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Existence of solutions for a fourth-order boundary value problem on the half-line via critical point theory

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Abstract. In this paper, a fourth-order boundary value problem on the half-line is considered and existence of solutions is proved using a minimization principle and the mountain pass theorem.

Keywords: fourth-order BVPs, unbounded interval, critical point, minimization principle, mountain-pass theorem.

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1 Introduction

We consider the existence of solutions for the following fourth-order boundary value problem set on the half-line

$$\begin{cases} u^{(4)}(t) - u''(t) + u(t) = f(t, u(t)), \quad t \in [0, +\infty), \\ u(0) = u(+\infty) = 0, \\ u''(0) = u''(+\infty) = 0, \end{cases} \tag{1.1}$$

where $f \in C([0, +\infty) \times \mathbb{R}, \mathbb{R})$.

Many authors used critical point theory to establish the existence of solutions for fourth-order boundary value problems on bounded intervals (see for example [8,9,13]), but there are only a few papers that consider the above problem on the half-line using critical point theory. We cite [5] where the authors consider the existence of solutions for a particular fourth-order BVP on the half-line using critical point theory.

We endow the following space

$$H_0^2(0, +\infty) = \Big\{ u \in L^2(0, +\infty),\ u' \in L^2(0, +\infty),\ u'' \in L^2(0, +\infty),\ u(0) = 0,\ u'(0) = 0 \Big\}$$

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with its natural norm

$$\|u\| = \left(\int_0^{+\infty} u''^2(t)dt + \int_0^{+\infty} u'^2(t)dt + \int_0^{+\infty} u^2(t)dt\right)^{\frac{1}{2}}.$$

Note that if $u \in H_0^2(0,+\infty)$, then $u(+\infty) = 0$, $u'(+\infty) = 0$, (see [3, Corollary 8.9]). Let $p, q : [0,+\infty) \longrightarrow (0,+\infty)$ be two continuously differentiable and bounded functions with

$$M_1 = \max(\|p\|_{L^2}, \|p'\|_{L^2}) < +\infty, \qquad M_2 = \max(\|q\|_{L^2}, \|q'\|_{L^2}) < +\infty.$$

We also consider the following spaces

$$C_{l,p}[0,+\infty) = \left\{u \in C([0,+\infty),\mathbb{R}) : \lim_{t\to+\infty} p(t)u(t) \text{ exists}\right\}$$

endowed with the norm

$$\|u\|_{\infty,p} = \sup_{t\in[0,+\infty)} p(t)|u(t)|,$$

and

$$C^1_{l,p,q}[0,+\infty) = \left\{u \in C^1([0,+\infty),\mathbb{R}) : \lim_{t\to+\infty} p(t)u(t),\ \lim_{t\to+\infty} q(t)u'(t) \text{ exist}\right\}$$

endowed with the natural norm

$$\|u\|_{\infty,p,q} = \sup_{t\in[0,+\infty)} p(t)|u(t)| + \sup_{t\in[0,+\infty)} q(t)|u'(t)|.$$

Let

$$C_l[0,+\infty) = \left\{u \in C([0,+\infty),\mathbb{R}) : \lim_{t\to+\infty} u(t) \text{ exists}\right\}$$

endowed with the norm $\|u\|_\infty = \sup_{t\in[0,+\infty)} |u(t)|$.

To prove that $H_0^2(0,+\infty)$ embeds compactly in $C^1_{l,p,q}[0,+\infty)$, we need the following Corduneanu compactness criterion.

Lemma 1.1 ([4]). *Let $D \subset C_l([0,+\infty),\mathbb{R})$ be a bounded set. Then D is relatively compact if the following conditions hold:*
(a) D is equicontinuous on any compact sub-interval of $\mathbb{R}^+$, i.e.

$$\forall J \subset [0,+\infty) \text{ compact}, \ \forall \varepsilon > 0, \exists \delta > 0, \ \forall t_1, t_2 \in J :$$
$$|t_1 - t_2| < \delta \Longrightarrow |u(t_1) - u(t_2)| \le \varepsilon, \ \forall u \in D;$$

(b) D is equiconvergent at $+\infty$ i.e.,

$$\forall \varepsilon > 0, \ \exists T = T(\varepsilon) > 0 \text{ such that}$$
$$\forall t : t \ge T(\varepsilon) \Longrightarrow |u(t) - u(+\infty)| \le \varepsilon, \ \forall u \in D.$$

Similar reasoning as in [6] yields the following compactness criterion in the space $C^1_{l,p,q}([0,+\infty),\mathbb{R})$.

Lemma 1.2. *Let $D \subset C^1_{l,p,q}([0,+\infty),\mathbb{R})$ be a bounded set. Then D is relatively compact if the following conditions hold:*
(a) D is equicontinuous on any compact sub-interval of $[0,+\infty)$, i.e.

$$\begin{aligned}
&\forall J \subset [0,+\infty) \text{ compact},\ \forall \varepsilon > 0,\ \exists \delta > 0,\ \forall t_1, t_2 \in J:\\
&|t_1 - t_2| < \delta \Longrightarrow |p(t_1)u(t_1) - p(t_2)u(t_2)| \le \varepsilon,\ \forall u \in D,\\
&|t_1 - t_2| < \delta \Longrightarrow |q(t_1)u'(t_1) - q(t_2)u'(t_2)| \le \varepsilon,\ \forall u \in D;
\end{aligned}$$

(b) D is equiconvergent at $+\infty$ i.e.,

$$\begin{aligned}
&\forall \varepsilon > 0, \exists T = T(\varepsilon) > 0 \text{ such that}\\
&\forall t : t \ge T(\varepsilon) \Longrightarrow |p(t)u(t) - (pu)(+\infty)| \le \varepsilon,\ \forall u \in D,\\
&\forall t : t \ge T(\varepsilon) \Longrightarrow |q(t)u'(t) - (qu')(+\infty)| \le \varepsilon,\ \forall u \in D.
\end{aligned}$$

Now we recall some essential facts from critical point theory (see [1,2,10]).

Definition 1.3. Let X be a Banach space, $\Omega \subset X$ an open subset, and $J : \Omega \longrightarrow \mathbb{R}$ a functional. We say that J is Gâteaux differentiable at $u \in \Omega$ if there exists $A \in X^*$ such that

$$\lim_{t\to 0} \frac{J(u+tv) - J(u)}{t} = Av,$$

for all $v \in X$. Now A, which is unique, is denoted by $A = J'_G(u)$.

The mapping which sends to every $u \in \Omega$ the mapping $J'_G(u)$ is called the Gâteaux differential of J and is denoted by J'_G.

We say that $J \in C^1$ if J is Gâteaux differential on Ω and J'_G is continuous at every $u \in \Omega$.

Definition 1.4. Let X be a Banach space. A functional $J : \Omega \longrightarrow \mathbb{R}$ is called coercive if, for every sequence $(u_k)_{k\in\mathbb{N}} \subset X$,

$$\|u_k\| \to +\infty \Longrightarrow |J(u_k)| \to +\infty.$$

Definition 1.5. Let X be a Banach space. A functional $J : X \longrightarrow (-\infty,+\infty]$ is said to be sequentially weakly lower semi-continuous (*swlsc* for short) if

$$J(u) \le \liminf_{n\to+\infty} J(u_n)$$

as $u_n \rightharpoonup u$ in X, $n \to \infty$.

Lemma 1.6 (Minimization principle [2])**.** *Let X be a reflexive Banach space and J a functional defined on X such that*

(1) $\lim_{\|u\|\to+\infty} J(u) = +\infty$ (coercivity condition),

(2) J is sequentially weakly lower semi-continuous.

Then J is lower bounded on X and achieves its lower bound at some point u_0.

Definition 1.7. Let X be a real Banach space, $J \in C^1(X,\mathbb{R})$. If any sequence $(u_n) \subset X$ for which $(J(u_n))$ is bounded in $\mathbb{R}$ and $J'(u_n) \longrightarrow 0$ as $n \to +\infty$ in X' possesses a convergent subsequence, then we say that J satisfies the Palais–Smale condition (PS condition for brevity).

Lemma 1.8 (Mountain Pass Theorem, [11, Theorem 2.2], [12, Theorem 3.1])**.** *Let X be a Banach space, and let $J \in C^1(X, \mathbb{R})$ satisfy $J(0) = 0$. Assume that J satisfies the (PS) condition and there exist positive numbers ρ and α such that*

(1) $J(u) \geq \alpha$ if $\|u\| = \rho$,

(2) there exists $u_0 \in X$ such that $\|u_0\| > \rho$ and $J(u_0) < \alpha$.

Then there exists a critical point. It is characterized by

$$J'(u) = 0, \quad J(u) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = u_0\}.$$

1.1 Variational setting

Take $v \in H_0^2(0, +\infty)$, and multiply the equation in Problem (1.1) by v and integrate over $(0, +\infty)$, so we get

$$\int_0^{+\infty} (u^{(4)}(t) - u''(t) + u(t))v(t)dt = \int_0^{+\infty} f(t, u(t))v(t)dt.$$

Hence

$$\int_0^{+\infty} \big(u''(t)v''(t) + u'(t)v'(t) + u(t)v(t)\big)dt = \int_0^{+\infty} f(t, u(t))v(t)dt.$$

This leads to the natural concept of a weak solution for Problem (1.1).

Definition 1.9. We say that a function $u \in H_0^2(0, +\infty)$ is a weak solution of Problem (1.1) if

$$\int_0^{+\infty} \big(u''(t)v''(t) + u'(t)v'(t) + u(t)v(t)\big)dt = \int_0^{+\infty} f(t, u(t))v(t)dt,$$

for all $v \in H_0^2(0, +\infty)$.

In order to study Problem (1.1), we consider the functional $J : H_0^2(0, +\infty) \longrightarrow \mathbb{R}$ defined by

$$J(u) = \frac{1}{2}\|u\|^2 - \int_0^{+\infty} F(t, u(t))dt,$$

where

$$F(t, u) = \int_0^u f(t, s)ds.$$

2 Some embedding results

We begin this section by proving some continuous and compact embeddings. Here p and q (and M_1, M_2) are as in Section 1.

Lemma 2.1. *$H_0^2(0, +\infty)$ embeds continuously in $C^1_{l,p,q}[0, +\infty)$.*

Proof. For $u \in H_0^2(0,+\infty)$, we have

$$
\begin{aligned}
|p(t)u(t)| &= |p(+\infty)u(+\infty) - p(t)u(t)| \\
&= \left| \int_t^{+\infty} (pu)'(s)ds \right| \\
&\leq \left| \int_t^{+\infty} p'(s)u(s)ds \right| + \left| \int_t^{+\infty} p(s)u'(s)ds \right| \\
&\leq \left(\int_0^{+\infty} p'^2(s)ds \right)^{\frac{1}{2}} \left(\int_0^{+\infty} u^2(s)ds \right)^{\frac{1}{2}} + \left(\int_0^{+\infty} p^2(s)ds \right)^{\frac{1}{2}} \left(\int_0^{+\infty} u'^2(s)ds \right)^{\frac{1}{2}} \\
&\leq \max(\|p'\|_{L^2}, \|p\|_{L^2})\|u\| \\
&\leq M_1\|u\|,
\end{aligned}
$$

and

$$
\begin{aligned}
|q(t)u'(t)| &= |q(+\infty)u'(+\infty) - q(t)u'(t)| \\
&= \left| \int_t^{+\infty} (qu')'(s)ds \right| \\
&\leq \left| \int_t^{+\infty} q'(s)u'(s)ds \right| + \left| \int_t^{+\infty} q(s)u''(s)ds \right| \\
&\leq \left(\int_0^{+\infty} q'^2(s)ds \right)^{\frac{1}{2}} \left(\int_0^{+\infty} u'^2(s)ds \right)^{\frac{1}{2}} + \left(\int_0^{+\infty} q^2(s)ds \right)^{\frac{1}{2}} \left(\int_0^{+\infty} u''^2(s)ds \right)^{\frac{1}{2}} \\
&\leq \max(\|q'\|_{L^2}, \|q\|_{L^2})\|u\| \\
&\leq M_2\|u\|.
\end{aligned}
$$

Hence $\|u\|_{\infty,p,q} \leq M\|u\|$, with $M = \max(M_1, M_2)$. □

The following compactness embedding is an important result.

Lemma 2.2. *The embedding $H_0^2(0,+\infty) \hookrightarrow C^1_{l,p,q}[0,+\infty)$ is compact.*

Proof. Let $D \subset H_0^2(0,+\infty)$ be a bounded set. Then it is bounded in $C^1_{l,p,q}[0,+\infty)$ by Lemma 2.1. Let $R > 0$ be such that for all $u \in D$, $\|u\| \leq R$. We will apply Lemma 1.2.
(a) D is equicontinuous on every compact interval of $[0,+\infty)$. Let $u \in D$ and $t_1, t_2 \in J \subset [0,+\infty)$ where J is a compact sub-interval. Using the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
|p(t_1)u(t_1) - p(t_2)u(t_2)| &= \left| \int_{t_2}^{t_1} (pu)'(s)ds \right| \\
&= \left| \int_{t_2}^{t_1} \left(p'(s)u(s) + u'(s)p(s)\right) ds \right| \\
&\leq \left(\int_{t_2}^{t_1} p'^2(s)ds \right)^{\frac{1}{2}} \left(\int_{t_2}^{t_1} u^2(s)ds \right)^{\frac{1}{2}} + \left(\int_{t_2}^{t_1} p^2(s)ds \right)^{\frac{1}{2}} \left(\int_{t_2}^{t_1} u'^2(s)ds \right)^{\frac{1}{2}} \\
&\leq \max\left[\left(\int_{t_2}^{t_1} p'^2(s)ds \right)^{\frac{1}{2}}, \left(\int_{t_2}^{t_1} p^2(s)ds \right)^{\frac{1}{2}} \right] \|u\| \\
&\leq R\max\left[\left(\int_{t_2}^{t_1} p'^2(s)ds \right)^{\frac{1}{2}}, \left(\int_{t_2}^{t_1} p^2(s)ds \right)^{\frac{1}{2}} \right] \longrightarrow 0,
\end{aligned}
$$

as $|t_1 - t_2| \to 0$, and

$$\begin{aligned}
|q(t_1)u'(t_1) - q(t_2)u'(t_2)| &= \left| \int_{t_2}^{t_1} (qu')'(s)ds \right| \\
&= \left| \int_{t_2}^{t_1} \left(q'(s)u'(s) + q(s)u''(s)\right) ds \right| \\
&\le \left(\int_{t_2}^{t_1} q'^2(s)ds \right)^{\frac{1}{2}} \left(\int_{t_2}^{t_1} u'^2(s)ds \right)^{\frac{1}{2}} \\
&\quad + \left(\int_{t_2}^{t_1} q^2(s)ds \right)^{\frac{1}{2}} \left(\int_{t_2}^{t_1} u''^2(s)ds \right)^{\frac{1}{2}} \\
&\le \max \left[\left(\int_{t_2}^{t_1} q'^2(s)ds \right)^{\frac{1}{2}}, \left(\int_{t_2}^{t_1} q^2(s)ds \right)^{\frac{1}{2}} \right] \|u\| \\
&\le R \max \left[\left(\int_{t_2}^{t_1} q'^2(s)ds \right)^{\frac{1}{2}}, \left(\int_{t_2}^{t_1} q^2(s)ds \right)^{\frac{1}{2}} \right] \longrightarrow 0,
\end{aligned}$$

as $|t_1 - t_2| \to 0$.
(b) D is equiconvergent at $+\infty$. For $t \in [0, +\infty)$ and $u \in D$, using the fact that $(pu)(+\infty) = 0, (qu')(+\infty) = 0$ (note that $u(\infty) = 0, u'(\infty) = 0$ and p, q are bounded) and using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
|(pu)(t) - (pu)(+\infty)| &= \left| \int_{t}^{+\infty} (pu)'(s)ds \right| \\
&= \left| \int_{t}^{+\infty} \left(p'(s)u(s) + u'(s)p(s)\right) ds \right| \\
&\le \max \left[\left(\int_{t}^{+\infty} p'^2(s)ds \right)^{\frac{1}{2}}, \left(\int_{t}^{+\infty} p^2(s)ds \right)^{\frac{1}{2}} \right] \|u\| \\
&\le R \max \left[\left(\int_{t}^{+\infty} p'^2(s)ds \right)^{\frac{1}{2}}, \left(\int_{t}^{+\infty} p^2(s)ds \right)^{\frac{1}{2}} \right] \longrightarrow 0,
\end{aligned}$$

as $t \to +\infty$, and

$$\begin{aligned}
|(qu')(t) - (qu')(+\infty)| &= \left| \int_{t}^{+\infty} (qu')'(s)ds \right| \\
&= \left| \int_{t}^{+\infty} \left(q'(s)u'(s) + q(s)u''(s)\right) ds \right| \\
&\le \max \left[\left(\int_{t}^{+\infty} q'^2(s)ds \right)^{\frac{1}{2}}, \left(\int_{t}^{+\infty} q^2(s)ds \right)^{\frac{1}{2}} \right] \|u\| \\
&\le R \max \left[\left(\int_{t}^{+\infty} q'^2(s)ds \right)^{\frac{1}{2}}, \left(\int_{t}^{+\infty} q^2(s)ds \right)^{\frac{1}{2}} \right] \longrightarrow 0,
\end{aligned}$$

as $t \to +\infty$. □

Corollary 2.3. *$C^1_{l,p,q}[0, +\infty)$ embeds continuously in $C_{l,p}[0, +\infty)$.*

Corollary 2.4. *The embedding $H^2_0(0, +\infty) \hookrightarrow C_{l,p}[0, +\infty)$ is continuous and compact.*

3 Existence results

Here p (and M_1) are as in Section 1.

Theorem 3.1. *Assume that F satisfy the following conditions.*

$(F1)$ *There exist two constants $1 < \alpha < \beta < 2$ and two functions a, b with $\frac{a}{p^\alpha} \in L^1([0,+\infty),[0,+\infty))$, $\frac{b}{p^\beta} \in L^1([0,+\infty),[0,+\infty))$ such that*

$$|F(t,x)| \le a(t)|x|^\alpha, \qquad \forall (t,x) \in [0,+\infty) \times \mathbb{R},\ |x| \le 1$$

and

$$|F(t,x)| \le b(t)|x|^\beta, \qquad \forall (t,x) \in [0,+\infty) \times \mathbb{R},\ |x| > 1.$$

$(F2)$ *There exist an open bounded set $I \subset [0,+\infty)$ and two constants $\eta > 0$ and $0 < \gamma < 2$ such that*

$$F(t,x) \ge \eta |x|^\gamma, \qquad \forall (t,x) \in I \times \mathbb{R},\ |x| \le 1.$$

Then Problem (1.1) has at least one nontrivial weak solution.

Proof.
Claim 1. We first show that J is well defined.
Let

$$\Omega_1 = \{t \ge 0,\ |u(t)| \le 1\}, \qquad \Omega_2 = \{t \ge 0,\ |u(t)| > 1\}.$$

Given $u \in H_0^2(0,+\infty)$, it follows from $(F1)$ and Corollary 2.4 that

$$\begin{aligned}
\int_0^{+\infty} |F(t,u(t))|dt &= \int_{\Omega_1} |F(t,u(t))|dt + \int_{\Omega_2} |F(t,u(t))|dt \\
&\le \int_{\Omega_1} a(t)|u(t)|^\alpha dt + \int_{\Omega_2} b(t)|u(t)|^\beta dt \\
&\le \int_{\Omega_1} \frac{a(t)}{p^\alpha(t)} |p(t)u(t)|^\alpha dt + \int_{\Omega_2} \frac{b(t)}{p^\beta(t)} |p(t)u(t)|^\beta dt \\
&\le \left|\frac{a}{p^\alpha}\right|_{L^1} \|u\|_{\infty,p}^\alpha + \left|\frac{b}{p^\beta}\right|_{L^1} \|u\|_{\infty,p}^\beta \\
&\le M_1^\alpha \left|\frac{a}{p^\alpha}\right|_{L^1} \|u\|^\alpha + M_1^\beta \left|\frac{b}{p^\beta}\right|_{L^1} \|u\|^\beta.
\end{aligned}$$

Thus

$$|J(u)| \le \frac{1}{2}\|u\|^2 + M_1^\alpha \left|\frac{a}{p^\alpha}\right|_{L^1} \|u\|^\alpha + M_1^\beta \left|\frac{b}{p^\beta}\right|_{L^1} \|u\|^\beta < +\infty.$$

Claim 2. J is coercive.
From $(F1)$ and Corollary 2.4, we have

$$\begin{aligned}
J(u) &= \frac{1}{2}\|u\|^2 - \int_{\Omega_1} F(t,u(t))dt - \int_{\Omega_2} F(t,u(t))dt \\
&\ge \frac{1}{2}\|u\|^2 - M_1^\alpha \left|\frac{a}{p^\alpha}\right|_{L^1} \|u\|^\alpha - M_1^\beta \left|\frac{b}{p^\beta}\right|_{L^1} \|u\|^\beta.
\end{aligned} \tag{3.1}$$

Now since $0 < \alpha < \beta < 2$, then (3.1) implies that

$$\lim_{\|u\| \to +\infty} J(u) = +\infty.$$

Consequently, J is coercive.

Claim 3. J is sequentially weakly lower semi-continuous.
Let (u_n) be a sequence in $H_0^2(0,+\infty)$ such that $u_n \rightharpoonup u$ as $n \longrightarrow +\infty$ in $H_0^2(0,+\infty)$. Then there exists a constant $A > 0$ such that $\|u_n\| \leq A$, for all $n \geq 0$ and $\|u\| \leq A$. Now (see Corollary 2.4) $(p(t)u_n(t))$ converges to $(p(t)u(t))$ as $n \longrightarrow +\infty$ for $t \in [0,+\infty)$. Since F is continuous, we have $F(t,u_n(t)) \longrightarrow F(t,u(t))$ as $n \longrightarrow +\infty$, and using $(F1)$ we have

$$\begin{aligned}|F(t,u_n(t))| &\leq a(t)|u_n(t)|^{\alpha} + b(t)|u_n(t)|^{\beta}\\ &\leq \frac{a(t)}{p^{\alpha}(t)}|p(t)u_n(t)|^{\alpha} + \frac{b(t)}{p^{\beta}(t)}|p(t)u_n(t)|^{\beta}\\ &\leq \frac{a(t)}{p^{\alpha}(t)}\|u_n\|_{\infty,p}^{\alpha} + \frac{b(t)}{p^{\beta}(t)}\|u_n\|_{\infty,p}^{\beta}\\ &\leq \frac{a(t)}{p^{\alpha}(t)}M_1^{\alpha}\|u_n\|^{\alpha} + \frac{b(t)}{p^{\beta}(t)}M_1^{\beta}\|u_n\|^{\beta}\\ &\leq \frac{a(t)}{p^{\alpha}(t)}M_1^{\alpha}A^{\alpha} + \frac{b(t)}{p^{\beta}(t)}M_1^{\beta}A^{\beta},\end{aligned}$$

so from the Lebesgue Dominated Convergence Theorem we have

$$\lim_{n\to+\infty}\int_0^{+\infty} F(t,u_n(t))dt = \int_0^{+\infty} F(t,u(t))dt.$$

The norm in the reflexive Banach space is sequentially weakly lower semi-continuous, so

$$\liminf_{n\to+\infty}\|u_n\| \geq \|u\|.$$

Thus one has

$$\begin{aligned}\liminf_{n\to+\infty} J(u_n) &= \liminf_{n\to+\infty}\left(\frac{1}{2}\|u_n\|^2 - \int_0^{+\infty} F(t,u_n(t))dt\right)\\ &\geq \frac{1}{2}\|u\|^2 - \int_0^{+\infty} F(t,u(t))dt = J(u).\end{aligned}$$

Then, J is sequentially weakly lower semi-continuous.

From Lemma 1.6, J has a minimum point u_0 which is a critical point of J.

Claim 4. We show that $u_0 \neq 0$.
Let $u_1 \in H_0^2(0,+\infty)\setminus\{0\}$ and $|u_1(t)| \leq 1$, for all $t \in I$. Then from $(F2)$, we have

$$\begin{aligned}J(su_1) &= \frac{s^2}{2}\|u_1\|^2 - \int_0^{+\infty} F(t,su_1(t))dt\\ &\leq \frac{s^2}{2}\|u_1\|^2 - \int_I \eta|su_1(t)|^{\gamma}dt\\ &\leq \frac{s^2}{2}\|u_1\|^2 - s^{\gamma}\eta\int_I |u_1(t)|^{\gamma}dt, \qquad 0 < s < 1.\end{aligned}$$

Since $0 < \gamma < 2$, it follows that $J(su_1) < 0$ for $s > 0$ small enough. Hence $J(u_0) < 0$, and therefore u_0 is a nontrivial critical point of J.

Finally, it is easy to see that under $(F1)$, the functional J is Gâteaux differentiable and the Gâteaux derivative at a point $u \in X$ is

$$(J'(u),v) = \int_0^{+\infty}\big(u''(t)v''(t) + u'(t)v'(t) + u(t)v(t)\big)dt - \int_0^{+\infty} f(t,u(t))v(t)dt, \tag{3.2}$$

for all $v \in H_0^2(0,+\infty)$. Therefore u is a weak solution of Problem (1.1). $\square$

Theorem 3.2. *Assume that f satisfies the following assumptions.*

$(F3)$ *There exist nonnegative functions φ, g such that $g \in C(\mathbb{R},[0,+\infty))$ with*

$$|f(t,x)| \leq \varphi(t)g(x), \quad \text{for all } t \in [0,+\infty) \text{ and all } x \in \mathbb{R},$$

and for any constant $R > 0$ there exists a nonnegative function ψ_R with $\varphi\psi_R \in L^1(0,+\infty)$ and

$$\sup\left\{g\left(\frac{y}{p(t)}\right) : y \in [-R,R]\right\} \leq \psi_R(t) \quad \text{for a.e. } t \geq 0.$$

$(F4)$

$$\frac{1}{a(t)}F(t,\frac{1}{p(t)}x) = o(|x|^2) \quad \text{as } x \longrightarrow 0$$

uniformly in $t \in [0,+\infty)$ for some function $a \in L^1(0,+\infty) \cap C[0,+\infty)$.

$(F5)$ *There exists a positive function c_1 and a nonnegative function c_2 with $c_1, c_2 \in L^1(0,\infty)$, and $\mu > 2$ such that*

$$(a)\ F(t,x) \geq c_1(t)|x|^\mu - c_2(t), \quad \text{for } t \geq 0,\ \forall\, x \in \mathbb{R}\setminus\{0\},$$
$$(b)\ \mu F(t,x) \leq x f(t,x), \quad \text{for } t \geq 0,\ \forall\, x \in \mathbb{R}.$$

Then Problem (1.1) has at least one nontrivial weak solution.

Proof. We have $J(0) = 0$.
Claim 1. J satisfies the (PS) condition.
Assume that $(u_n)_{n\in\mathbb{N}} \subset H_0^2(0,+\infty)$ is a sequence such that $(J(u_n))_{n\in\mathbb{N}}$ is bounded and $J'(u_n) \longrightarrow 0$ as $n \longrightarrow +\infty$. Then there exists a constant $d > 0$ such that

$$|J(u_n)| \leq d, \quad \|J'(u_n)\|_{E'} \leq d\mu, \quad \forall n \in \mathbb{N}.$$

From $(F5)(b)$ we have

$$\begin{aligned}
2d + 2d\|u_n\| &\geq 2J(u_n) - \frac{2}{\mu}(J'(u_n),u_n) \\
&\geq \left(1-\frac{2}{\mu}\right)\|u_n\|^2 + 2\left[\int_0^{+\infty}\left(\frac{1}{\mu}u_n(t)f(t,u_n(t)) - F(t,u_n(t))\right)dt\right] \\
&\geq \left(1-\frac{2}{\mu}\right)\|u_n\|^2.
\end{aligned}$$

Since $\mu > 2$, then $(u_n)_{n\in\mathbb{N}}$ is bounded in $H_0^2(0,+\infty)$.

Now, we show that (u_n) converges strongly to some u in $H_0^2(0,+\infty)$. Since (u_n) is bounded in $H_0^2(0,+\infty)$, there exists a subsequence of (u_n) still denoted by (u_n) such that (u_n) converges weakly to some u in $H_0^2(0,+\infty)$. There exists a constant $c > 0$ such that $\|u_n\| \leq c$. Now (see Corollary 2.4) $(p(t)u_n(t))$ converges to $p(t)u(t)$ on $[0,+\infty)$. We have $f(t,u_n(t)) \longrightarrow f(t,u(t))$ and

$$\begin{aligned}
|f(t,u_n(t))| &= \left|f(t,\frac{1}{p(t)}p(t)u_n(t))\right| \\
&\leq \varphi(t)g\left(\frac{1}{p(t)}p(t)u_n(t)\right) \\
&\leq \varphi(t)\psi_{cM_1}(t),
\end{aligned}$$

and using the Lebesgue Dominated Convergence Theorem, we have

$$\lim_{n\to+\infty}\int_0^{+\infty}\left(f(t,u_n(t))-f(t,u(t))\right)\left(u_n(t)-u(t)\right)dt=0. \tag{3.3}$$

Since $\lim_{n\to+\infty} J'(u_n)=0$ and (u_n) converges weakly to some u, we have

$$\lim_{n\to+\infty}\langle J'(u_n)-J'(u),u_n-u\rangle=0. \tag{3.4}$$

It follows from (3.2) that

$$(J'(u_n)-J'(u),u_n-u)=\|u_n-u\|^2-\int_0^{+\infty}(f(t,u_n(t))-f(t,u(t)))(u_n(t)-u(t))dt.$$

Hence $\lim_{n\to+\infty}\|u_n-u\|=0$. Thus (u_n) converges strongly to u in $H_0^2(0,+\infty)$, so J satisfies the (PS) condition.

Claim 2. J satisfies assumption (1) *of Lemma 1.8.*
Let $0<\varepsilon<\frac{1}{|a|_{L^1}M_1^2}$. From $(F4)$, there exists $0<\delta<1$ such that

$$\left|\frac{1}{a(t)}F\big(t,\frac{1}{p(t)}x\big)\right|\le\frac{\varepsilon}{2}|x|^2,\quad \text{for } t\in[0,+\infty) \text{ and } |x|\le\delta.$$

Using Corollary 2.4, we have

$$\begin{aligned}\int_0^{+\infty}|F(t,u(t))dt|&=\int_0^{+\infty}\left|F\left(t,\frac{1}{p(t)}p(t)u(t)\right)dt\right|\\&\le\int_0^{+\infty}\frac{\varepsilon}{2}|a(t)|p^2(t)|u(t)|^2dt\\&\le\frac{\varepsilon}{2}M_1^2|a|_{L^1}\|u\|^2,\end{aligned}$$

whenever $\|u\|_{\infty,p}\le\delta$.
Let $0<\rho\le\frac{\delta}{M_1}$ and $\alpha=\frac{1}{2}(1-\varepsilon|a|_{L^1}M_1^2)\rho^2$. Then for $\|u\|=\rho$ (note $\|u\|_{\infty,p}\le\delta$), we have

$$\begin{aligned}J(u)&=\frac{1}{2}\|u\|^2-\int_0^{+\infty}F(t,u(t))dt\\&\ge\frac{1}{2}(1-\varepsilon|a|_{L^1}M_1^2)\|u\|^2=\alpha,\end{aligned}$$

so assumption (1) in Lemma 1.8 is satisfied.

Claim 3. J satisfies assumption (2) *of Lemma 1.8.*
By $(F5)(a)$ we have for some $v_0\in H_0^2(0,+\infty)$, $v_0\neq0$,

$$\begin{aligned}J(\xi v_0)&=\frac{1}{2}\xi^2\|v_0\|^2-\int_0^{+\infty}F(t,\xi v_0(t))dt\\&\le\frac{1}{2}\xi^2\|v_0\|^2-|\xi|^\mu\int_0^{+\infty}c_1(t)|v_0(t)|^\mu dt+\int_0^{+\infty}c_2(t)dt.\end{aligned}$$

Now since $\mu>2$, then for $u_0=\xi v_0$, $J(u_0)\le0$, as $\xi\to+\infty$, so assumption (2) in Lemma 1.8 is satisfied. From Lemma 1.8, J possesses a critical point which is a nontrivial weak solution of Problem (1.1). □

As an example of the above theorem, take $f(t,x) = \frac{5}{2}\exp(-t)|x|^{\frac{1}{2}}x$. To see this take

$$c_1(t) = \exp(-t), \qquad c_2(t) = 0,$$
$$\mu = \frac{5}{2}, \qquad a(t) = \frac{1}{(1+t)^2}, \qquad p(t) = \frac{1}{1+t},$$
$$\varphi(t) = \frac{5}{2}e^{-t}, \qquad g(x) = |x|^{\frac{3}{2}} \quad \text{and} \quad \psi_R(t) = (1+t)^{\frac{3}{2}}R^{\frac{3}{2}}.$$

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