model are usually periodically time-varying.

---

 Email: longzw2005@126.com

Therefore, (1.1) has been frequently generalized into the following non-autonomous Nicholson's blowflies model:

$$x'(t) = -a(t)x(t) + \sum_{j=1}^m \beta_j(t)x(t - \tau_j(t))e^{-\gamma_j(t)x(t - \tau_j(t))}, \quad (1.2)$$

where  $m$  is a given positive integer,  $a : \mathbb{R} \rightarrow \mathbb{R}$  and  $\beta_j, \gamma_j, \tau_j : \mathbb{R} \rightarrow [0, +\infty)$  are bounded and continuous functions, and  $j = 1, 2, \dots, m$ . In particular, there have been extensive results on the problem of the convergence and persistence of model (1.2) in the literature. We refer the reader to [1, 3, 4, 7–10] and the references cited therein. Moreover, in these known results in [1, 3, 4, 6–11], we find the following condition that the coefficient function  $a(t)$  in the death rate is not oscillating, i.e.,

$$a(t) > 0 \quad \text{for all } t \in \mathbb{R}, \quad (1.3)$$

has been adopted as fundamental for the considered dynamic behaviors of (1.1) and (1.2).

However, as pointed out in [2, 12], equations with oscillating coefficients appear in linearizations of population dynamics models with seasonal fluctuations, where during some seasons the death or harvesting rates may be greater or lesser than the birth rate, and therefore, it is more reasonable to assume that the death rate in (1.2) is oscillating. This motivates us to establish criteria on the global exponential convergence of the zero equilibrium point for (1.2) without condition (1.3).

The remaining of this paper is organized as follows. In Section 2, we give a lemma, which tells us that some kinds of solutions to (1.2) are bounded and permanent. This result plays an important role in Section 3 to establish the global exponential convergence for (1.2) with an oscillating death rate. The paper concludes with an example to illustrate the effectiveness of the obtained results by numerical simulation.

## 2 Preliminaries

Let  $C = C([- \tau, 0], \mathbb{R})$  be the continuous functions space equipped with the supremum norm  $\|\cdot\|$ , where  $\tau = \max_{1 \leq j \leq m} \sup_{t \in \mathbb{R}} \tau_j(t)$ . Denote  $C_+ = C([- \tau, 0], \mathbb{R}_+)$  and  $\mathbb{R}_+ = [0, +\infty)$ . If  $x(t)$  is continuous and defined on  $[-\tau + t_0, \rho)$  with  $t_0, \rho \in \mathbb{R}$  and  $t_0 < \rho$ , then, for all  $t \in [t_0, \rho)$ , we define  $x_t \in C$ , in which  $x_t(\theta) = x(t + \theta)$  for all  $\theta \in [-\tau, 0]$ . Given a bounded continuous function  $g$  defined on  $\mathbb{R}$ , let  $g^+$  and  $g^-$  be defined as

$$g^+ = \sup_{t \in \mathbb{R}} |g(t)|, \quad g^- = \inf_{t \in \mathbb{R}} |g(t)|.$$

According to the biological interpretation of (1.2), only positive solutions are meaningful and therefore admissible. Consequently, the initial conditions are given by

$$x_{t_0} = \varphi, \quad \varphi \in C_+ \quad \text{and} \quad \varphi(0) > 0. \quad (2.1)$$

Denote  $x_t(t_0, \varphi)(x(t; t_0, \varphi))$  for a solution of the admissible initial value problem (1.2) and (2.1) with  $x_{t_0}(t_0, \varphi) = \varphi \in C_+$  and  $t_0 \in \mathbb{R}$ . Moreover, let  $[t_0, \eta(\varphi))$  be the maximal right-interval of existence of  $x_t(t_0, \varphi)$ .

**Lemma 2.1.** *Let  $a^* : \mathbb{R} \rightarrow (0, +\infty)$  be a bounded and continuous function with  $a^{*-} > 0$ , and  $M$  be a nonnegative constant such that*

$$\int_s^t (a^*(u) - a(u)) du \leq M \quad \text{for all } t, s \in \mathbb{R} \text{ and } t - s \geq 0, \quad (2.2)$$

then for any  $t_0 \in \mathbb{R}$ , the solution  $x(t; t_0, \varphi)$  satisfies

$$x(t; t_0, \varphi) > 0 \quad \text{for all } t \in [t_0, \eta(\varphi)),$$

and  $\eta(\varphi) = +\infty$ .

*Proof.* Since  $\varphi \in C_+$ , using Theorem 5.2.1 [13, p. 46] we have  $x_t(t_0, \varphi) \in C_+$  for all  $t \in [t_0, \eta(\varphi))$ . For the sake of convenience, we denote  $x(t; t_0, \varphi)$  by  $x(t)$ . Multiplying both sides of (1.2) by  $e^{\int_{t_0}^t a(v)dv}$ , and integrating it on  $[t_0, t]$ , by virtue of (2.1), we have

$$x(t) = e^{-\int_{t_0}^t a(v)dv} x(t_0) + \int_{t_0}^t e^{-\int_s^t a(v)dv} \sum_{j=1}^m \beta_j(s) x(s - \tau_j(s)) e^{-\gamma_j(s)x(s - \tau_j(s))} ds, \quad (2.3)$$

for all  $t \in [t_0, \eta(\varphi))$ .

We first claim that

$$x(t) > 0 \quad \text{for all } t \in [t_0, \eta(\varphi)). \quad (2.4)$$

If not, then there exists  $t_1 \in (t_0, \eta(\varphi))$  such that

$$x(t_1) = 0 \quad \text{and} \quad x(t) > 0 \quad \text{for all } t \in [t_0 - \tau, t_1).$$

Observe that

$$\beta_j(t)x(t - \tau_j(t)) \geq 0 \quad \text{for all } t \in [t_0, t_1],$$

(2.3) and the fact that  $x(t_0) = \varphi(0) > 0$  yield

$$\begin{aligned} 0 &= x(t_1) \\ &= e^{-\int_{t_0}^{t_1} a(v)dv} x(t_0) + \int_{t_0}^{t_1} e^{-\int_s^{t_1} a(v)dv} \sum_{j=1}^m \beta_j(s) x(s - \tau_j(s)) e^{-\gamma_j(s)x(s - \tau_j(s))} ds \\ &\geq e^{-\int_{t_0}^{t_1} a(v)dv} x(t_0) \\ &> 0, \end{aligned}$$

which is a contradiction and proves (2.4).

Next, we prove the global existence of  $x(t; t_0, \varphi)$ , which means  $\eta(\varphi) = +\infty$ . It follows from (2.2) that

$$e^{-\int_s^t a(u)du} \leq e^M e^{-\int_s^t a^*(u)du} \quad \text{for all } t, s \in \mathbb{R} \text{ and } t - s \geq 0.$$

In particular,

$$e^{-\int_{t_0}^t a(u)du} \leq e^M e^{-\int_{t_0}^t a^*(u)du} \quad \text{for all } t > t_0. \quad (2.5)$$

By (2.3) and (2.5), and using the fact that  $\sup_{u \geq 0} ue^{-u} = \frac{1}{e}$ , we obtain

$$\begin{aligned} x(t) &= e^{-\int_{t_0}^t a(v)dv} x(t_0) + \int_{t_0}^t e^{-\int_s^t a(v)dv} \sum_{j=1}^m \beta_j(s) x(s - \tau_j(s)) e^{-\gamma_j(s)x(s - \tau_j(s))} ds \\ &\leq e^M e^{-\int_{t_0}^t a^*(v)dv} x(t_0) + \int_{t_0}^t e^M e^{-\int_s^t a^*(v)dv} \sum_{j=1}^m \beta_j(s) x(s - \tau_j(s)) e^{-\gamma_j(s)x(s - \tau_j(s))} ds \\ &\leq e^M x(t_0) + e^{M-1} \sum_{j=1}^m \int_{t_0}^t e^{-\int_s^t a^*(v)dv} \frac{\beta_j(s)}{\gamma_j(s)} ds \\ &\leq e^M x(t_0) + e^{M-1} \sum_{j=1}^m \frac{(\beta_j / \gamma_j)^+}{a^* -}, \quad \text{for all } t \in [t_0, \eta(\varphi)), \end{aligned}$$

which, combining with (2.4) and the continuation theorem (see Theorem 2.3.1 in [5]), implies that  $\eta(\varphi) = +\infty$ . This ends the proof of Lemma 2.1.  $\square$

### 3 Main result

We are now in a position to establish new criteria on the global exponential convergence of the zero equilibrium point for (1.2) with an oscillating death rate.

**Theorem 3.1.** *Under the assumptions of Lemma 2.1, and suppose further that*

$$\sup_{t \in \mathbb{R}} \left[ -a^*(t) + e^M \sum_{j=1}^m \beta_j(t) \right] < 0, \quad (3.1)$$

then there exist two positive constants  $L$  and  $\lambda$  such that

$$|x(t; t_0, \varphi)| \leq Le^{-\lambda t} \quad \text{for all } t \geq t_0,$$

where  $x(t; t_0, \varphi)$  is the solution of (1.2) with initial condition (2.1).

*Proof.* From (3.1), we can choose a constant  $\lambda \in (0, \inf_{t \in \mathbb{R}} a^*(t))$  such that

$$\sup_{t \in \mathbb{R}} \left[ \lambda - a^*(t) + e^M \sum_{j=1}^m \beta_j(t) e^{\lambda \tau_j(t)} \right] < 0. \quad (3.2)$$

Let  $K = e^M + 1$ , for any  $\varepsilon > 0$ , it is clear that

$$|x(t)| < \|x\| + \varepsilon < K(\|\varphi\| + \varepsilon) e^{\lambda t_0} e^{-\lambda t} \quad \text{for all } t \in [t_0 - \tau, t_0].$$

We claim that

$$|x(t)| < K(\|\varphi\| + \varepsilon) = K(\|\varphi\| + \varepsilon) e^{\lambda t_0} e^{-\lambda t} \quad \text{for all } t > t_0. \quad (3.3)$$

Otherwise, there exists  $T > t_0$ , such that

$$\begin{cases} |x(T)| = K(\|\varphi\| + \varepsilon) e^{\lambda t_0} e^{-\lambda T}, \\ |x(t)| < K(\|\varphi\| + \varepsilon) e^{\lambda t_0} e^{-\lambda t} \quad \text{for all } t \in [t_0, T). \end{cases} \quad (3.4)$$

On the other hand, in view of (2.3) and (2.5), we have

$$\begin{aligned} |x(T)| &= e^{-\int_{t_0}^T a(v)dv} x(t_0) + \int_{t_0}^T e^{-\int_s^T a(v)dv} \sum_{j=1}^m \beta_j(s) x(s - \tau_j(s)) e^{-\gamma_j(s)x(s - \tau_j(s))} ds \\ &\leq e^M e^{-\int_{t_0}^T a^*(v)dv} x(t_0) + \int_{t_0}^T e^M e^{-\int_s^T a^*(v)dv} \sum_{j=1}^m \beta_j(s) x(s - \tau_j(s)) e^{-\gamma_j(s)x(s - \tau_j(s))} ds \\ &\leq K(\|\varphi\| + \varepsilon) \frac{e^M}{K} e^{-\int_{t_0}^T a^*(v)dv} \\ &\quad + \int_{t_0}^T e^M e^{-\int_s^T a^*(v)dv} \sum_{j=1}^m \beta_j(s) K(\|\varphi\| + \varepsilon) e^{\lambda t_0} e^{-\lambda(s - \tau_j(s))} e^{-\gamma_j(s)x(s - \tau_j(s))} ds \\ &\leq K(\|\varphi\| + \varepsilon) e^{\lambda t_0} e^{-\lambda T} \left[ \frac{e^M}{K} e^{-\int_{t_0}^T (a^*(v) - \lambda)dv} + \int_{t_0}^T e^{-\int_s^T (a^*(v) - \lambda)dv} e^M \sum_{j=1}^m \beta_j(s) e^{\lambda \tau_j(s)} ds \right] \end{aligned}$$

$$\begin{aligned}
 &\leq K(\|\varphi\| + \varepsilon)e^{\lambda t_0}e^{-\lambda T} \left[ \frac{e^M}{K}e^{-\int_{t_0}^T(a^*(v)-\lambda)dv} + \int_{t_0}^T e^{-\int_s^T(a^*(v)-\lambda)dv}(a^*(s) - \lambda)ds \right] \\
 &= K(\|\varphi\| + \varepsilon)e^{\lambda t_0}e^{-\lambda T} \left[ 1 - \left(1 - \frac{e^M}{K}\right)e^{-\int_{t_0}^T(a^*(v)-\lambda)dv} \right] \\
 &< K(\|\varphi\| + \varepsilon)e^{\lambda t_0}e^{-\lambda T},
 \end{aligned} \tag{3.5}$$

which contradicts to the first equation in (3.4). Hence, (3.3) holds. Letting  $\varepsilon \rightarrow 0^+$ , it follows from (3.3) that

$$|x(t)| \leq Le^{-\lambda t} \quad \text{for all } t > t_0,$$

where  $L = K\|\varphi\|e^{\lambda t_0}$ . The proof is complete.  $\square$

## 4 An example

In this section, we give an example and its numerical simulations to demonstrate the result obtained in Section 3.

**Example 4.1.** Consider the following Nicholson's blowflies model with an oscillating death rate:

$$\begin{aligned}
 x'(t) = &-(8 + 10 \cos 2000t)x(t) \\
 &+ \left(\frac{1}{2} + \frac{1}{2}|\sin \sqrt{2}t|\right)x(t - |\sin 2t|)e^{-(1+\frac{1}{10}|\sin \sqrt{7}t|)x(t-|\sin 2t|)} \\
 &+ \left(\frac{1}{2} + \frac{1}{2}|\sin \sqrt{3}t|\right)x(t - |\sin 3t|)e^{-(1+\frac{1}{10}|\sin \sqrt{5}t|)x(t-|\sin 3t|)},
 \end{aligned} \tag{4.1}$$

where  $a(t) = 8 + 10 \cos 2000t$ ,  $\beta_1(t) = \frac{1}{2} + \frac{1}{2}|\sin \sqrt{2}t|$ ,  $\beta_2(t) = \frac{1}{2} + \frac{1}{2}|\sin \sqrt{3}t|$ ,  $\tau_1(t) = |\sin 2t|$ ,  $\tau_2(t) = |\sin 3t|$ ,  $\gamma_1(t) = 1 + \frac{1}{10}|\sin \sqrt{7}t|$ ,  $\gamma_2(t) = 1 + \frac{1}{10}|\sin \sqrt{5}t|$ . Clearly,  $\beta_j(t) \leq 1$ ,  $\tau_j(t) \leq 1$ ,  $j = 1, 2$ . Let  $a^*(t) = 8$ ,  $M = \frac{1}{100}$ , then

$$\int_s^t (a^*(u) - a(u))du \leq M, \quad \text{for all } t, s \in \mathbb{R} \text{ and } t - s \geq 0.$$

Moreover, let  $\lambda = \frac{99}{100}$ , a simple calculation shows that

$$\left[ \lambda - a^*(t) + e^M \sum_{j=1}^m \beta_j(t) e^{\lambda \tau_j(t)} \right] < \frac{99}{100} + 2e - 8 < 0.$$

Then (4.1) satisfies all the conditions in Theorem 3.1. It follows that all solutions of (4.1) with initial conditions in (2.1) converge to the zero equilibrium point as  $t \rightarrow +\infty$ . This fact is verified by the numerical simulations in Figure 4.1.

**Remark 4.2.** To the best of our knowledge, no results on the dynamics of (1.2) with an oscillating death rate have been reported up to now and we also mention that none of the results in the references [1–4, 7–10, 12] can be applied to (4.1), which implies that the obtained results in the present paper are completely new and extend previously known results to some extent.

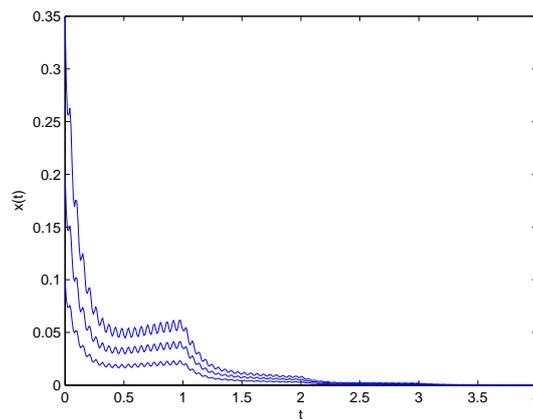


Figure 4.1: Numerical solutions  $x(t)$  of (4.1) with initial values  $x_0 \equiv 0.1, 0.2, 0.35$ , respectively.

## Acknowledgements

The author would like to thank the associate editor and the anonymous reviewer for their valuable comments and constructive suggestions, which helped to enrich the content and greatly improve the presentation of this paper. The research is supported by National Natural Science Foundation of China (11171098).

## References

- [1] P. AMSTER, A. DÉBOLI, Existence of positive  $T$ -periodic solutions of a generalized Nicholson's blowflies model with a nonlinear harvesting term, *Appl. Math. Lett.* **25**(2012), 1203–1207. [MR2930746](#); [url](#)
- [2] L. BEREZANSKY, E. BRAVERMAN, On exponential stability of a linear delay differential equation with an oscillating coefficient, *Appl. Math. Lett.* **22**(2009), 1833–1837. [MR2558549](#); [url](#)
- [3] L. BEREZANSKY, E. BRAVERMAN, L. IDELS, Nicholson's blowflies differential equations revisited: Main results and open problems, *Appl. Math. Modelling* **34**(2010), 1405–1417. [MR2592579](#); [url](#)
- [4] L. DUAN, L. HUANG, Pseudo almost periodic dynamics of delay Nicholson's blowflies model with a linear harvesting term, *Math. Methods Appl. Sci.* **38**(2015), 1178–1189. [MR3338142](#); [url](#)
- [5] J. K. HALE, S. M. VERDUYN LUNEL, *Introduction to functional differential equations*, Applied Mathematical Sciences, Vol. 3, Springer-Verlag, New York, 1977. [MR1243878](#)
- [6] W. GURNEY, S. BLYTHE, R. NISBET, Nicholson's blowflies revisited, *Nature* **287**(1980), 17–21.
- [7] L. V. HIEN, Global asymptotic behaviour of positive solutions to a non-autonomous Nicholson's blowflies model with delays, *J. Biol. Dyn.* **8**(2014), 135–144. [MR3271022](#); [url](#)

- [8] X. HOU, L. DUAN, New results on periodic solutions of delayed Nicholson's blowflies models, *Electron. J. Qual. Theory Differ. Equ.* **24**(2012), 1–11. [MR2900484](#)
- [9] X. HOU, L. DUAN, Z. HUANG, Permanence and periodic solutions for a class of delay Nicholson's blowflies models, *Appl. Math. Modelling* **37**(2013), 1537–1544. [MR3002241](#); [url](#)
- [10] B. LIU, Global exponential stability of positive periodic solutions for a delayed Nicholson's blowflies model, *J. Math. Anal. Appl.* **412**(2014), 212–221. [MR3145795](#); [url](#)
- [11] A. NICHOLSON, An outline of the dynamics of animal populations, *Aust. J. Zool.* **2**(1954), 19–65.
- [12] J. SHAO, Pseudo almost periodic solutions for a Lasota–Ważewska model with an oscillating death rate, *Appl. Math. Lett.* **43**(2015), 90–95. [MR3305635](#); [url](#)
- [13] H. L. SMITH, *Monotone dynamical systems: an introduction to the theory of competitive and cooperative systems*, Mathematical Surveys and Monographs, Vol. 41, Providence, RI, 1995. [MR1319817](#)