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Existence of weak quasi-periodic solutions for a second order Hamiltonian system with damped term via a PDE approach

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Abstract. In this paper, we investigate the existence of weak quasi-periodic solutions for the second order Hamiltonian system with damped term:

$$\ddot{u}(t) + q(t)\dot{u}(t) + DW(u(t)) = 0, \qquad t \in \mathbb{R}, \tag{HSD}$$

where $u : \mathbb{R} \to \mathbb{R}^n$, $q : \mathbb{R} \to \mathbb{R}$ is a quasi-periodic function, $W : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable, DW denotes the gradient of W, $W(x) = -K(x) + F(x) + H(x)$ for all $x \in \mathbb{R}^n$ and W is concave and satisfies the Lipschitz condition. Under some reasonable assumptions on q, K, F, H, we obtain that system has at least one weak quasi-periodic solution. Motivated by Berger et al. (1995) and Blot (2009), we transform the problem of seeking a weak quasi-periodic solution of system (HSD) into a problem of seeking a weak solution of some partial differential system. We construct the variational functional which corresponds to the partial differential system and then by using the least action principle, we obtain the partial differential system has at least one weak solution. Moreover, we present two propositions which are related to the working space and the variational functional, respectively.

Keywords: weak quasi-periodic solution, second order Hamiltonian system, damped term, variational method, PDE approach.

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1 Introduction

Assume that $\omega = (\omega_1, \dots, \omega_m)$ is a list of linearly independent real numbers over the rationals.

Definition 1.1 ([20]). *$u : \mathbb{R} \to \mathbb{R}^n$ is said to be quasi-periodic with m basic frequencies if there exists a function $x \to \Phi(x) \in \mathbb{R}^n$ which is Lipschitz continuous for $x \in \mathbb{R}^m$ and periodic of period 1 in each of its arguments, and m real numbers $\omega_1, \dots, \omega_m$ linearly independent over the rationals, such that $u(t) = \Phi(\omega_1 t, \dots, \omega_m t)$. Any such choice of $\omega_1, \dots, \omega_m$ will be called a set of basic frequencies for $u(t)$.*

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In this paper, we are concerned with the second order Hamiltonian system with damped term:

$$\ddot{u}(t) + q(t)\dot{u}(t) + DW(u(t)) = 0, \qquad t \in \mathbb{R}, \tag{HSD}$$

where DW denotes the gradient of W, $q : \mathbb{R} \to \mathbb{R}$ is a quasi-periodic function with module of frequencies generated by $\omega = (\omega_1, \ldots, \omega_m)$, and satisfies the following condition:

($\mathcal{Q}$) *$q(t) = \sum_{j=1}^{m} q_j(t)$, where $q_j(t)$ is continuous on $\mathbb{R}$ and $\frac{1}{\omega_j}$-periodic, $j = 1, \ldots, m$;*

and $W : \mathbb{R}^n \to \mathbb{R}$, $W(x) = -K(x) + F(x) + H(x)$ for all $x \in \mathbb{R}^n$ and satisfies the following condition:

($\mathcal{W}$) *$W \in C^1(\mathbb{R}^n, \mathbb{R})$, and there exists a positive constant L such that*

$$|DW(x) - DW(y)| \le L|x - y|, \quad \textit{for all } x, y \in \mathbb{R}^n.$$

⋆ **Next, we present the definition of weak ω-quasi-periodic solution for system (HSD).**

For the purpose, we need to recall some function spaces which can be seen in [15], [16], [6] and [7] for more details.

Define

$$AP^0(\mathbb{R}^n) = \{u : \mathbb{R} \to \mathbb{R}^n \mid u \text{ is Bohr almost periodic}\}$$

endowed with the norm $\|u\|_\infty = \sup_{t\in\mathbb{R}} |u(t)|$. Then $AP^0(\mathbb{R}^n)$ is a Banach space. Let $f \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^n)$, that is f is locally Lebesgue-integrable from $\mathbb{R}$ to $\mathbb{R}^n$. Then the mean value of f is the limit (when it exists) $\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)dt$. A fundamental property of Bohr almost periodic function u is that such function has convergent mean, that is, the limit $\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} u(t)dt$ exists. When $u \in AP^0(\mathbb{R}^n)$, define

$$a(u, \lambda) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-i\lambda t} u(t)dt$$

which is a complex vector and is called Fourier–Bohr coefficient of u. Let

$$\Lambda(u) = \{\lambda \in \mathbb{R} \mid a(u, \lambda) \neq 0\}$$

and $\text{Mod}(u)$ the $\mathbb{Z}$-module generated by $\Lambda(u)$. Denote $\mathbb{Z}\langle\omega\rangle$ by the $\mathbb{Z}$-module generated by ω in $\mathbb{R}$.

Define

$$QP^0_\omega(\mathbb{R}^n) := \{u \mid u \in AP^0(\mathbb{R}^n) \text{ and } \text{Mod}(u) \subset \mathbb{Z}\langle\omega\rangle\}.$$

$B^2_\omega(\mathbb{R}^n)$ is the completion of $QP^0_\omega(\mathbb{R}^n)$ with respect to the inner product

$$(u, v)_{B^2} := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} u(t)v(t)dt.$$

For $u \in B^2_\omega(\mathbb{R}^n)$, if $\lim_{r\to 0} \frac{u(t+r)-u(t)}{r}$ exists, then define $\nabla u = \lim_{r\to 0} \frac{u(t+r)-u(t)}{r}$ and $\nabla^2 u = \nabla(\nabla u)$. Define

$$B^{1,2}_\omega(\mathbb{R}^n) := \{u \mid u \in B^2_\omega(\mathbb{R}^n) \text{ and } \nabla u \in B^2_\omega(\mathbb{R}^n)\}$$

endowed with the inner product

$$(u, v)_{B^{1,2}} := (u, v)_{B^2} + (\nabla u, \nabla v)_{B^2}.$$

Then $B^{1,2}_\omega(\mathbb{R}^n)$ is a Hilbert space. Define

$$B^{2,2}_\omega(\mathbb{R}^n) := \{u \mid u \in B^{1,2}_\omega(\mathbb{R}^n) \text{ and } \nabla^2 u \in B^2_\omega(\mathbb{R}^n)\}.$$

Definition 1.2. *If $u \in B^{2,2}_{\omega}(\mathbb{R}^n)$ and satisfies*

$$\nabla^2 u(t) + q(t)\nabla u(t) + DW(u(t)) = 0, \qquad t \in \mathbb{R}.$$

Then u is a weak ω-quasi-periodic solution of system (HSD).

Hamiltonian system is a very important model in physics and it has also extensively appeared in other subjects such as life science, social science, bioengineering, space science and so on. Hence, the theory of Hamiltonian system has been focused on for a long time by mathematicians and physicists. Especially, over the past 40 years, the existence and multiplicity of various solutions have attracted lots of mathematicians. Since Hamiltonian system possesses the variational structure, variational method becomes a very effective tool to deal with those problems on the existence and multiplicity of solutions for Hamiltonian systems. There have been many contributions on periodic solutions, subharmonic solutions and homoclinic solutions (for example, see [10, 14, 17, 21–25, 30] and reference therein). For the investigation about almost periodic solutions of Hamiltonian system, there are less works. Joël Blot and co-authors made some important contributions and had a list of papers (see [1–7]). We refer the reader to [8, 9, 11–13, 18, 28, 29] for some other known results. Next we only recall two works which have a direct relationship with our problem investigated in this paper.

In 1995, via a PDE approach and the least action principle, Berger and Zhang [11] investigated the existence of quasi-periodic solutions of fixed frequencies for the nondissipative second order Duffing equation:

$$\ddot{u}(t) + au(t) - bu^3(t) = f(t),$$

where $u : \mathbb{R} \to \mathbb{R}$, $a > 0, b > 0$ and $f : \mathbb{R} \to \mathbb{R}$ is a quasi-periodic function with frequencies ω.

For system case, in 2009, Blot [7] investigated the existence of ω-quasi-periodic solution for the second order Hamiltonian system without the damped term:

$$\ddot{u}(t) + DW(u(t)) = e(t), \qquad t \in \mathbb{R}, \tag{HS}$$

via a PDE viewpoint which is partially similar to [11], where $u : \mathbb{R} \to \mathbb{R}^n$, $W \in C^2(\mathbb{R}^n)$ and is concave, and

$$\|D^2 W\|_\infty := \sup_{z \in \mathbb{R}^n} |D^2 W(z)| < \min\left\{\frac{1}{C_*}, \frac{1}{C_*^2}\right\}, \tag{1.1}$$

C_* is defined by (2.3) below and $e \in B^{1,2}_{\omega}(\mathbb{R}^n)$ which satisfies $\sum_{\nu \in \mathbb{Z}^m} |a(e, \nu \cdot \omega)|^2 (1 + |\nu|^2) < \infty$. In order to obtain the ω-quasi-periodic solution of system (HS), the author first investigated the existence of weak ω-quasi-periodic solution for system (HS) via a PDE approach. To be precise, the author transformed the problem into seeking a weak solution of the partial differential system:

$$\begin{cases} \sum_{j=1}^{m} \sum_{i=1}^{m} \omega_i \omega_j \dfrac{\partial^2 U}{\partial x_i \partial x_j} + DW(U(x)) = E(x), & \text{on } \Omega \\ U = 0, & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

where $\Omega := (-\pi, \pi)^m \subset \mathbb{R}^m$, $U : \mathbb{R}^m \to \mathbb{R}^n$, $E : \mathbb{R}^m \to \mathbb{R}^n$ and $E(t\omega) = e(t)$. Furthermore, in order to obtain weak solution of system (1.2), the author first investigated the existence of weak solution for the partial differential system:

$$\begin{cases} \sum_{j=1}^{m} \sum_{i=1}^{m} \omega_i \omega_j \dfrac{\partial^2 U}{\partial x_i \partial x_j} + \dfrac{1}{k} \sum_{j=1}^{m} \dfrac{\partial^2 U}{\partial x_j^2} + DW(U(x)) = E(x), & \text{on } \Omega \\ U = 0, & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

Then by careful analysis, as $k \to +\infty$, the sequence $\{U_k\}$ which consists of weak solutions of system (1.3) converges to a weak solution of system (1.2).

Following ideas in [11] and [7], in this paper, when W satisfies some reasonable growth conditions, we investigate the existence of weak ω-quasi-periodic solutions for system (HSD) via a similar PDE approach.

$\star$ **Next, we transform the problem of seeking a weak ω-quasi-periodic solution of (HSD) into a problem of seeking a weak solution of partial differential system (PDS*) below.**

By ($\mathcal{Q}$), define $\hat{q} : \mathbb{R}^m \to \mathbb{R}$ by $\hat{q}(x) = \sum_{j=1}^m \hat{q}_j(x_j)$, where $\hat{q}_j : \mathbb{R} \to \mathbb{R}$ defined by $\hat{q}_j(x_j) := q_j(\frac{x_j}{2\pi\omega_j})$ which satisfies

$$\hat{q}_j(2\pi\omega_j t) = q_j\left(\frac{2\pi\omega_j t}{2\pi\omega_j}\right) = q_j(t).$$

It is easy to verify that $\hat{q}_j$ is 2π-periodic.

Let $Q_j(t) = \int_0^t q_j(t)dt$. Then $Q_j \in C^1(\mathbb{R}, \mathbb{R})$ and is $\frac{1}{\omega_j}$-periodic. Define $\hat{Q} : \mathbb{R}^m \to \mathbb{R}$ by $\hat{Q}(x) = \sum_{j=1}^m \hat{Q}_j(x_j)$, where $\hat{Q}_j : \mathbb{R} \to \mathbb{R}$ defined by $\hat{Q}_j(x_j) := Q_j(\frac{x_j}{2\pi\omega_j})$ which is continuously differentiable and satisfies

$$\hat{Q}_j(2\pi\omega_j t) = Q_j\left(\frac{2\pi\omega_j t}{2\pi\omega_j}\right) = Q_j(t).$$

Then we have

$$\frac{\partial \hat{Q}(x)}{\partial x_j} = \frac{d\hat{Q}_j(x_j)}{dx_j} = \frac{dQ_j(\frac{x_j}{2\pi\omega_j})}{dx_j} = q_j\left(\frac{x_j}{2\pi\omega_j}\right)\frac{1}{2\pi\omega_j} = \frac{1}{2\pi\omega_j}\hat{q}_j(x_j)$$

and

$$\hat{Q}(2\pi\omega t) = \sum_{j=1}^m \hat{Q}_j(2\pi\omega_j t) = \sum_{j=1}^m Q_j(t),$$

which implies that

$$q_j(t) = \frac{dQ_j(t)}{dt} = \frac{d\hat{Q}_j(2\pi\omega_j t)}{dt} = \frac{d\hat{Q}_j(x_j)}{dx_j}\cdot\frac{dx_j}{dt} = \frac{1}{2\pi\omega_j}\hat{q}_j(x_j)\cdot 2\pi\omega_j = \hat{q}_j(x_j)$$

if $x_j = 2\pi\omega_j t, j = 1, \ldots, m$.

Consider the second order elliptic partial differential system:

$$\sum_{j=1}^m\sum_{i=1}^m (2\pi)^2\omega_i\omega_j\frac{\partial^2 U}{\partial x_i\partial x_j} + \hat{q}(x)\sum_{i=1}^m 2\pi\omega_i\frac{\partial U}{\partial x_i} - DK(U(x)) + DF(U(x)) + DH(U(x)) = 0, \quad \text{(PDS)}$$

where $U : \mathbb{R}^m \to \mathbb{R}^n$. Let $\overline{U}(s) := U(2\pi s), s \in \mathbb{R}^m$. Then it is easy to verify that $\overline{U}$ is 1-periodic in each of its arguments if U is 2π-periodic in each of its arguments. Hence, if U is a weak 2π-periodic solution of system (PDS), then $u(t) = U(2\pi\omega t) = \overline{U}(t\omega)$ is a weak ω-quasi-periodic solution of system (HSD). Furthermore, in order to obtain a 2π-periodic solution of system (PDS), following the idea of Blot [7], we seek a weak solution U of the Dirichlet boundary value problem

$$\begin{cases} \sum_{j=1}^m\sum_{i=1}^m (2\pi)^2\omega_i\omega_j\frac{\partial^2 U}{\partial x_i\partial x_j} + \hat{q}(x)\sum_{i=1}^m 2\pi\omega_i\frac{\partial U}{\partial x_i} \\ \qquad -DK(U(x)) + DF(U(x)) + DH(U(x)) = 0, \quad \text{on } \Omega \\ U = 0, \quad \text{on } \partial\Omega \end{cases} \qquad \text{(PDS*)}$$

where $\Omega := (-\pi, \pi)^m \subset \mathbb{R}^m$, and the solution can be extendable into a 2π-periodic solution U of system (PDS) on $\mathbb{R}^m$.

We organize our paper as follows. In Section 2, we introduce the working spaces. In Section 3, we present the variational functional which corresponds to system (PDS*) and then by using the least action principle, give two existence theorems. Finally, we present two propositions which are related to the working space and the variational functional, respectively.

2 Working spaces

In this section, we present the working spaces which were established in [6] and [7].

Let $\mathbb{T}^m := \mathbb{R}^m/2\pi\mathbb{Z}^m$ be the m-dimensional torus and $\bar{\Omega} := [-\pi, \pi]^m \subset \mathbb{R}^m$. Define

$$L^2(\mathbb{T}^m) := \{U : \mathbb{R}^m \to \mathbb{R} \mid U(x + 2\pi\nu) = U(x) \text{ for all } x \in \mathbb{R}^m \text{ and for all } \nu \in \mathbb{Z}^m, \\ \text{and } |U|^2 \text{ is locally Lebesgue-integrable on } \mathbb{R}^m\}$$

and

$$\mathcal{L}^2 := L^2(\mathbb{T}^m)^n := \overbrace{L^2(\mathbb{T}^m) \times \cdots \times L^2(\mathbb{T}^m)}^{n}$$

with the inner product

$$(U, V)_{\mathcal{L}^2} = \int_{\mathbb{T}^m} U \cdot V dx, \quad \text{for all } U, V \in \mathcal{L}^2,$$

and

$$\|U\|_{\mathcal{L}^2} = \int_{\mathbb{T}^m} |U|^2 dx, \quad \text{for all } U \in \mathcal{L}^2,$$

where $U = (U^1, \ldots, U^n)$, $V = (V^1, \ldots, V^n)$, $U \cdot V = \sum_{j=1}^n U^j V^j$ and

$$\int_{\mathbb{T}^m} U(x) dx = \frac{1}{(2\pi)^m} \int_{\bar{\Omega}} U(x) dx, \quad \text{for all } U \in \mathcal{L}^2. \tag{2.1}$$

Define

$$\mathcal{H}^1 := \left\{ U = (U^1, \ldots, U^n) \;\middle|\; U \in \mathcal{L}^2 \text{ and } \frac{\partial U^j}{\partial x_k} \in L^2(\mathbb{T}^m) \text{ for all } j = 1, \ldots, n, k = 1, \ldots, m \right\}$$

and

$$\mathcal{H}^2 := \left\{ U = (U^1, \ldots, U^n) \;\middle|\; U \in \mathcal{H}^1 \text{ and } \frac{\partial^2 U^j}{\partial x_l \partial x_k} \in L^2(\mathbb{T}^m) \text{ for all } j = 1, \ldots, n, l, k = 1, \ldots, m \right\}.$$

For $U \in \mathcal{H}^1$, define

$$\|U\|^2_{\mathcal{H}^1} := \|U\|^2_{\mathcal{L}^2} + \sum_{k=1}^m \left\| \frac{\partial U}{\partial x_k} \right\|^2_{\mathcal{L}^2}.$$

For $U \in \mathcal{L}^2$, define

$$D_\omega U := \lim_{s \to 0} \frac{U(x + s\omega) - U(x)}{s}.$$

Then

$$D_\omega U = \sum_{j=1}^m \omega_j \frac{\partial U}{\partial x_j}. \tag{2.2}$$

By (2.2), it is easy to obtain that

$$D_\omega^2 U := D_\omega(D_\omega U) = \sum_{j=1}^{m}\sum_{i=1}^{m}\omega_j\omega_i\frac{\partial^2 U}{\partial x_i \partial x_j}$$

if it exists. Let

$$\mathcal{H}_\omega^1 := \{U \mid U \in \mathcal{L}^2 \text{ and } D_\omega U \in \mathcal{L}^2\}$$

with the inner product

$$(U,V)_{\mathcal{H}_\omega^1} = (U,V)_{\mathcal{L}^2} + (D_\omega U, D_\omega V)_{\mathcal{L}^2}.$$

Then $\mathcal{H}_\omega^1$ is a Hilbert space and

$$\|U\|_{\mathcal{H}_\omega^1} = \|U\|_{\mathcal{L}^2} + \|D_\omega U\|_{\mathcal{L}^2}.$$

Let

$$\mathcal{H}_\omega^2 := \left\{U \mid U \in \mathcal{H}_\omega^1 \text{ and } D_\omega^2 U \in \mathcal{L}^2\right\}$$

with the inner product

$$(U,V)_{\mathcal{H}_\omega^2} = (U,V)_{\mathcal{L}^2} + (D_\omega U, D_\omega V)_{\mathcal{L}^2} + (D_\omega^2 U, D_\omega^2 V)_{\mathcal{L}^2}.$$

$\mathcal{H}_\omega^2$ is also a Hilbert space.

Let

$$\Omega := (-\pi,\pi)^m = \overbrace{(-\pi,\pi)\times\cdots\times(-\pi,\pi)}^{m} \subset \mathbb{R}^m.$$

Define

$$L^2(\Omega) := \left\{U : \overline{\Omega} \to \mathbb{R} \;\Big|\; \int_\Omega |U(x)|^2 dx < +\infty\right\}$$

and

$$\mathcal{L}^2(\Omega) := L^2(\Omega)^n := \overbrace{L^2(\Omega)\times\cdots\times L^2(\Omega)}^{n}$$

with the inner product

$$(U,V)_{\mathcal{L}^2(\Omega)} = \int_\Omega U\cdot V dx, \quad \text{for all } U,V \in \mathcal{L}^2(\Omega)$$

and the norm

$$\|U\|_{\mathcal{L}^2(\Omega)} = \int_\Omega |U|^2 dx, \quad \text{for all } U \in \mathcal{L}^2(\Omega).$$

Obviously, $\mathcal{L}^2(\Omega) = \mathcal{L}^2$.

Define

$$\begin{aligned}\mathcal{C}^0(\Omega) &:= C_0^0(\Omega)^n \\ &= \{U : \Omega \to \mathbb{R}^n \mid U \text{ is continuous and has a compact support included in } \Omega\}\end{aligned}$$

and for integer $1 \le k < +\infty$, define

$$\mathcal{C}^k(\Omega) := C^k(\Omega)^n = \left\{U : \Omega \to \mathbb{R}^n \mid U \text{ is of class } C^k \text{ on } \Omega\right\}.$$

Let $\mathcal{C}_0^k(\Omega) = \mathcal{C}^0(\Omega) \cap \mathcal{C}^k(\Omega)$ and then define $\mathcal{H}_0^1(\Omega) = \overline{\mathcal{C}_0^1(\Omega)}$ which is the closure of $\mathcal{C}_0^1(\Omega)$ with the inner product

$$(U,V)_{\mathcal{H}_0^1} = (U,V)_{\mathcal{L}^2(\Omega)} + \sum_{j=1}^{m} \left(\frac{\partial U}{\partial x_j}, \frac{\partial V}{\partial x_j} \right)_{\mathcal{L}^2(\Omega)}.$$

Then $(\mathcal{H}_0^1(\Omega), (\cdot,\cdot)_{\mathcal{H}_0^1})$ is a Hilbert space. Following Blot [7], one can extend a function $U \in \mathcal{H}_0^1(\Omega)$ to a function $\tilde{U} \in \mathcal{H}^1$ and by a trace theorem, one can give sense to $U = 0$ on $\partial\Omega$ if $U \in \mathcal{H}_0^1(\Omega)$ so that $\mathcal{H}_0^1(\Omega) \subset \mathcal{H}^1$ and for $U \in \mathcal{H}_0^1(\Omega)$, the following inequality holds: there exists $C_* > 0$ such that

$$\|U\|_{\mathcal{L}^2(\Omega)} \leq C_* \|D_\omega U\|_{\mathcal{L}^2(\Omega)}, \quad \text{for all } U \in \mathcal{H}_0^1(\Omega). \tag{2.3}$$

Let $CL_\omega \mathcal{H}_0^1(\Omega)$ be the closure of $\mathcal{H}_0^1(\Omega)$ in $\mathcal{H}_\omega^1$ with the norm $\|\cdot\|_{\mathcal{H}_\omega^1}$. Then, obviously, $(CL_\omega \mathcal{H}_0^1(\Omega), (\cdot,\cdot)_{\mathcal{H}_\omega^1})$ is also a Hilbert space. We refer the reader for more details about the above working spaces to [6] and [7].

3 Main results

In [26], Wu and Chen directly construct a variational functional which corresponds to the second order Hamiltonian system like (HSD) in order to investigate the existence of periodic solutions. Motivated by [26], we define a functional $\mathcal{J} : CL_\omega \mathcal{H}_0^1(\Omega) \to \mathbb{R}$ by

$$\begin{aligned} \mathcal{J}(U) &= \int_\Omega e^{\hat{Q}(x)} \left[\frac{1}{2} \left(\sum_{i=1}^{m} 2\pi\omega_i \frac{\partial U}{\partial x_i} \right)^2 - W(U(x)) \right] dx \\ &= \int_\Omega e^{\hat{Q}(x)} \left[\frac{1}{2} \left(\sum_{i=1}^{m} 2\pi\omega_i \frac{\partial U}{\partial x_i} \right)^2 + K(U(x)) - F(U(x)) - H(U(x)) \right] dx. \end{aligned}$$

When $(\mathcal{Q})$ and $(\mathcal{W})$ hold, a standard argument can be made easily so that $\mathcal{J}$ is of class C^1 and

$$\begin{aligned} \langle \mathcal{J}'(U), V \rangle &= \int_\Omega e^{\hat{Q}(x)} \left[\left(\sum_{i=1}^{m} 2\pi\omega_i \frac{\partial U}{\partial x_i}, \sum_{j=1}^{m} 2\pi\omega_j \frac{\partial V}{\partial x_j} \right) - (DW(U(x)), V(x)) \right] dx \\ &= \int_\Omega e^{\hat{Q}(x)} \Bigg[\left(\sum_{i=1}^{m} 2\pi\omega_i \frac{\partial U}{\partial x_i}, \sum_{j=1}^{m} 2\pi\omega_j \frac{\partial V}{\partial x_j} \right) + (DK(U(x)), V(x)) \\ &\qquad - (DF(U(x)), V(x)) - (DH(U(x)), V(x)) \Bigg] dx \end{aligned} \tag{3.1}$$

for all $V \in CL_\omega \mathcal{H}_0^1(\Omega)$.

Remark 3.1. When $n = 1$, in [19] and [27], there have been more general functionals which correspond to more general partial differential equations. In some sense, when $n = 1$, the functional $\mathcal{J}(U)$ can be seen as a special case of those in [19] and [27] if we choose $a_{i,j}(x,u) \equiv e^{\hat{Q}(x)}$, $i,j = 1, \ldots, m$, where the details of $a_{i,j}(x,u)$ can be seen in [19] and [27].

Lemma 3.2. *Assume that $\mathcal{J}'(U^*) = 0$ for some $U^* \in CL_\omega \mathcal{H}_0^1(\Omega)$. Then $u^*(t) := U^*(2\pi\omega t)$ is a weak ω-quasi-periodic solution of system* (HSD).

Proof. For any given $V \in CL_\omega \mathcal{H}_0^1(\Omega)$, there exists a sequence $\{V_k\} \subset \mathcal{H}_0^1(\Omega)$ such that

$$\|V_k - V\|_{\mathcal{H}^1_\omega} \to 0$$

which implies that

$$\|D_\omega V_k - D_\omega V\|_{\mathcal{L}^2} \to 0 \quad \text{and} \quad \|V_k - V\|_{\mathcal{L}^2} \to 0, \quad \text{as } k \to \infty \tag{3.2}$$

and so $V_k(x) \to V(x)$ for a.e. $x \in \Omega$. Let

$$M(V_k) := \int_\Omega e^{\hat{Q}(x)} \left(\sum_{i=1}^m 2\pi\omega_i \frac{\partial U^*}{\partial x_i}, \sum_{j=1}^m 2\pi\omega_j \frac{\partial V}{\partial x_j} - \sum_{j=1}^m 2\pi\omega_j \frac{\partial V_k}{\partial x_j} \right) dx.$$

Then it follows from Hölder's inequality and (3.2) that $M(V_k) \to 0$ as $k \to \infty$,

$$\begin{aligned} &\int_\Omega e^{\hat{Q}(x)} \left(\sum_{j=1}^m \sum_{i=1}^m (2\pi)^2 \omega_j \omega_i \frac{\partial^2 U^*}{\partial x_i \partial x_j}, V_k(x) \right) dx \\ &\quad \to \int_\Omega e^{\hat{Q}(x)} \left(\sum_{j=1}^m \sum_{i=1}^m (2\pi)^2 \omega_j \omega_i \frac{\partial^2 U^*}{\partial x_i \partial x_j}, V(x) \right) dx \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} &\int_\Omega e^{\hat{Q}(x)} \left(\sum_{j=1}^m \hat{q}_j(x_j) \sum_{i=1}^m 2\pi\omega_i \frac{\partial U^*}{\partial x_i}, V_k(x) \right) dx \\ &\quad \to \int_\Omega e^{\hat{Q}(x)} \left(\sum_{j=1}^m \hat{q}_j(x_j) \sum_{i=1}^m 2\pi\omega_i \frac{\partial U^*}{\partial x_i}, V(x) \right) dx. \end{aligned} \tag{3.4}$$

By integration by parts and noting that $V_k = 0$ on $\partial\Omega$, we have

$$\begin{aligned} &\int_\Omega e^{\hat{Q}(x)} \left(\sum_{i=1}^m 2\pi\omega_i \frac{\partial U^*}{\partial x_i}, \sum_{j=1}^m 2\pi\omega_j \frac{\partial V}{\partial x_j} \right) dx \\ &= \int_\Omega e^{\hat{Q}(x)} \left(\sum_{i=1}^m 2\pi\omega_i \frac{\partial U^*}{\partial x_i}, \sum_{j=1}^m 2\pi\omega_j \frac{\partial V}{\partial x_j} - \sum_{j=1}^m 2\pi\omega_j \frac{\partial V_k}{\partial x_j} + \sum_{j=1}^m 2\pi\omega_j \frac{\partial V_k}{\partial x_j} \right) dx \\ &= \int_\Omega e^{\hat{Q}(x)} \left(\sum_{i=1}^m 2\pi\omega_i \frac{\partial U^*}{\partial x_i}, \sum_{j=1}^m 2\pi\omega_j \frac{\partial V_k}{\partial x_j} \right) dx + M(V_k) \\ &= -\int_\Omega \left(\sum_{j=1}^m 2\pi\omega_j \sum_{i=1}^m \left(2\pi\omega_i e^{\hat{Q}(x)} \frac{\partial U^*}{\partial x_i} \right)'_{x_j}, V_k(x) \right) dx + M(V_k) \\ &= -\int_\Omega \left(\sum_{j=1}^m 2\pi\omega_j \left(\sum_{i=1}^m 2\pi\omega_i e^{\hat{Q}(x)} \frac{\partial^2 U^*}{\partial x_i \partial x_j} \right), V_k(x) \right) dx \\ &\quad - \int_\Omega \left(\sum_{j=1}^m 2\pi\omega_j \left(\sum_{i=1}^m 2\pi\omega_i e^{\hat{Q}(x)} \frac{\partial \hat{Q}(x)}{\partial x_j} \frac{\partial U^*}{\partial x_i} \right), V_k(x) \right) dx + M(V_k) \\ &= -\int_\Omega \left(\sum_{j=1}^m 2\pi\omega_j \left(\sum_{i=1}^m 2\pi\omega_i e^{\hat{Q}(x)} \frac{\partial^2 U^*}{\partial x_i \partial x_j} \right), V_k(x) \right) dx \end{aligned}$$

$$- \int_\Omega \left(\sum_{j=1}^m 2\pi\omega_j \left(\sum_{i=1}^m 2\pi\omega_i e^{\hat{Q}(x)} \frac{1}{2\pi\omega_j} \hat{q}_j(x_j) \frac{\partial U^*}{\partial x_i} \right), V_k(x) \right) dx + M(V_k)$$
$$= - \int_\Omega e^{\hat{Q}(x)} \left(\sum_{j=1}^m \sum_{i=1}^m (2\pi)^2 \omega_j\omega_i \frac{\partial^2 U^*}{\partial x_i \partial x_j}, V_k(x) \right) dx$$
$$- \int_\Omega e^{\hat{Q}(x)} \left(\sum_{j=1}^m \hat{q}_j(x_j) \sum_{i=1}^m 2\pi\omega_i \frac{\partial U^*}{\partial x_i}, V_k(x) \right) dx + M(V_k). \tag{3.5}$$

Let $k \to \infty$. (3.3), (3.4) and (3.5) imply that

$$\int_\Omega e^{\hat{Q}(x)} \left(\sum_{i=1}^m 2\pi\omega_i \frac{\partial U^*}{\partial x_i}, \sum_{j=1}^m 2\pi\omega_j \frac{\partial V}{\partial x_j} \right) dx$$
$$= - \int_\Omega e^{\hat{Q}(x)} \left(\sum_{j=1}^m \sum_{i=1}^m (2\pi)^2 \omega_j\omega_i \frac{\partial^2 U^*}{\partial x_i \partial x_j}, V(x) \right) dx$$
$$- \int_\Omega e^{\hat{Q}(x)} \left(\sum_{j=1}^m \hat{q}_j(x_j) \sum_{i=1}^m 2\pi\omega_i \frac{\partial U^*}{\partial x_i}, V(x) \right) dx$$

for all $V \in CL_\omega \mathcal{H}_0^1(\Omega)$. If $\mathcal{J}'(U^*) = 0$, then (3.1) and the above equality imply that

$$\begin{aligned} 0 = &- \int_\Omega e^{\hat{Q}(x)} \left(\sum_{j=1}^m \sum_{i=1}^m (2\pi)^2 \omega_j\omega_i \frac{\partial^2 U^*}{\partial x_i \partial x_j}, V(x) \right) dx \\ &- \int_\Omega e^{\hat{Q}(x)} \left(\sum_{j=1}^m \hat{q}_j(x_j) \sum_{i=1}^m 2\pi\omega_i \frac{\partial U^*}{\partial x_i}, V(x) \right) dx \\ &+ \int_\Omega e^{\hat{Q}(x)} \left[(DK(U^*(x)), V(x)) - (DF(U^*(x)), V(x)) - (DH(U^*(x)), V(x)) \right] dx \end{aligned} \tag{3.6}$$

for all $V \in CL_\omega \mathcal{H}_0^1(\Omega)$. Following the idea of Blot [7], (3.6) implies that

$$\begin{aligned} &- \sum_{j=1}^m \sum_{i=1}^m (2\pi)^2 \omega_j\omega_i \frac{\partial^2 U^*}{\partial x_i \partial x_j} - \sum_{j=1}^m \hat{q}_j(x_j) \sum_{i=1}^m 2\pi\omega_i \frac{\partial U^*}{\partial x_i} \\ &\qquad + DK(U^*(x)) - DF(U^*(x)) - DH(U^*(x)) = 0 \end{aligned} \tag{3.7}$$

in $D'(\Omega)^n$, where $D'(\Omega)$ denotes the space of the distributions in the sense of Schwartz on Ω and $D'(\Omega)^n$ is the n-times product of $D'(\Omega)$. Note that $DK(U^*)$, $DF(U^*)$, $DH(U^*)$, and $\sum_{i=1}^m \omega_i \frac{\partial U^*}{\partial x_i} (= D_\omega U^*)$ belong to $\mathcal{L}^2(\Omega)(= \mathcal{L}^2)$ and $\hat{q}_j \in L^2(\Omega)$, $j = 1, \ldots, m$. Hence, $D_\omega^2 U^* := \sum_{j=1}^m \sum_{i=1}^m \omega_j\omega_i \frac{\partial^2 U^*}{\partial x_i \partial x_j} \in \mathcal{L}^2(\Omega)(= \mathcal{L}^2)$. Hence, (3.7) holds in $\mathcal{L}^2$, which shows that U^* is a weak 2π-periodic solution of system (PDS) and $U^* \in \mathcal{H}_\omega^2$. Then $u^*(t) := U^*(2\pi\omega t) \in B_\omega^{2,2}(\mathbb{R}^n)$ and $u^*(t)$ is a weak ω-quasi-periodic solution of system (HSD). □

Lemma 3.3 (see [21]). *If φ is weakly lower semi-continuous on a reflexive Banach space X and has a bounded minimizing sequence, then φ has minimum on X.*

Theorem 3.4. *Suppose that W is concave, $(\mathcal{Q})$, $(\mathcal{W})$, and the following conditions hold:*

$(\mathcal{K})$ there exists a positive constant b such that

$$K(x) \geq b|x|^2, \qquad \forall x \in \mathbb{R}^n;$$

($\mathcal{F}$) *there exist positive constants* d_1, d_2 *and* $\alpha \in [0,2)$ *such that*

$$F(x) \le d_1|x|^\alpha + d_2, \qquad \forall x \in \mathbb{R}^n;$$

($\mathcal{H}$) *there exists a positive constant* $l < \frac{2\pi^2}{C_*} + b$ *such that*

$$|DH(x) - DH(y)| \le l|x-y|, \quad \text{for all } x, y \in \mathbb{R}^n.$$

Then system (HSD) *has at least one weak* ω*-quasi-periodic solution* $u^* \in B^{2,2}_\omega(\mathbb{R}^n)$.

Proof. By ($\mathcal{H}$), there exists $\theta \in (0,1)$ such that

$$\begin{aligned} |H(x)| - |H(0)| &\le |H(x) - H(0)| \\ &\le |DH(\theta x)||x| \\ &\le (|DH(0)| + l|\theta x|)|x| \\ &\le |DH(0)||x| + l|x|^2 \end{aligned} \tag{3.8}$$

for all $x \in \mathbb{R}^m$. Then it follows from ($\mathcal{K}$), ($\mathcal{F}$), (3.8) and Hölder's inequality that

$$\begin{aligned} \mathcal{J}(U) &= \int_\Omega e^{\hat{Q}(x)} \left[\frac{1}{2} \left(\sum_{i=1}^m 2\pi\omega_i \frac{\partial U}{\partial x_i} \right)^2 + K(U(x)) - F(U(x)) - H(U(x)) \right] dx \\ &\ge \int_\Omega e^{\hat{Q}(x)} \left[\frac{1}{2} \left(\sum_{i=1}^m 2\pi\omega_i \frac{\partial U}{\partial x_i} \right)^2 + b|U(x)|^2 - d_1|U(x)|^\alpha - d_2 \right. \\ &\qquad \left. - |H(0)| - |DH(0)||U(x)| - l|U(x)|^2 \right] dx \\ &\ge e^{M_*} \Big[\min\{2\pi^2, b-l\} \|U\|^2_{\mathcal{H}^1_\omega} - d_1 (\text{mes}\Omega)^{\frac{2-\alpha}{2}} \|U\|^\alpha_{\mathcal{L}^2} \\ &\qquad -(d_2 + |H(0)|)\,\text{mes}\,\Omega - |DH(0)|\sqrt{\text{mes}\,\Omega}\|U\|_{\mathcal{L}^2} \Big] \\ &\ge e^{M_*} \Big[\min\{2\pi^2, b-l\} \|U\|^2_{\mathcal{H}^1_\omega} - d_1 (\text{mes}\Omega)^{\frac{2-\alpha}{2}} \|U\|^\alpha_{\mathcal{H}^1_\omega} \\ &\qquad -(d_2 + |H(0)|)\,\text{mes}\,\Omega - |DH(0)|\sqrt{\text{mes}\,\Omega}\|U\|_{\mathcal{H}^1_\omega} \Big] \end{aligned} \tag{3.9}$$

for all $U \in CL_\omega\mathcal{H}^1_0(\Omega)$, where $M_* = \min_{x\in\bar{\Omega}} \hat{Q}(x)$. Note that $\alpha \in [0,2)$. The above inequality implies that $\mathcal{J}(U)$ is coercive, that is,

$$\mathcal{J}(U) \to +\infty \quad \text{as } \|U\|_{\mathcal{H}^1_\omega} \to +\infty. \tag{3.10}$$

If $\{U_k\}$ is a minimizing sequence in $CL_\omega\mathcal{H}^1_0(\Omega)$, that is,

$$\mathcal{J}(U_k) \to \inf_{x\in CL_\omega\mathcal{H}^1_0(\Omega)} \mathcal{J}(U), \quad \text{as } k \to \infty,$$

then there exists a positive constant C such that $|\mathcal{J}(U_k)| \le C$, which, together with (3.10), implies that $\{U_k\}$ is bounded in $CL_\omega\mathcal{H}^1_0(\Omega)$. Since W is concave, $\mathcal{J}$ is convex. Moreover, note that $\mathcal{J}$ is of class C^1. Hence, $\mathcal{J}$ is lower semi-continuous on $CL_\omega\mathcal{H}^1_0(\Omega)$. By Theorem 1.2 in [21], $\mathcal{J}$ is weakly lower semi-continuous on $CL_\omega\mathcal{H}^1_0(\Omega)$. Then Lemma 3.3 implies that $\mathcal{J}$ has a critical point U^* in $CL_\omega\mathcal{H}^1_0(\Omega)$. Finally, by Lemma 3.2, we complete the proof. □

Example 3.5. Let $K(x) = b_1|x|^2$ with $b_1 \geq b$, $F(x) = d(1+|x|)^{3/2}$ with $d > 0$ and $H(x) = c\ln(1+|x|^2)$ with $c > 0$ and $-b_1 + c + \frac{3}{8}d < 0$. It is easy to verify that K, F and H satisfy those assumptions in Theorem 3.1.

When $\alpha = 2$, by (3.9), it is easy to obtain the following theorem.

Theorem 3.6. *Suppose that W is concave, $(\mathcal{Q})$, $(\mathcal{W})$, $(\mathcal{K})$, $(\mathcal{H})$ and the following condition holds:*
$(\mathcal{F})'$ there exist constants $0 < d_1 < \min\{2\pi^2, b-l\}$, and $d_2 > 0$ such that

$$F(x) \leq d_1|x|^2 + d_2, \qquad \forall x \in \mathbb{R}^n.$$

Then system (HSD) has at least one weak ω-quasi-periodic solution $u^* \in B^{2,2}_{\omega}(\mathbb{R}^n)$.

Next, we present two propositions which are related to the working space and the variational functional, respectively.

Proposition 3.7. *For $U \in CL_\omega\mathcal{H}^1_0(\Omega)$, (2.3) also holds.*

Proof. Since $CL_\omega\mathcal{H}^1_0(\Omega)$ is the closure of $\mathcal{H}^1_0(\Omega)$ in $\mathcal{H}^1_\omega$, for $U \in CL_\omega\mathcal{H}^1_0(\Omega)$, there exists a sequence $\{U_k\} \subset \mathcal{H}^1_0(\Omega)$ such that $\|U - U_k\|_{\mathcal{H}^1_\omega} \to 0$ which implies that $\|D_\omega U - D_\omega U_k\|_{\mathcal{L}^2} \to 0$ and then by (2.1), $\|D_\omega U - D_\omega U_k\|_{\mathcal{L}^2(\Omega)} \to 0$. Note that $U_k \in \mathcal{H}^1_0(\Omega)$. Hence, (2.3) implies that

$$\begin{aligned} C_*\|D_\omega U\|_{\mathcal{L}^2(\Omega)} &= C_*\|D_\omega U - D_\omega U_k + D_\omega U_k\|_{\mathcal{L}^2(\Omega)} \\ &\geq C_*\|D_\omega U_k\|_{\mathcal{L}^2(\Omega)} - C_*\|D_\omega U - D_\omega U_k\|_{\mathcal{L}^2(\Omega)} \\ &\geq \|U_k\|_{\mathcal{L}^2(\Omega)} - C_*\|D_\omega U - D_\omega U_k\|_{\mathcal{L}^2(\Omega)}. \end{aligned}$$

Let $k \to \infty$. We have

$$C_*\|D_\omega U\|_{\mathcal{L}^2(\Omega)} \geq \|U\|_{\mathcal{L}^2(\Omega)}.$$

□

Proposition 3.8. *Assume that $\mathcal{J}'(U^*) = 0$ for some $U^* \in \mathcal{H}^1_0(\Omega)$. Then U^* is a critical point of $\mathcal{J}$ in $CL_\omega\mathcal{H}^1_0(\Omega)$, that is, $\langle \mathcal{J}'(U^*), V\rangle = 0$, for all $V \in CL_\omega\mathcal{H}^1_0(\Omega)$.*

Proof. For an arbitrary $V \in CL_\omega\mathcal{H}^1_0(\Omega)$, there exists a sequence $\{V_k\} \subset \mathcal{H}^1_0(\Omega)$ such that $\|V_k - V\|_{\mathcal{H}^1_\omega} \to 0$. Then by Hölder's inequality and (2.1), we have

$$\begin{aligned} &|\langle \mathcal{J}'(U^*), V - V_k\rangle| \\ &= \left| \int_\Omega e^{\hat{Q}(x)} \left[\left(\sum_{i=1}^m 2\pi\omega_i \frac{\partial U^*}{\partial x_i}, \sum_{j=1}^m 2\pi\omega_j \left(\frac{\partial V}{\partial x_j} - \frac{\partial V_k}{\partial x_j} \right) \right) - (DW(U^*(x)), V(x) - V_k(x)) \right] dx \right| \\ &\leq e^{M^*} \left[\left(\int_\Omega \left| \sum_{i=1}^m 2\pi\omega_i \frac{\partial U^*}{\partial x_i} \right|^2 dx \right)^{1/2} \left(\int_\Omega \left| \sum_{j=1}^m 2\pi\omega_j \left(\frac{\partial (V - V_k)}{\partial x_j} \right) \right|^2 dx \right)^{1/2} \right. \\ &\qquad \left. + \left(\int_\Omega |DW(U^*)|^2 dx \right)^{1/2} \left(\int_\Omega |V - V_k|^2 \right)^{1/2} dx \right] \\ &= e^{M^*} \left[4\pi^2 \|D_\omega U^*\|_{\mathcal{L}^2(\Omega)} \|D_\omega V - D_\omega V_k\|_{\mathcal{L}^2(\Omega)} + \|DW(U^*)\|_{\mathcal{L}^2(\Omega)} \|V - V_k\|_{\mathcal{L}^2(\Omega)} \right] \\ &= e^{M^*} \left[(2\pi)^{(2m+2)} \|D_\omega U^*\|_{\mathcal{L}^2} \|D_\omega V - D_\omega V_k\|_{\mathcal{L}^2} + (2\pi)^{(2m)} \|DW(U^*)\|_{\mathcal{L}^2} \|V - V_k\|_{\mathcal{L}^2} \right] \\ &\leq e^{M^*} \left[(2\pi)^{(2m+2)} \|D_\omega U^*\|_{\mathcal{L}^2} + (2\pi)^{(2m)} \|DW(U^*)\|_{\mathcal{L}^2} \right] \|V - V_k\|_{\mathcal{H}^1_\omega} \to 0 \quad \text{as } k \to \infty, \end{aligned} \tag{3.11}$$

where $M^* = \max_{x\in\bar{\Omega}} e^{\hat{Q}(x)}$. Note that $\langle \mathcal{J}'(U^*), V\rangle = 0$ for all $V \in \mathcal{H}_0^1(\Omega)$. Hence, by (3.11), we have

$$|\langle \mathcal{J}'(U^*), V\rangle| \le |\langle \mathcal{J}'(U^*), V - V_k\rangle| + |\langle \mathcal{J}'(U^*), V_k\rangle| = |\langle \mathcal{J}'(U^*), V - V_k\rangle| \to 0 \quad \text{as } k \to \infty.$$

□

Remark 3.9. Proposition 3.7 and Proposition 3.8 are maybe useful for one to seek the critical points of the functional $\mathcal{J}$ by using those abstract critical point theorems with Palais–Smale condition. We try to do such things by using Ekeland variational principle so that the restriction on concavity of W can be deleted. However, we come across a difficulty whether the embedding $CL_\omega\mathcal{H}_0^1(\Omega) \hookrightarrow \mathcal{L}^2(\Omega)$ is compact. Note that the embedding $\mathcal{H}_0^1(\Omega) \hookrightarrow \mathcal{L}^2(\Omega)$ is compact. So maybe one can reduce our problem from $CL_\omega\mathcal{H}_0^1(\Omega)$ to $\mathcal{H}_0^1(\Omega)$ by Proposition 3.8. However, a new difficulty whether (PS) sequence of $\mathcal{J}$ is bounded in $\mathcal{H}_0^1(\Omega)$ appears. This is a problem that is worthy of consideration.

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References

[1] J. Blot, Calculus of variations in mean and convex Lagrangians, *J. Math. Anal. Appl.* **134**(1988), 312–321. MR961340; url

[2] J. Blot, Calculus of variations in mean and convex Lagrangians, II, *Bull. Austral. Math. Soc.* **40**(1989), 457–463. MR1037643; url

[3] J. Blot, Calculus of variations in mean and convex Lagrangians, III, *Israel J. Math.*, **67**(1989), 337–344. MR1029907; url

[4] J. Blot, Une approche variationnelle des orbites quasi-périodiques des systèmes hamiltoniens (in French) [A variational approach to the quasiperiodic orbits of Hamiltonian systems], *Ann. Sci. Math. Québec*, **13**(1989), No. 2, 7–32. MR1038364

[5] J. Blot, Almost-periodic solutions of forced second order Hamiltonian systems, *Ann. Fac. Sci. Toulouse Math. (5)* **12**(1991), 351–363. MR1189445

[6] J. Blot, D. Pennequin, Spaces of quasi-periodic functions and oscillations in differential equations, *Acta Appl. Math.* **65**(2001), 83–113. MR1843787; url

[7] J. Blot, Quasi-periodic oscillations in second-order systems via semilinear elliptic equations, in: *Proceedings of the 6th International Conference on Differential Equations and Dynamical Systems*, DCDIS A Supplement, Watam Press, 2009, pp. 135–140.

[8] M. S. Berger, Y. Y. Chen, Forced quasiperiodic and almost periodic oscillations of nonlinear Duffing equations, *Nonlinear Anal.* **19**(1992), 249–257. MR1176061; url

[9] M. S. Berger, Y. Y. Chen, Forced quasiperiodic and almost periodic solution for nonlinear systems, *Nonlinear Anal.* **21**(1993), 949–965. MR1249212; url

[10] G. Bonanno, R. Livrea, Existence and multiplicity of periodic solutions for second order Hamiltonian systems depending on a parameter, *J. Convex Anal.* **20**(2013), 1075–1094. MR3184297

[11] M. S. Berger, L. Zhang, New method for large quasiperiodic nonlinear oscillations with fixed frequencies for the nondissipative second type duffing equation, *Topol. Method Nonlinear Anal.* **6**(1995), 283-293. MR1399541

[12] C. Carminati, Forced systems with almost periodic and quasiperiodic forcing term, *Nonlinear Anal.* **32**(1998), 727–739. MR1612110; url

[13] P. Cieutat, Bounded and almost periodic solutions of convex Lagrangian systems, *J. Differential Equations* **190**(2003), 108–130. MR1970958; url

[14] Y. Ding, *Variational methods for strongly indefinite problems*, Singapore, Interdisciplinary Mathematical Sciences, Vol. 7, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007. MR2389415; url

[15] A. M. Fink, *Almost periodic differential equations*, Lecture Notes in Mathematics, Vol. 377, Springer, Berlin, Germany, 1974. MR0460799

[16] C. Y. He, *Almost periodic differential equations*, Higher Education Publishing House, Beijing, China, 1992.

[17] M. Izydorek, J. Janczewska, Homoclinic solutions for a class of the second order Hamiltonian systems, *J. Differential Equations*, **219**(2005), 375–389. MR2183265; url

[18] J. Kuang, Variational approach to quasi-periodic solution of nonautonomous second-order Hamiltonian systems, *Abstr. Appl. Anal.* **2012**, Art. ID 271616, 14 pp. MR2889075; url

[19] X. Q. Liu, J. Q. Liu, Z. Q. Wang, Quasilinear elliptic equations via perturbation method, *Proc. Amer. Math. Soc.* **141**(2013), 253–263. MR2988727; url

[20] M. Y. Li , J. S. Muldowney, Poincaré's stability condition for quasi-periodic orbits, *Canad. Appl. Math. Quart.* **6**(1998), 367–381. MR1668038

[21] J. Mawhin, M. Willem, *Critical point theory and Hamiltonian systems*, Applied Mathematical Sciences, Vol. 74, Springer-Verlag, New York, 1989. MR982267; url

[22] P. H. Rabinowitz, Periodic solutions of Hamiltonian systems, *Comm. Pure Appl. Math.* **31**(1978), 157–184. MR0467823

[23] P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conference Series in Mathematics, Vol. 65, American Mathematical Society, Providence, RI, 1986. MR845785; url

[24] M. Schechter, *Minimax systems and critical point theory*, Birkhäuser, Boston, 2009. MR2512303; url

[25] M. Schechter, Homoclinic solutions of nonlinear second-order Hamiltonian systems, *Ann. Mat. Pura Appl.* **195**(2016), 1665–1683. MR3537968; url

[26] X. Wu, J. Chen, Existence theorems of periodic solutions for a class of damped vibration problems, *Appl. Math. Comput.* **207**(2009), 230–235. MR2492737; url

[27] X. Wu, K. Wu, Existence of positive solutions, negative solutions and high energy solutions for quasi-linear elliptic equations on $\mathbb{R}^N$, *Nonlinear Anal. Real World Appl.* **16**(2014), 48–64. MR3123800; url

[28] X. Zhang, Existence and multiplicity of weak quasi-periodic solutions for second order Hamiltonian system with a forcing term, *Electron. J. Qual. Theory Differ. Equ.* **2014**, No. 63, 1–19. MR3296524; url

[29] S. F. Zakharin, I. O. Parasyuk, Generalized and classical almost periodic solutions of Lagrangian systems, *Funkcial. Ekvac.* **42**(1999), 325–338. MR1745307

[30] X. Zhang, X. Tang, Subharmonic solutions for a class of non-quadratic second order Hamiltonian systems, *Nonlinear Anal. Real World Appl.* **13**(2012), 113–130. MR2846824; url