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Existence of extremal solutions of a three-point boundary value problem for a general second order $p-$Laplacian integro-differential equation

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Abstract

In this paper, we prove the existence of extremal positive, concave and pseudo-symmetric solutions for a general three-point second order $p-$Laplacian integro-differential boundary value problem by using an abstract monotone iterative technique.
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1 Introduction

The subject of multi-point second order boundary value problems, initiated by Il'in and Moiseev [9, 10], has been extensively addressed by many authors, for instance, see [6, 7, 13, 16, 17]. There has also been a considerable attention on $p-$Laplacian boundary value problems [3, 8, 12, 19] as $p-$Laplacian appears in the study of flow through porous media ($p = 3/2$), nonlinear elasticity ($p \geq 2$), glaciology ($1 \leq p \leq 4/3$), etc. Recently, Sun and Ge [18] discussed the existence of positive pseudo-symmetric solutions for a second order three-point boundary value problem involving $p-$Laplacian operator given by

$$(\psi_p(u'(t)))' + q(t)f(t, u(t), u'(t)) = 0, \ t \in (0, 1),$$

$$u(0) = 0, \qquad u(\eta) = u(1), \quad 0 < \eta < 1.$$

Ahmad and Nieto [1] studied a three-point second order $p-$Laplacian integro-differential boundary value problem with the non-integral term of the form $f(t, x(t))$. In this paper, we allow the nonlinear function f to depend on x' along with x and consider a more general three-point second order $p-$Laplacian integro-differential boundary value problem of the form

$$(\psi_p(x'(t)))' + a(t)\left(f(t, x(t), x'(t)) + \int_t^{(1+\eta)/2} K(t, \zeta, x(\zeta))d\zeta \right) = 0, \ t \in (0, 1), \quad (1.1)$$

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$$x(0) = 0, \qquad x(\eta) = x(1), \quad 0 < \eta < 1, \tag{1.2}$$

where $p > 1$, $\psi_p(s) = s|s|^{p-2}$. Let ψ_q be the inverse of ψ_p.

We apply an abstract monotone iterative technique due to Amann [2] to prove the existence of extremal positive, concave and pseudo-symmetric solutions for (1.1)-(1.2). For the details of the abstract monotone iterative method, we refer the reader to the papers [1, 4-5, 14-15, 18]. The importance of the work lies in the fact that integro-differential equations are encountered in many areas of science where it is necessary to take into account aftereffect or delay. Especially, models possessing hereditary properties are described by integro-differential equations in practice. Also, the governing equations in the problems of biological sciences such as spreading of disease by the dispersal of infectious individuals, the reaction-diffusion models in ecology to estimate the speed of invasion, etc. are integro-differential equations.

2 Preliminaries

Let $E = C^1[0,1]$ be the Banach space equipped with norm $\|x\| = \max_{0\le t\le 1}[x^2(t) + (x'(t))^2]^{1/2}$ and let P be a cone in E defined by $P = \{x \in E :$ x is nonnegative, concave on $[0,1]$ and pseudo-symmetric about $(1+\eta)/2$ on $[0,1]\}$. Further, for $\theta > 0$, let $\overline{P}_\theta = \{x \in P : \|x\| \le \theta\}$.
A functional γ is said to be concave on $[0,1]$ if

$$\gamma(tx + (1-t)y) \ge t\gamma(x) + (1-t)\gamma(y), \ \forall x, y \in [0,1] \ \text{ and } t \in [0,1].$$

A function x is said to be pseudo-symmetric about $(1+\eta)/2$ on $[0,1]$ if x is symmetric on the interval $[\eta, 1]$, that is, $x(t) = x(1-(t-\eta))$ for $t \in [\eta, 1]$.

Throughout the paper, we assume that

($\mathbf{A_1}$) $f(t,x,y) : [0,1] \times [0,\infty) \times R \to [0,\infty)$ is continuous with $f(t,x_1,y_1) \le f(t,x_2,y_2)$, for any $0 \le t \le 1$, $0 \le x_1 \le x_2 \le \theta$, $0 \le |y_1| \le |y_2| \le \theta$ (f is nondecreasing in x and $|y|$) and $f(t,x,y)$ is pseudo-symmetric in t about $(1+\eta)/2$ on $(0,1)$ for any fixed $x \in [0,\infty)$, $y \in R$. Moreover, $f(t,0,0)$ is not identically equal to zero on any subinterval of $(0,1)$.

($\mathbf{A_2}$) $K(t,\zeta,x) : [0,1] \times [0,1] \times [0,\infty) \to [0,\infty)$ is continuous, nondecreasing in x and for any fixed $(\zeta,x) \in [0,1] \times [0,\infty)$, $K(t,\zeta,x)$ is pseudo-symmetric in t and ζ about $(1+\eta)/2$ on $(0,1)$. Further, $K(t,\zeta,0)$ is not identically equal to zero for $0 \le t, \zeta \le 1$.

($\mathbf{A_3}$) $a(t) \in L(0,1)$ is nonnegative on $(0,1)$ and pseudo-symmetric in t about $(1+\eta)/2$ on $(0,1)$. Further, $a(t)$ is not identically zero on any nontrivial compact subinterval of $(0,1)$.

$(\mathbf{A_4})$ $\max_{0\le t\le 1}\left\{f(t,\theta,\theta)+\int_t^{(1+\eta)/2}K(t,\zeta,\theta)d\zeta\right\}\le\psi_p(\theta/\Theta)$,
where $\Theta=\max\{\sqrt{2}\Theta_1,\sqrt{2}\Theta_2\}$,

$$\Theta_1=\int_0^{(1+\eta)/2}\psi_q\Big(\int_w^{(1+\eta)/2}a(\nu)d\nu\Big)dw,\quad \Theta_2=\psi_q\Big(\int_0^{(1+\eta)/2}a(\nu)d\nu\Big).$$

Definition 2.1. Let us define an operator $\mathcal{G}:P\to E$ as follows

$$(\mathcal{G}x)(t)=\begin{cases}\int_0^t\psi_q\Big[\int_w^{(1+\eta)/2}a(\nu)\Big(f(\nu,x(\nu),x'(\nu))+\int_\nu^{(1+\eta)/2}K(\nu,\zeta,x(\zeta))d\zeta\Big)d\nu\Big]dw,\\ \hfill t\in[0,(1+\eta)/2];\\ \int_0^\eta\psi_q\Big[\int_w^{(1+\eta)/2}a(\nu)\Big(f(\nu,x(\nu),x'(\nu))+\int_\nu^{(1+\eta)/2}K(\nu,\zeta,x(\zeta))d\zeta\Big)d\nu\Big]dw\\ \quad+\int_t^1\psi_q\Big[\int_{(1+\eta)/2}^w a(\nu)\Big(f(\nu,x(\nu),x'(\nu))+\int_{(1+\eta)/2}^\nu K(\nu,\zeta,x(\zeta))d\zeta\Big)d\nu\Big]dw,\\ \hfill t\in[(1+\eta)/2,1].\end{cases}$$

By the definition of $\mathcal{G}$, it follows that $\mathcal{G}x\in C^1[0,1]$ and is nonnegative for each $x\in P$, and is a solution of (1.1) and (1.2) if and only if $\mathcal{G}x=x$.
In order to develop the iteration schemes for (1.1) and (1.2), we establish some properties of the operator $\mathcal{G}x$.
Since

$$(\mathcal{G}x)'(t)=\begin{cases}\psi_q\Big[\int_t^{(1+\eta)/2}a(\nu)\Big(f(\nu,x(\nu),x'(\nu))+\int_\nu^{(1+\eta)/2}K(\nu,\zeta,x(\zeta))d\zeta\Big)d\nu\Big],\\ \hfill t\in[0,(1+\eta)/2];\\ -\psi_q\Big[\int_{(1+\eta)/2}^t a(\nu)\Big(f(\nu,x(\nu),x'(\nu))+\int_{(1+\eta)/2}^\nu K(\nu,\zeta,x(\zeta))d\zeta\Big)d\nu\Big],\\ \hfill t\in[(1+\eta)/2,1],\end{cases}$$

is continuous and nonincreasing on $[0,1]$ with $(\mathcal{G}x)'((1+\eta)/2)=0$, therefore, it follows that $\mathcal{G}x$ is concave. The nondecreasing nature of $\mathcal{G}x$ in x and $|x'|$ follows from the assumptions (A_1) and (A_2). Now, we show that $\mathcal{G}x$ is pseudo-symmetric about $(1+\eta)/2$ on $[0,1]$. For that, we note that $(1-(t-\eta))\in[(1+\eta)/2,1]$ for all $t\in[\eta,(1+\eta)/2]$. Thus,

$$\begin{aligned}&(\mathcal{G}x)(1-(t-\eta))\\ =&\int_0^\eta\psi_q\Big[\int_w^{(1+\eta)/2}a(\nu)\Big(f(\nu,x(\nu),x'(\nu))+\int_\nu^{(1+\eta)/2}K(\nu,\zeta,x(\zeta))d\zeta\Big)d\nu\Big]dw\\ +&\int_{1-(t-\eta)}^1\psi_q\Big[\int_{(1+\eta)/2}^w a(\nu)\Big(f(\nu,x(\nu),x'(\nu))+\int_{(1+\eta)/2}^\nu K(\nu,\zeta,x(\zeta))d\zeta\Big)d\nu\Big]dw\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\eta \psi_q\Big[\int_w^{(1+\eta)/2} a(\nu)\Big(f(\nu,x(\nu),x'(\nu)) + \int_\nu^{(1+\eta)/2} K(\nu,\zeta,x(\zeta))d\zeta\Big)d\nu\Big]dw \\
&- \int_t^\eta \psi_q\Big[\int_{(1+\eta)/2}^{1-(w-\eta)} a(\nu)\Big(f(\nu,x(\nu),x'(\nu)) + \int_{(1+\eta)/2}^{\nu} K(\nu,\zeta,x(\zeta))d\zeta\Big)d\nu\Big]dw \\
&= \int_0^\eta \psi_q\Big[\int_w^{(1+\eta)/2} a(\nu)\Big(f(\nu,x(\nu),x'(\nu)) + \int_\nu^{(1+\eta)/2} K(\nu,\zeta,x(\zeta))d\zeta\Big)d\nu\Big]dw \\
&+ \int_\eta^t \psi_q\Big[\int_w^{(1+\eta)/2} a(\nu)\Big(f(\nu,x(\nu),x'(\nu)) + \int_{(1+\eta)/2}^{1-(\nu-\eta)} K(\nu,\zeta,x(\zeta))d\zeta\Big)d\nu\Big]dw \\
&= \int_0^\eta \psi_q\Big[\int_w^{(1+\eta)/2} a(\nu)\Big(f(\nu,x(\nu),x'(\nu)) + \int_{(1+\eta)/2}^{\nu} K(\nu,\zeta,x(\zeta))d\zeta\Big)d\nu\Big]dw \\
&+ \int_\eta^t \psi_q\Big[\int_w^{(1+\eta)/2} a(\nu)\Big(f(\nu,x(\nu),x'(\nu)) + \int_\nu^{(1+\eta)/2} K(\nu,\zeta,x(\zeta))d\zeta\Big)d\nu\Big]dw \\
&= \int_0^t \psi_q\Big[\int_w^{(1+\eta)/2} a(\nu)\Big(f(\nu,x(\nu),x'(\nu)) + \int_\nu^{(1+\eta)/2} K(\nu,\zeta,x(\zeta))d\zeta\Big)d\nu\Big]dw \\
&= (\mathcal{G}x)(t).
\end{aligned}
$$

Now, $\forall t \in [(1+\eta)/2, 1]$, we have $(1-(t-\eta)) \in [\eta, (1+\eta)/2]$. Thus,

$$
\begin{aligned}
&(\mathcal{G}x)(1-(t-\eta)) \\
&= \int_0^{1-(t-\eta)} \psi_q\Big[\int_w^{(1+\eta)/2} a(\nu)\Big(f(\nu,x(\nu),x'(\nu)) + \int_\nu^{(1+\eta)/2} K(\nu,\zeta,x(\zeta))d\zeta\Big)d\nu\Big]dw \\
&= \int_0^\eta \psi_q\Big[\int_w^{(1+\eta)/2} a(\nu)\Big(f(\nu,x(\nu),x'(\nu)) + \int_\nu^{(1+\eta)/2} K(\nu,\zeta,x(\zeta))d\zeta\Big)d\nu\Big]dw \\
&+ \int_\eta^{1-(t-\eta)} \psi_q\Big[\int_w^{(1+\eta)/2} a(\nu)\Big(f(\nu,x(\nu),x'(\nu)) + \int_\nu^{(1+\eta)/2} K(\nu,\zeta,x(\zeta))d\zeta\Big)d\nu\Big]dw \\
&= \int_0^\eta \psi_q\Big[\int_w^{(1+\eta)/2} a(\nu)\Big(f(\nu,x(\nu),x'(\nu)) + \int_\nu^{(1+\eta)/2} K(\nu,\zeta,x(\zeta))d\zeta\Big)d\nu\Big]dw \\
&+ \int_t^1 \psi_q\Big[\int_{1-(w-\eta)}^{(1+\eta)/2} a(\nu)\Big(f(\nu,x(\nu),x'(\nu)) + \int_\nu^{(1+\eta)/2} K(\nu,\zeta,x(\zeta))d\zeta\Big)d\nu\Big]dw \\
&= \int_0^\eta \psi_q\Big[\int_w^{(1+\eta)/2} a(\nu)\Big(f(\nu,x(\nu),x'(\nu)) + \int_{(1+\eta)/2}^{\nu} K(\nu,\zeta,x(\zeta))d\zeta\Big)d\nu\Big]dw \\
&+ \int_t^1 \psi_q\Big[\int_{(1+\eta)/2}^{w} a(\nu)\Big(f(\nu,x(\nu),x'(\nu)) + \int_{1-(\nu-\eta)}^{(1+\eta)/2} K(\nu,\zeta,x(\zeta))d\zeta\Big)d\nu\Big]dw \\
&= \int_0^\eta \psi_q\Big[\int_w^{(1+\eta)/2} a(\nu)\Big(f(\nu,x(\nu),x'(\nu)) + \int_{(1+\eta)/2}^{\nu} K(\nu,\zeta,x(\zeta))d\zeta\Big)d\nu\Big]dw \\
&+ \int_t^1 \psi_q\Big[\int_{(1+\eta)/2}^{w} a(\nu)\Big(f(\nu,x(\nu),x'(\nu)) + \int_{(1+\eta)/2}^{\nu} K(\nu,\zeta,x(\zeta))d\zeta\Big)d\nu\Big]dw \\
&= (\mathcal{G}x)(t).
\end{aligned}
$$

So $(\mathcal{G}x)$ is pseudo-symmetric about $(1+\eta)/2$ on $[0,1]$. Hence we conclude that $\mathcal{G}: P \to P$. Also, it follows by the standard arguments [1, 18] that $\mathcal{G}: P \to P$ is completely continuous.

Next, we show that $\mathcal{G}: \overline{P}_\theta \to \overline{P}_\theta$. For $u \in \overline{P}_\theta$, it follows that $|u| \le \theta$ and

$$0 \le u(t) \le \max_{0\le t\le 1} |u(t)| \le \|u\| \le \theta, \quad 0 \le |u'(t)| \le \max_{0\le t\le 1} |u'(t)| \le \|u\| \le \theta.$$

By the assumptions (A_1), (A_2) and (A_4), we have

$$\begin{aligned}
&0 \le f(t, x(t), x'(t)) + \int_t^{(1+\eta)/2} K(t,\zeta,x(\zeta))d\zeta \\
&\le f(t,\theta,\theta) + \int_t^{(1+\eta)/2} K(t,\zeta,\theta)d\zeta \\
&\le \max_{0\le t\le 1}\left\{ f(t,\theta,\theta) + \int_t^{(1+\eta)/2} K(t,\zeta,\theta)d\zeta \right\} \le \psi_p(\theta/\Theta),\ t \in [0,1].
\end{aligned} \tag{2.1}$$

Clearly

$$\begin{aligned}
\|\mathcal{G}x\| &= \max_{0\le t\le 1}[(\mathcal{G}x)^2(t) + ((\mathcal{G}x)')^2(t)]^{1/2} \\
&\le \sqrt{2}\max_{0\le t\le 1}\{|\mathcal{G}x(t)|,\ |(\mathcal{G}x)'(t)|\} = \sqrt{2}\max\{(\mathcal{G}x)((1+\eta)/2),\ \max_{0\le t\le 1}|(\mathcal{G}x)'|(t)\}.
\end{aligned}$$

By the definition of $\mathcal{G}x$ and (2.1), we obtain

$$\begin{aligned}
&\mathcal{G}x((1+\eta)/2) \\
= &\int_0^{(1+\eta)/2} \psi_q\left[\int_w^{(1+\eta)/2} a(\nu)\left(f(\nu,x(\nu),x'(\nu)) + \int_\nu^{(1+\eta)/2} K(\nu,\zeta,x(\zeta))d\zeta\right)d\nu\right]dw \\
\le &\int_0^{(1+\eta)/2} \psi_q\left[\int_w^{(1+\eta)/2} a(\nu)d\nu\psi_p(\theta/\Theta)\right]dw = \theta\Theta_1/\Theta \le \theta/\sqrt{2},
\end{aligned}$$

$$\begin{aligned}
&\max_{0\le t\le 1}|(\mathcal{G}x)'(t)| = \max\{(\mathcal{G}x)'(0), -(\mathcal{G}x)'(1)\} \\
= &\max\left\{\psi_q\left[\int_0^{(1+\eta)/2} a(\nu)\left(f(\nu,x(\nu),x'(\nu)) + \int_\nu^{(1+\eta)/2} K(\nu,\zeta,x(\zeta))d\zeta\right)d\nu\right],\right. \\
&\left.\psi_q\left[\int_{(1+\eta)/2}^1 a(\nu)\{f(\nu,x(\nu),x'(\nu)) + \int_{(1+\eta)/2}^\nu K(\nu,\zeta,x(\zeta))d\zeta\}d\nu\right]\right\} \\
\le &\max\left\{\psi_q\left[\int_0^{(1+\eta)/2} a(\nu)d\nu\psi_p(\theta/\Theta)\right],\ \psi_q\left[\int_{(1+\eta)/2}^1 a(\nu)d\nu\psi_p(\theta/\Theta)\right]\right\} \\
= &\theta\Theta_2/\Theta \le \theta/\sqrt{2}.
\end{aligned}$$

Consequently, we have $\|\mathcal{G}x\| \le \theta$. Hence we conclude that $\mathcal{G}: \overline{P}_\theta \to \overline{P}_\theta$.

3 Existence of extremal solutions

Theorem 3.1 Assume that $(A_1)-(A_4)$ hold. Then there exist extremal positive, concave and pseudo-symmetric solutions α^*, β^* of (1.1) and (1.2) with $0 < \alpha^* \leq \theta_1/\sqrt{2},\ 0 < |(\alpha^*)'| \leq \theta_1/\sqrt{2}$ and $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\mathcal{G}^n\alpha_0 = \alpha^*,\ \lim_{n\to\infty}(\alpha_n)' = \lim_{n\to\infty}(\mathcal{G}^n\alpha_0)' = (\alpha^*)'$, where

$$\alpha_0(t) = \begin{cases} \theta_1 t/\sqrt{2}, & \text{if } 0 \leq t \leq (1+\eta)/2, \\ \theta_1|1-(t-\eta)|/\sqrt{2}, & \text{if } (1+\eta)/2 \leq t \leq 1, \end{cases}$$

and $0 < \beta^* \leq \theta_1,\ 0 < |(\beta^*)'| \leq \theta_1$ with $\lim_{n\to\infty}\beta_n = \lim_{n\to\infty}\mathcal{G}^n\beta_0 = \beta^*,\ \lim_{n\to\infty}(\beta_n)' = \lim_{n\to\infty}(\mathcal{G}^n\beta_0)' = (\beta^*)'$, where $\beta_0(t) = 0,\ t \in [0,1]$.

Proof. We define the iterative schemes

$$\alpha_1 = \mathcal{G}\alpha_0,\ \alpha_{n+1} = \mathcal{G}\alpha_n = \mathcal{G}^{n+1}\alpha_0,\quad n = 1,2,...,$$

$$\beta_1 = \mathcal{G}\beta_0 = \mathcal{G}0,\ \beta_{n+1} = \mathcal{G}\beta_n = \mathcal{G}^{n+1}\beta_0,\ n = 1,2,....$$

In view of the fact that $\mathcal{G} : \overline{P}_\theta \to \overline{P}_\theta$, it follows that $\alpha_n \in \mathcal{G}\overline{P}_\theta \subseteq \overline{P}_\theta$ for $n = 1,2,....$ As $\mathcal{G}$ is completely continuous, therefore $\{\alpha_n\}_{n=1}^\infty$ is sequentially compact. By the assumption (A_3), it easily follows that $\alpha_1(t) = \mathcal{G}\alpha_0(t) \leq \alpha_0(t),\ t \in [0,1]$. Also

$$\begin{aligned} |\alpha_1'(t)| &= |(\mathcal{G}\alpha_0)'(t)| \leq \max\{(\mathcal{G}(\theta_1 t/\sqrt{2}))'(0), -(\mathcal{G}(\theta_1(1-(t-\eta))/\sqrt{2}))'(1)\} \\ &\leq \theta_1/\sqrt{2} = |\alpha_0'(t)|,\quad t \in [0,1]. \end{aligned}$$

Now, by the nondecreasing nature of $(\mathcal{G}x)$, we obtain

$$\alpha_2(t) = \mathcal{G}\alpha_1(t) \leq \mathcal{G}\alpha_0(t) = \alpha_1(t),\quad |\alpha_2'(t)| = |(\mathcal{G}\alpha_1)'(t)| \leq |(\mathcal{G}\alpha_0)'(t)| = |\alpha_1'(t)|,$$

for $t \in [0,1]$. Thus, by induction, we have

$$\alpha_{n+1}(t) \leq \alpha_n(t),\quad |\alpha_{n+1}'(t)| \leq |\alpha_n'(t)|,\quad n = 1,2,....,\quad t \in [0,1].$$

Hence there exists $\alpha^* \in \overline{P}_\theta$ such that $\alpha_n \to \alpha^*$. Applying the continuity of $\mathcal{G}$ and $\alpha_{n+1} = \mathcal{G}\alpha_n$, we get $\mathcal{G}\alpha^* = \alpha^*$ [18]. In view of the fact that $f(t,0,0)$ and $K(t,\zeta,0)$ are not identically equal to zero for $0 \leq t,\zeta \leq 1$, we find that the zero function is not the solution of (1.1) and (1.2). Thus, $\|\alpha^*\| > 0$ and hence $\alpha^* > 0,\ t \in (0,1)$. Now, let $\beta_1 = \mathcal{G}\beta_0 = \mathcal{G}0,\ \beta_2 = \mathcal{G}^2\beta_0 = \mathcal{G}^2 0 = \mathcal{G}\beta_1$. Since $\mathcal{G} : \overline{P}_\theta \to \overline{P}_\theta$, it follows that $\beta_n \in \mathcal{G}\overline{P}_\theta \subseteq \overline{P}_\theta$ for n=1,2,.... As $\mathcal{G}$ is completely continuous, therefore $\{\beta_n\}_{n=1}^\infty$ is sequentially compact. Since $\beta_1 = \mathcal{G}\beta_0 = \mathcal{G}0 \in \mathcal{G}\overline{P}_\theta$, we have $\beta_1(t) = \mathcal{G}\beta_0(t) = (\mathcal{G}0)(t) \geq 0$ and $|\beta_1'(t)| = |(\mathcal{G}\beta_0)'(t) = |(\mathcal{G}0)'(t)| \geq 0,\quad t \in [0,1]$. Thus, $\beta_2(t) = \mathcal{G}\beta_1(t) \geq (\mathcal{G}0)(t) = \beta_1(t)$ and $|\beta_2'(t)| = |(\mathcal{G}\beta_1)'(t)| = |(\mathcal{G}0)'(t)| = |\beta_1'(t)|,\quad t \in [0,1]$. As before, by induction, it follows that

$$\beta_{n+1}(t) \geq \beta_n(t),\quad |\beta_{n+1}'(t)| \geq |\beta_n'(t)|,\quad n = 1,2,....,\quad t \in [0,1].$$

Hence there exists $\beta^* \in \overline{P}_\theta$ such that $\beta_n \to \beta^*$ and $\mathcal{G}\beta^* = \beta^*$ with $\beta^*(t) > 0,\ t \in (0,1)$.
Now, using the well known fact that a fixed point of the operator $\mathcal{G}$ in P must be a solution of (1.1) and (1.2) in P, it follows from the monotone iterative technique [11] that α^* and β^* are the extremal positive, concave and pseudo-symmetric solutions of (1.1) and (1.2). This completes the proof.

Example. Consider the boundary value problem

$$x''(t) + f(t, x(t), x'(t)) + \int_t^{\frac{3}{4}} K(t, \zeta, x(\zeta))d\zeta = 0,\ t \in (0,1), \tag{3.1}$$

$$x(0) = 0, \qquad x(\frac{1}{2}) = x(1), \tag{3.2}$$

where $a(t) = 1,\ f(t, x, x') = -\frac{16}{3}t^2 + 8t + \frac{1}{2\sqrt{2}}x + \frac{1}{36}(x')^2,\ K(t, \zeta, x) = -\frac{4}{3}t^2 + 2t + x$. Clearly the functions $f(t, x, x')$ and $K(t, \zeta, x)$ satisfy the assumptions (A_1) and (A_2). In relation to (A_3), we choose $\theta = 6\sqrt{2}$ so that $\Theta = 3/(2\sqrt{2})$ and $\max_{0\le t\le 1}\{f(t, \theta, \theta) + \int_t^{(1+\eta)/2} K(t, \zeta, \theta)d\zeta\} = 8 = \psi_2(\theta/\Theta)$. Hence the conclusion of Theorem 3.1 applies to the problem (3.1)-(3.2).

4 Conclusions

The extremal positive, concave and pseudo-symmetric solutions for a nonlocal three-point $p-$Laplacian integro-differential boundary value problem are obtained by applying an abstract monotone iterative technique. The consideration of $p-$Laplacian boundary value problems is quite interesting and important as it covers a wide range of problems for various values of p occurring in applied sciences (as indicated in the introduction). The nonlocal three-point boundary conditions further enhance the scope of $p-$Laplacian boundary value problems as such boundary conditions appear in certain problems of thermodynamics and wave propagation where the controller at the end $t = 1$ dissipates or adds energy according to a censor located at a position $t = \eta$ $(0 < \eta < 1)$ where as the other end $t = 0$ is maintained at a constant level of energy. The results presented in this paper are new and extend some earlier results. For $p = 2$, our results correspond to a three-point second order quasilinear integro-differential boundary value problem. The results of [1] are improved as the nonlinear function f is allowed to depend on x' together with x in this paper whereas it only depends on x in [1].

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