# EXISTENCE OF A POSITIVE SOLUTION TO A RIGHT FOCAL BOUNDARY VALUE PROBLEM

RICHARD I. AVERY, JOHNNY HENDERSON AND DOUGLAS R. ANDERSON

ABSTRACT. In this paper we apply the recent extension of the Leggett-Williams Fixed Point Theorem which requires neither of the functional boundaries to be invariant to the second order right focal boundary value problem. We demonstrate a technique that can be used to deal with a singularity and provide a non-trivial example.

## 1. INTRODUCTION

The recent topological proof and extension of the Leggett-Williams fixed point theorem [3] does not require either of the functional boundaries to be invariant with respect to a functional wedge and its proof uses topological methods instead of axiomatic index theory. Functional fixed point theorems (including [2, 4, 5, 6, 8]) can be traced back to Leggett and Williams [7] when they presented criteria which guaranteed the existence of a fixed point for a completely continuous map that did not require the operator to be invariant with regard to the concave functional boundary of a functional wedge. Avery, Henderson, and O'Regan [1], in a dual of the Leggett-Williams fixed point theorem, gave conditions which guaranteed the existence of a fixed point for a completely continuous map that did not require the operator to be invariant et a fixed point for a completely continuous map that did not require the operator to be invariant et a technique to take advantage of the added flexibility of the new fixed point theorem for a right focal boundary value problem.

#### 2. Preliminaries

In this section we will state the definitions that are used in the remainder of the paper.

**Definition 1.** Let E be a real Banach space. A nonempty closed convex set  $P \subset E$  is called a cone if it satisfies the following two conditions:

(i)  $x \in P, \lambda \ge 0$  implies  $\lambda x \in P$ ; (ii)  $x \in P, -x \in P$  implies x = 0.

Every cone  $P \subset E$  induces an ordering in E given by

 $x \leq y$  if and only if  $y - x \in P$ .

**Definition 2.** An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

<sup>2000</sup> Mathematics Subject Classification. 34B10.

Key words and phrases. Fixed-point theorems, positive solutions, singularities, cone.

**Definition 3.** A map  $\alpha$  is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E if  $\alpha : P \to [0, \infty)$  is continuous and

$$\alpha(tx + (1-t)y) \ge t\alpha(x) + (1-t)\alpha(y)$$

for all  $x, y \in P$  and  $t \in [0, 1]$ . Similarly we say the map  $\beta$  is a nonnegative continuous convex functional on a cone P of a real Banach space E if  $\beta : P \to [0, \infty)$  is continuous and

$$\beta(tx + (1-t)y) \le t\beta(x) + (1-t)\beta(y)$$

for all  $x, y \in P$  and  $t \in [0, 1]$ .

Let  $\alpha$  and  $\psi$  be non-negative continuous concave functionals on P and  $\delta$  and  $\beta$  be non-negative continuous convex functionals on P; then, for non-negative real numbers a, b, c and d, we define the following sets:

(1) 
$$A := A(\alpha, \beta, a, d) = \{ x \in P : a \le \alpha(x) \text{ and } \beta(x) \le d \},\$$

(2) 
$$B := B(\alpha, \delta, \beta, a, b, d) = \{x \in A : \delta(x) \le b\},\$$

and

(3) 
$$C := C(\alpha, \psi, \beta, a, c, d) = \{x \in A : c \le \psi(x)\}.$$

We say that A is a functional wedge with concave functional boundary defined by the concave functional  $\alpha$  and convex functional boundary defined by the convex functional  $\beta$ . We say that an operator  $T : A \to P$  is invariant with respect to the concave functional boundary, if  $a \leq \alpha(Tx)$  for all  $x \in A$ , and that T is invariant with respect to the convex functional boundary, if  $\beta(Tx) \leq d$  for all  $x \in A$ . Note that A is a convex set. The following theorem is an extension of the original Leggett-Williams fixed point theorem [7].

**Theorem 4.** [Extension of Leggett-Williams] Suppose P is a cone in a real Banach space E,  $\alpha$  and  $\psi$  are non-negative continuous concave functionals on P,  $\delta$  and  $\beta$  are non-negative continuous convex functionals on P, and for non-negative real numbers a, b, c and d the sets A, B and C are as defined in (1), (2) and (3). Furthermore, suppose that A is a bounded subset of P, that  $T : A \to P$  is completely continuous and that the following conditions hold:

- (A1)  $\{x \in A : c < \psi(x) \text{ and } \delta(x) < b\} \neq \emptyset \text{ and } \{x \in P : \alpha(x) < a \text{ and } d < \beta(x)\} = \emptyset;$
- (A2)  $\alpha(Tx) \ge a \text{ for all } x \in B;$
- (A3)  $\alpha(Tx) \ge a$  for all  $x \in A$  with  $\delta(Tx) > b$ ;
- (A4)  $\beta(Tx) \leq d$  for all  $x \in C$ ; and,
- (A5)  $\beta(Tx) \leq d$  for all  $x \in A$  with  $\psi(Tx) < c$ .

Then T has a fixed point  $x^* \in A$ .

## 3. RIGHT FOCAL BOUNDARY VALUE PROBLEM

In this section we will illustrate the key techniques for verifying the existence of a positive solution for a boundary value problem using the newly developed extension of the Leggett-Williams fixed point theorem, applying the properties of a Green's function, bounding the EJQTDE, 2010 No. 5, p. 2

nonlinearity by constants over some intervals, and using concavity to deal with a singularity. Consider the second order nonlinear focal boundary value problem

(4) 
$$x''(t) + f(x(t)) = 0, \quad t \in (0, 1).$$

(5) 
$$x(0) = 0 = x'(1),$$

where  $f: \mathbb{R} \to [0, \infty)$  is continuous. If x is a fixed point of the operator T defined by

$$Tx(t) := \int_0^1 G(t,s)f(x(s))ds,$$

where

$$G(t,s) = \begin{cases} t & : t \leq s, \\ s & : s \leq t, \end{cases}$$

is the Green's function for the operator L defined by

$$Lx(t) := -x''$$

with right-focal boundary conditions

$$x(0) = 0 = x'(1),$$

then it is well known that x is a solution of the boundary value problem (4), (5). Throughout this section of the paper we will use the facts that G(t, s) is nonnegative, and for each fixed  $s \in [0, 1]$ , the Green's function is nondecreasing in t.

Define the cone  $P \subset E = C[0, 1]$  by

 $P := \{x \in E : x \text{ is nonnegative, nondecreasing, and concave}\}.$ 

For fixed  $\nu, \tau, \mu \in [0, 1]$  and  $x \in P$ , define the concave functionals  $\alpha$  and  $\psi$  on P by

$$\alpha(x) := \min_{t \in [\tau, 1]} x(t) = x(\tau), \ \psi(x) := \min_{t \in [\mu, 1]} x(t) = x(\mu),$$

and the convex functionals  $\delta$  and  $\beta$  on P by

$$\delta(x) := \max_{t \in [0,\nu]} x(t) = x(\nu), \quad \beta(x) := \max_{t \in [0,1]} x(t) = x(1).$$

In the following theorem, we demonstrate how to apply the Extension of the Leggett-Williams Fixed Point Theorem (Theorem 4), to prove the existence of at least one positive solution to (4), (5).

**Theorem 5.** If  $\tau, \nu, \mu \in (0, 1]$  are fixed with  $\tau \leq \mu < \nu \leq 1$ , d and m are positive real numbers with  $0 < m \leq d\mu$  and  $f : [0, \infty) \to [0, \infty)$  is a continuous function such that

 $\begin{array}{l} (a) \ f(w) \geq \frac{d}{\nu - \tau} \ for \ w \in [\tau d, \nu d], \\ (b) \ f(w) \ is \ decreasing \ for \ w \in [0, m] \ with \ f(m) \geq f(w) \ for \ w \in [m, d], \ and \\ (c) \ \int_0^\mu \ s \ f\left(\frac{ms}{\mu}\right) \ ds \leq \frac{2d - f(m)(1 - \mu^2)}{2}, \end{array}$ 

then the operator T has at least one positive solution  $x^* \in A(\alpha, \beta, \tau d, d)$ .

*Proof.* Let  $a = \tau d$ ,  $b = \nu d = \frac{a\nu}{\tau}$ , and  $c = d\mu$ . Let  $x \in A(\alpha, \beta, a, d)$  then if  $t \in (0, 1)$ , by the properties of the Green's function (Tx)''(t) = -f(x(t)) and Tx(0) = 0 = (Tx)'(1), thus

$$T: A(\alpha, \beta, a, d) \to P.$$

We will also take advantage of the following property of the Green's function. For any  $y, w \in [0, 1]$  with  $y \leq w$  we have

(6) 
$$\min_{s \in [0,1]} \frac{G(y,s)}{G(w,s)} \ge \frac{y}{w}.$$

By the Arzela-Ascoli Theorem it is a standard exercise to show that T is a completely continuous operator using the properties of G and f, and by the definition of  $\beta$ , we have that A is a bounded subset of the cone P. Also, if  $x \in P$  and  $\beta(x) > d$ , then by the properties of the cone P,

$$\alpha(x) = x(\tau) \ge \left(\frac{\tau}{1}\right) x(1) = \tau\beta(x) > \tau d = a.$$

Therefore,

$$\{x \in P : \alpha(x) < a \text{ and } d < \beta(x)\} = \emptyset.$$

For any  $K \in \left(\frac{2d}{2-\mu}, \frac{2d}{2-\nu}\right)$  the function  $x_K$  defined by

$$x_K(t) \equiv \int_0^1 KG(t,s)ds = \frac{Kt(2-t)}{2} \in A$$

since

$$\alpha(x_K) = x_K(\tau) = \frac{K\tau(2-\tau)}{2} > \frac{d\tau(2-\tau)}{2-\mu} \ge d\tau = a,$$
  
$$\beta(x_K) = x_K(1) = \frac{K}{2} < \frac{d}{2-\nu} \le d,$$

and  $x_K$  has the properties that

$$\psi(x_K) = x_K(\mu) = \frac{K\mu(2-\mu)}{2} > \left(\frac{2d}{2-\mu}\right) \left(\frac{\mu(2-\mu)}{2}\right) = d\mu = c$$

and

$$\delta(x_K) = x_K(\nu) = \frac{K\nu(2-\nu)}{2} < \left(\frac{2d}{2-\nu}\right)\left(\frac{\nu(2-\nu)}{2}\right) = d\nu = b.$$

Hence

$$\{x \in A : c < \psi(x) \text{ and } \delta(x) < b\} \neq \emptyset.$$

Claim 1:  $\alpha(Tx) \ge a$  for all  $x \in B$ .

Let  $x \in B$ . Thus by condition (a),

$$\alpha(Tx) = \int_0^1 G(\tau, s) f(x(s)) ds \ge \left(\frac{a}{\tau(\nu - \tau)}\right) \int_{\tau}^{\nu} G(\tau, s) ds$$
$$= \left(\frac{a}{\tau(\nu - \tau)}\right) (\tau(\nu - \tau)) = a.$$

Claim 2:  $\alpha(Tx) \ge a$ , for all  $x \in A$  with  $\delta(Tx) > b$ .

Let  $x \in A$  with  $\delta(Tx) > b$ . Thus by the properties of G (6),

$$\begin{aligned} \alpha(Tx) &= \int_0^1 G(\tau, s) \ f(x(s)) \ ds \ge \left(\frac{\tau}{\nu}\right) \int_0^1 \ G(\nu, s) \ f(x(s)) \ ds \\ &= \left(\frac{\tau}{\nu}\right) \delta(Tx) > \left(\frac{\tau}{\nu}\right) (d\nu) = a. \end{aligned}$$

Claim 3:  $\beta(Tx) \leq d$ , for all  $x \in C$ .

Let  $x \in C$ , thus by the concavity of x, for  $s \in [0, \mu]$  we have

$$x(s) \ge \frac{cs}{\mu} \ge \frac{ms}{\mu}$$

Hence by properties (b) and (c),

$$\begin{split} \beta(Tx) &= \int_0^1 G(1,s) \ f(x(s)) \ ds = \int_0^1 \ s \ f(x(s)) \ ds \\ &= \int_0^\mu \ s \ f(x(s)) \ ds + \int_\mu^1 \ s \ f(x(s)) \ ds \\ &\leq \int_0^\mu \ s \ f\left(\frac{ms}{\mu}\right) \ ds + f(m) \int_\mu^1 \ s \ ds \\ &\leq \frac{2d - f(m)(1 - \mu^2)}{2} + \frac{f(m)(1 - \mu^2)}{2} = d. \end{split}$$

Claim 4:  $\beta(Tx) \leq d$ , for all  $x \in A$  with  $\psi(Tx) < c$ .

Let  $x \in A$  with  $\psi(Tx) < c$ . Thus by the properties of G (6),

$$\beta(Tx) = \int_0^1 G(1,s) f(x(s)) ds \le \left(\frac{1}{\mu}\right) \int_0^1 G(\mu,s) f(x(s)) ds$$
$$= \left(\frac{1}{\mu}\right) Tx(\mu) = \left(\frac{1}{\mu}\right) \psi(Tx) \le \left(\frac{1}{\mu}\right) c = d.$$

Therefore, the hypotheses of Theorem 4 have been satisfied; thus the operator T has at least one positive solution  $x^* \in A(\alpha, \beta, a, d)$ .

We note that because of the concavity of solutions, the proof of Theorem 5 remains valid for certain singular nonlinearities as presented in this example.

Example: Let

$$d = \frac{5}{4}, \ \tau = \frac{1}{16}, \ \mu = \frac{3}{4}, \ \text{and} \ \nu = \frac{15}{16}.$$
  
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Then the boundary value problem

$$x'' + \frac{1}{\sqrt{x}} + \sqrt{x} = 0,$$

with right-focal boundary conditions

$$x(0) = 0 = x'(1),$$

has at least one positive solution  $x^*$  which can be verified by the above theorem, with

$$5/64 \le x^*(1/16)$$
 and  $x^*(1) \le 5/4$ .

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(Received November 23, 2009)

COLLEGE OF ARTS AND SCIENCES, DAKOTA STATE UNIVERSITY, MADISON, SOUTH DAKOTA 57042 USA *E-mail address*: rich.avery@dsu.edu

DEPARTMENT OF MATHEMATICS, BAYLOR UNIVERSITY, WACO, TEXAS 76798 USA *E-mail address*: Johnny\_Henderson@baylor.edu

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, CONCORDIA COLLEGE, MOORHEAD, MN 56562 USA

 $E\text{-}mail\ address: \ \texttt{andersod} \texttt{@cord.edu}$