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Spectral asymptotics for inverse nonlinear Sturm-Liouville problems

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Abstract

We consider the nonlinear Sturm-Liouville problem

$$-u''(t) + f(u(t), u'(t)) = \lambda u(t), \;\; u(t) > 0, \quad t \in I := (-1/2, 1/2), \;\; u(\pm 1/2) = 0,$$

where $f(x, y) = |x|^{p-1}x - |y|^m$, $p > 1, 1 \le m < 2$ are constants and $\lambda > 0$ is an eigenvalue parameter. To understand well the global structure of the bifurcation branch of positive solutions in $\mathbf{R}_+ \times L^q(I)$ $(1 \le q < \infty)$ from a viewpoint of inverse problems, we establish the precise asymptotic formulas for the eigenvalue $\lambda = \lambda_q(\alpha)$ as $\alpha := \|u_\lambda\|_q \to \infty$, where u_λ is a solution associated with given $\lambda > \pi^2$.

1 Introduction

We consider the following nonlinear Sturm-Liouville problem

$$-u''(t) + f(u(t), u'(t)) = \lambda u(t), \quad t \in I := (-1/2, 1/2), \tag{1.1}$$

$$u(t) > 0, \quad t \in I, \tag{1.2}$$

$$u(-1/2) = u(1/2) = 0, \tag{1.3}$$

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where $f(x,y) = |x|^{p-1}x - |y|^m$, $p > 1, 1 \le m < 2$ are constants and $\lambda > 0$ is an eigenvalue parameter. Let $1 \le q < \infty$ be fixed. Then we know from [1] that for each given $\lambda > \pi^2$, there exists a solution $(\lambda, u) = (\lambda, u_\lambda) \in \mathbf{R}_+ \times C^2(\bar{I})$ of (1.1)–(1.3) with $\|u_\lambda\|_\infty < \lambda^{1/(p-1)}$. We denote by $\alpha = \alpha_\lambda = \|u_\lambda\|_q$ and $\lambda = \lambda_q(\alpha)$. Here, $\|\cdot\|_q$ denotes the usual L^q-norm.

The purpose of this paper is to study precisely how the damping term $|u'(t)|^m$ gives effect on the global behavior of the bifurcation branch $\lambda_q(\alpha)$ from a viewpoint of *inverse eigenvalue problems*. To this end, we establish the precise asymptotic formulas for $\lambda_q(\alpha)$ as $\alpha \to \infty$.

To explain the background and the motivation of our problem here, we recall the known and related facts of our problems. The equation (1.1)–(1.3) without damping term (e.g. $f(u(t), u'(t)) = |u(t)|^{p-1}u(t)$) is well known as the diffusive equation of population dynamics (the solution is denoted by $u_{0,\lambda}$), and has been studied extensively by many authors by local and global L^∞ bifurcation theory. We refer to [2] and [7–10]. In particular, as a basic asymptotic behavior of $u_{0,\lambda}$ as $\lambda \to \infty$, the following formula is well known.

$$\|u_{0,\lambda}\|_\infty^{p-1} = \lambda - \lambda e^{-\sqrt{(p-1)\lambda}(1+o(1))/2}. \tag{1.4}$$

We also mention that relationship between nonlinear term and the property of eigenvalues has been studied in [14] with emphasis on the uniqueness of $f(u)$. Furthermore, since (1.1)–(1.3) without damping term is regarded as a nonlinear eigenvalue problem, it is quite important to study (1.1)–(1.3) from a viewpoint of L^2-theory, that is the case where $q = 2$. For the works in this direction, we refer to [4–6] and the references therein. It should be mentioned that one of the chief concern in this field is to investigate the local and global shape of the L^2-bifurcation branch $\lambda_2(\alpha)$, and the asymptotic behavior of $\lambda_2(\alpha)$ as $\alpha \to 0$ has been studied in [4], [5]. Besides, it seems important to study the asymptotic behavior of $\lambda_q(\alpha)$ as $\alpha \to \infty$ for general $1 \le q < \infty$ $(q \ne 2)$. In particular, it is meaningful to consider this problem in L^1-framework, since (1.1)–(1.3) without damping term comes from the equation of population dynamics.

The leading term of $\lambda_q(\alpha)$ without damping term in (1.1) can be obtained easily as follows. Since it is known from [2] that

$$\frac{u_{0,\lambda}(t)}{\lambda^{1/(p-1)}} \to 1 \tag{1.5}$$

locally uniformly on I as $\lambda \to \infty$, we obtain that for $\lambda \gg 1$ (i.e. $\alpha \gg 1$),

$$\lambda_q(\alpha) = \alpha^{p-1} + o(\alpha^{p-1}).$$

Recently, more precise asymptotic formula for $\lambda_q(\alpha)$ has been obtained in [11].

Theorem 1.1 ([11]). *Consider (1.1)–(1.3) with $f(u,u') = |u|^{p-1}u$. Then as $\alpha \to \infty$*

$$\lambda_q(\alpha) = \alpha^{p-1} + C_1\alpha^{(p-1)/2} + a_0 + o(1), \tag{1.6}$$

where

$$C_1 = \frac{p-1}{q}C(q), \quad C(q) := 2\int_0^1 \frac{1-s^q}{\sqrt{1-s^2-2(1-s^{p+1})/(p+1)}}ds, \quad a_0 = \frac{p-1}{2q}C(q)^2.$$

Since such a quite precise asymptotic formula for $\lambda_q(\alpha)$ as (1.6) has been obtained, from the standpoint of the better understanding of the global structure of the bifurcation branch of the positive solutions, the following problem from a view point of inverse problem was proposed in [12]. Let $f(u,u') = f(u)$ in (1.1). We assume that $f(u)$ is an unknown function, but it is known that it satisfies the following conditions (A.1) and (A.2).

(A.1) $f(u)$ is a positive function of C^1 for $u \ge 0$ satisfying $f(0) = f'(0) = 0$.

(A.2) $pf(u) \ge uf'(u) \ge pu^p$ for $u \ge 0$.

The inverse problem here means whether we can reconstruct the unknown nonlinear term $f(u)$ from the information of the asymptotic behavior of $\lambda_q(\alpha)$ as $\alpha \to \infty$ or not.

Theorem 1.2 ([12]). *Let $f(u,u') = f(u)$ in (1.1) and consider the problem (1.1)–(1.3), where $f(u)$ is known to satisfy (A.1)–(A.2), and suppose*

$$\lambda_q(\alpha) = \alpha^{p-1} + C_1\alpha^{(p-1)/2} + r_1(\alpha) + O(1) \qquad \text{as } \alpha \to \infty, \tag{1.7}$$

where

$$\lim_{\alpha\to\infty}\left|\frac{r_1(\alpha)}{\alpha^{(p-1)/2}}\right| = 0, \quad \lim_{\alpha\to\infty} r_1(\alpha) = \infty. \tag{1.8}$$

Furthermore, assume that r_1 satisfies

$$\left|\frac{r_1(u)-r_1(v)}{u-v}\right| \le C\left|\frac{r_1(u)}{u}\right| \tag{1.9}$$

for sufficiently large $u > v$. Then,

$$f(u) = u^p + r_1(u)u + O(u) \qquad as \ u \to \infty. \tag{1.10}$$

Roughly speaking, Theorem 1.2 implies that if $f = f(x, y)$ does not depend on y and satisfies the positivity and growth conditions (A.1) and (A.2), then the inverse problem is solved successfully.

Motivated by Theorem 1.2, it is natural to extend the result of Theorem 1.2 to two directions, the first one is as follows.

Problem A: Under some suitable conditions on f, Theorem 1.2 holds even if $r_1(u)$ in (1.7) is negative function like $r_1(u) = -u^\beta$ $(0 < \beta < (p-1)/2)$.

To consider Problem A, we note the results obtained recently.

Theorem 1.3 ([13, Corollary 1.5]). *Let $f(u) = u^p - u^\beta$ with $1 < \beta < (p+1)/2$. Then as $\alpha \to \infty$*

$$\lambda_q(\alpha) \ = \ \alpha^{p-1} + \frac{p-1}{q} C(q) \alpha^{(p-1)/2} - \alpha^{\beta-1} + a_0 + O(\alpha^{(2\beta-p-1)/2}). \tag{1.11}$$

Therefore, from a viewpoint of Theorem 1.2, it is reasonable to expect that the results like Theorem 1.2 holds for Problem A.

Another direction we would like to consider is:

Problem B: Treat the case where $f = f(u, u')$, and consider the inverse problem.

The best way for us is to consider Problems A and B in the same framework and establish unified results for the inverse problems. The results in this paper, however, gives us the difficulty how to treat Problems A and B at the same time.

Now we state our result. We put

$$\eta := \frac{2p - (p+1)m}{2(p-1)}. \tag{1.12}$$

We note that $0 < \eta < 1/2$ if $1 \le m < 2p/(p+1)$. Furthermore, let

$$C_2 := 2 \int_0^1 \frac{(1 - s^q) \int_s^1 [(1 - x^2) - 2(1 - x^{p+1})/(p+1)]^{m/2} dx}{[(1 - s^2) - 2(1 - s^{p+1})/(p+1)]^{3/2}} ds.$$

Theorem 1.4. *Assume that* $1 \le m < 2p/(p+1)$. *Let* $m_0 := (-1+\sqrt{1+2(p+1)^2})/(p+1)$. *Then as* $\alpha \to \infty$, *the following asymptotic formulas for* $\lambda_q(\alpha)$ *hold:*
(i) If $1 \le m < m_0$, *then*

$$\lambda_q(\alpha) = \alpha^{p-1} + C_1\alpha^{(p-1)/2} - \frac{p-1}{q}C_2\alpha^{(p-1)(1/2-\eta)} + a_0 + o(1). \qquad (1.13)$$

(ii) If $m = m_0$, *then*

$$\lambda_q(\alpha) = \alpha^{p-1} + C_1\alpha^{(p-1)/2} - \frac{p-1}{q}C_2\alpha^{(p-1)(1/2-\eta)} + O(1). \qquad (1.14)$$

(iii) If $m_0 < m < 2p/(p+1)$, *then*

$$\lambda_q(\alpha) = \alpha^{p-1} + C_1\alpha^{(p-1)/2} - \frac{p-1}{q}C_2\alpha^{(p-1)(1/2-\eta)} + O(\alpha^{(p-1)(1/2-(m+2)\eta/2)}). \qquad (1.15)$$

We see from (1.11) and (1.13) that we should be careful to treat the Problem A and B at the same time, since we may not determine the unknown nonlinear term from the third term of $\lambda_q(\alpha)$ if $\beta - 1 = (p-1)(1/2-\eta)$. Therefore, as a next step of this work, before treating the Problem A and B simultaneously, we should find reasonable conditions for $f(u,u')$ to reconstruct $f(u,u')$ from the asymptotic formula for $\lambda_q(\alpha)$.

Remark 1.5. (i) $m = m_0$ is uniquely determined by the equation

$$h(m) := \frac{1}{2} - \frac{(m+1)\eta}{2} = 0,$$

which implies that the exponent of α of the fourth term in (1.15) is equal to 0. In this case, we obtain (1.14). Clearly, $1 \le m < m_0$ (resp. $m_0 < m < 2p/(p+1)$ implies $h(m) < 0$ (resp. $h(m) > 0$), and according to these two cases, we obtain (1.13) and (1.15), respectively.
(ii) The case $2p/(p+1) \le m < 2$ is worth considering by the methods developed here. However, for instance, the case $2p/((p+1) = m$ should be handled more carefully, and quite a long and complicated calculation will be necessary. From this point of view, we may go on to an more detailed study of the case $2p/(p+1) \le m < 2$.

2 Proof of Theorem 1.4

We begin with notations and the fundamental properties of u_λ. In what follows, C and k denote various positive constants independent of $\lambda \gg 1$ for simplicity. We write $\lambda = \lambda_q(\alpha)$.

Since $U := \lambda^{1/(p-1)}$ and $u_{0,\lambda}$ are super-solution and sub-solution of (1.1)–(1.3) with $u_{0,\lambda}(t) < \lambda^{1/p-1}$, respectively, we know from [1] and [3] that

$$u_\lambda(t) = u_\lambda(-t), \quad t \in I, \tag{2.1}$$

$$u_\lambda(0) = \max_{t \in \bar{I}} u_\lambda(t) = \|u_\lambda\|_\infty, \tag{2.2}$$

$$u'_\lambda(t) > 0, \quad -1/2 < t < 0, \tag{2.3}$$

$$u_{0,\lambda}(t) < u_\lambda(t) < \lambda^{1/(p-1)}. \tag{2.4}$$

By (1.5) and (2.4), we see that $\lambda \to \infty$ is equivalent to $\alpha \to \infty$. By (1.4) and (2.4), we see that for $\lambda \gg 1$,

$$0 < \eta_\lambda := \lambda - \|u_\lambda\|_\infty^{p-1} = O(\lambda e^{-k\sqrt{\lambda}}), \tag{2.5}$$

where $k > 0$ is a constant.

The proof of Theorem 1.4 is based on the following Proposition 2.1.

Proposition 2.1. *For $\lambda \gg 1$*

$$\begin{aligned} \|u_\lambda\|_\infty^q - \|u_\lambda\|_q^q &= C(q)\|u_\lambda\|_\infty^{q-(p-1)/2} - C_2\|u_\lambda\|_\infty^{q-(p-1)/2-(p-1)\eta} \\ &\quad + O(\|u_\lambda\|_\infty^{q-(p-1)/2-(m+2)(p-1)\eta/2})(1+o(1)). \end{aligned} \tag{2.6}$$

We accept Proposition 2.1 here tentatively. The proof will be given in Section 3. Once it is obtained, Theorem 1.4 is proved by direct calculation as follows.

Proof of Theorem 1.4. We have only to consider the case $1 \le m < m_0$ and show (1.13), since the other cases can be treated by the same calculation as that of the case $1 \le m < m_0$. Note that, in this case, $-1/2 - (m+2)\eta/2 < -1$ by Remark 1.5 (i).

Step 1: We calculate the second term of $\lambda_q(\alpha)$. We put $r(\alpha) := \lambda - \alpha^{p-1}$. By (1.5) and (2.4), $r(\alpha) = o(\alpha^{p-1})$ for $\alpha \gg 1$. By this and (2.6), for $\lambda \gg 1$

$$\begin{aligned} \|u_\lambda\|_q^{p-1} &= \|u_\lambda\|_\infty^{p-1}\Big\{1 - C(q)\|u_\lambda\|_\infty^{-(p-1)/2} + C_2\|u_\lambda\|_\infty^{-(p-1)/2-(p-1)\eta} \\ &\quad + O(\|u_\lambda\|_\infty^{-(p-1)/2-(m+2)(p-1)\eta/2})(1+o(1))\Big\}^{(p-1)/q}. \end{aligned} \tag{2.7}$$

This along with (2.5) and Taylor expansion implies that

$$\alpha^{p-1} = \left(\lambda - O(\lambda e^{-k\sqrt{\lambda}})\right)\left(1 - \frac{p-1}{q}C(q)\lambda^{-1/2} + o(\lambda^{-1/2})\right)$$

$$
\begin{aligned}
&= \lambda - C_1\lambda^{1/2} + o(\lambda^{1/2}) \\
&= \alpha^{p-1} + r(\alpha) - C_1\alpha^{(p-1)/2}(1+o(1)).
\end{aligned}
$$

By this, for $\alpha \gg 1$,

$$
r(\alpha) = C_1\alpha^{(p-1)/2} + o(\alpha^{(p-1)/2}).
$$

Step 2: We calculate the third and fourth terms. Let

$$
R(\alpha) := \lambda - \alpha^{p-1} - C_1\alpha^{(p-1)/2}. \tag{2.8}
$$

By Step 1, $R(\alpha) = o(\alpha^{(p-1)/2})$ for $\alpha \gg 1$. By (2.5), (2.7), Remark 1.5 (i) and Taylor expansion, for $\lambda \gg 1$

$$
\begin{aligned}
\alpha^{p-1} &= \left(\lambda - O(\lambda e^{-k\sqrt{\lambda}})\right) \qquad (2.9)\\
&\quad \times \left\{1 + \frac{p-1}{q}\left(-C(q)\lambda^{-1/2} + C_2\lambda^{-1/2-\eta} + o(\lambda^{-1})\right)\right. \\
&\qquad \left. + \frac{(p-1)(p-1-q)}{2q^2}\left(-C(q)\lambda^{-1/2} + C_2\lambda^{-1/2-\eta}\right)^2 (1+o(1))\right\} \\
&= \lambda - C_1\lambda^{1/2} + \frac{p-1}{q}C_2\lambda^{1/2-\eta} + \frac{(p-1)(p-1-q)}{2q^2}C(q)^2 + o(1).
\end{aligned}
$$

By this, (2.8) and Taylor expansion,

$$
\begin{aligned}
\alpha^{p-1} &= \alpha^{p-1} + C_1\alpha^{(p-1)/2} + R(\alpha) - C_1(\alpha^{p-1} + C_1\alpha^{(p-1)/2} + R(\alpha))^{1/2} \\
&\quad + \frac{p-1}{q}C_2(\alpha^{p-1} + C_1\alpha^{(p-1)/2} + R(\alpha))^{1/2-\eta} + \frac{(p-1)(p-1-q)}{2q^2}C(q)^2 + o(1) \\
&= \alpha^{p-1} + C_1\alpha^{(p-1)/2} + R(\alpha) \\
&\quad - C_1\alpha^{(p-1)/2}\left(1 + \frac{1}{2}\left(C_1\alpha^{-(p-1)/2} + R(\alpha)\alpha^{-(p-1)}\right) + O(\alpha^{-(p-1)})\right) \\
&\quad + \frac{p-1}{q}C_2\alpha^{(p-1)(1/2-\eta)}(1 + C_1\alpha^{-(p-1)/2} + R(\alpha)\alpha^{-(p-1)})^{1/2-\eta} \\
&\quad + \frac{(p-1)(p-1-q)}{2q^2}C(q)^2 + o(1).
\end{aligned}
$$

This along with direct calculation implies that

$$
\begin{aligned}
R(\alpha) &= -\frac{p-1}{q}C_2\alpha^{(p-1)(1/2-\eta)} + \frac{1}{2}C_1^2 - \frac{(p-1)(p-1-q)}{2q^2}C(q)^2 + o(1) \qquad (2.10)\\
&= -\frac{p-1}{q}C_2\alpha^{(p-1)(1/2-\eta)} + a_0 + o(1).
\end{aligned}
$$

By this, we obtain (1.13). Thus the proof is complete. ∎

3 Proof of Proposition 2.1

We start with the fundamental equality. By (1.1) and (2.3), for $-1/2 \le t \le 0$,

$$\{u_\lambda''(t) + \lambda u_\lambda(t) - u_\lambda(t)^p + u_\lambda'(t)^m\}\, u_\lambda'(t) = 0.$$

By this, for $-1/2 \le t \le 0$,

$$\begin{aligned}
&\frac{1}{2}u_\lambda'(t)^2 + \frac{1}{2}\lambda u_\lambda(t)^2 - \frac{1}{p+1}u_\lambda(t)^{p+1} + \int_{-1/2}^{t} u_\lambda'(s)^{m+1} ds = \text{constant} \qquad (3.1)\\
&= \frac{1}{2}\lambda \|u_\lambda\|_\infty^2 - \frac{1}{p+1}\|u_\lambda\|_\infty^{p+1} + \int_{-1/2}^{0} u_\lambda'(s)^{m+1} ds\\
&= \frac{1}{2}u_\lambda'\left(-\frac{1}{2}\right)^2.
\end{aligned}$$

We put

$$A_\lambda(\theta) := \lambda(\|u_\lambda\|_\infty^2 - \theta^2) - \frac{2}{p+1}(\|u_\lambda\|_\infty^{p+1} - \theta^{p+1}), \tag{3.2}$$

$$A_{0,\lambda}(\theta) := \|u_\lambda\|_\infty^{p-1}\|(\|u_\lambda\|_\infty^2 - \theta^2) - \frac{2}{p+1}(\|u_\lambda\|_\infty^{p+1} - \theta^{p+1}), \tag{3.3}$$

$$B_\lambda(t) := 2\int_t^0 u_\lambda'(s)^{m+1} ds, \tag{3.4}$$

$$R_\lambda(s) := \frac{\lambda}{\|u_\lambda\|_\infty^{p-1}}(1 - s^2) - \frac{2}{p+1}(1 - s^{p+1}), \tag{3.5}$$

$$S_\lambda(s) := 1 - s^2 - \frac{2}{p+1}(1 - s^{p+1}). \tag{3.6}$$

We note that if we put $u_\lambda(t) = \|u_\lambda\|_\infty s$, then

$$A_{0,\lambda}(u_\lambda(t)) = \|u_\lambda\|_\infty^{p+1} S_\lambda(s), \quad A_\lambda(u_\lambda(t)) = \|u_\lambda\|_\infty^{p+1} R_\lambda(s). \tag{3.7}$$

Further, by (2.5), (3.2) and (3.3), $A_\lambda(\theta) > A_{0,\lambda}(\theta)$ for $\theta > 0$. By (2.3), (3.1), (3.2) and (3.4), for $-1/2 \le t \le 0$

$$u_\lambda'(t) = \sqrt{A_\lambda(u_\lambda(t)) + B_\lambda(t)}. \tag{3.8}$$

By this and (2.1), we obtain

$$\begin{aligned}
\|u_\lambda\|_\infty^q - \|u_\lambda\|_q^q &= 2\int_{-1/2}^{0} (\|u_\lambda\|_\infty^q - u_\lambda^q(t)) \frac{u_\lambda'(t)}{\sqrt{A_\lambda(u_\lambda(t)) + B_\lambda(t)}} dt \qquad (3.9)\\
&= 2\int_{-1/2}^{0} (\|u_\lambda\|_\infty^q - u_\lambda^q(t)) \frac{u_\lambda'(t)}{\sqrt{A_\lambda(u_\lambda(t))}} dt
\end{aligned}$$

$$
\begin{aligned}
&+\left(2\int_{-1/2}^{0}(\|u_\lambda\|_\infty^q - u_\lambda^q(t))\frac{u_\lambda'(t)}{\sqrt{A_\lambda(u_\lambda(t))+B_\lambda(t)}}dt\right.\\
&\qquad\left.-2\int_{-1/2}^{0}(\|u_\lambda\|_\infty^q - u_\lambda^q(t))\frac{u_\lambda'(t)}{\sqrt{A_\lambda(u_\lambda(t))}}dt\right)\\
=&\quad I+II.
\end{aligned}
$$

By direct calculation, we obtain

$$
II=-2\int_{-1/2}^{0}\frac{B_\lambda(t)(\|u_\lambda\|_\infty^q - u_\lambda^q(t))u_\lambda'(t)dt}{\sqrt{A_\lambda(u_\lambda(t))+B_\lambda(t)}\sqrt{A_\lambda(u_\lambda(t))}(\sqrt{A_\lambda(u_\lambda(t))+B_\lambda(t)}+\sqrt{A_\lambda(u_\lambda(t))})}. \quad (3.10)
$$

Proposition 2.1 is obtained directly from the following Lemma 3.1, and Lemmas 3.2–3.3.

Lemma 3.1. As $\lambda \to \infty$,

$$
I = C(q)\|u_\lambda\|_\infty^{q-(p-1)/2} + O(\lambda^{q/(p-1)}e^{-k\sqrt{\lambda}}). \quad (3.11)
$$

Lemma 3.2. As $\lambda \to \infty$,

$$
|II| \;\le\; C_2\|u_\lambda\|_\infty^{q-(p-1)/2-(2p-(p+1)m)/2} + O(\|u_\lambda\|_\infty^{q-(p-1)/2-(m+2)(2p-(p+1)m)/4}). \quad (3.12)
$$

Lemma 3.3. As $\lambda \to \infty$,

$$
|II| \;\ge\; C_2\|u_\lambda\|_\infty^{q-(p-1)/2-(2p-(p+1)m)/2} - O(\|u_\lambda\|_\infty^{q-(p-1)/2-(2p-(p+1)m)}). \quad (3.13)
$$

If we do not have the term $B_\lambda(t)$, then the situation here is the same as that of [11]. Therefore, to prove Proposition 2.1, the most important part is to calculate II in Lemmas 3.2 and 3.3 precisely. Clearly, the calculation in Lemmas 3.2 and 3.3 deeply depend on the estimates of $B_\lambda(t)$ as $\lambda \to \infty$, which is the main part of the calculation of this paper. Since it is accomplished by long and tedious calculations, we fulfill it in the next section.

On the contrary, Lemma 3.1 can be proved by applying the argument in [11] to our case. This is rather an easy part, so the proof will be given in Appendix.

4 Proof of Lemmas 3.2 and 3.3

It is clear from (3.10) that the estimate of $B_\lambda(t)$ plays an crucial role for the proofs of Lemmas 3.2 and 3.3. We put

$$D_\lambda(\theta) = 2\int_{\theta/\|u_\lambda\|_\infty}^1 \left[1 - x^2 - \frac{2}{p+1}(1 - x^{p+1})\right]^{m/2} dx, \tag{4.1}$$

$$\xi_\lambda = \|u_\lambda\|_\infty^{mp/2} \exp\left(\frac{-mk\sqrt{\lambda}}{2}\right). \tag{4.2}$$

Lemma 4.1. *For $-1/2 < t < 0$ and $\lambda \gg 1$*

$$B_\lambda(t) \geq D_\lambda(u_\lambda(t))\|u_\lambda\|_\infty^{1+m(p+1)/2} - C\xi_\lambda(\|u_\lambda\|_\infty - u_\lambda(t))^{m/2+1}. \tag{4.3}$$

Proof. Since $1/2 \leq m/2 < 1$, we know that for $0 \leq b \leq a$,

$$0 \leq a^{m/2} - b^{m/2} \leq C(a-b)^{m/2}. \tag{4.4}$$

By this, (2.5), (3.2) and (3.3), we have

$$\begin{aligned} 0 &< \int_{u_\lambda(t)}^{\|u_\lambda\|_\infty} \left(A_\lambda(\theta)^{m/2} - A_{0,\lambda}(\theta)^{m/2}\right) d\theta \qquad (4.5)\\ &\leq C\int_{u_\lambda(t)}^{\|u_\lambda\|_\infty} \left[(\lambda - \|u_\lambda\|_\infty^{p-1})(\|u_\lambda\|_\infty^2 - \theta^2)\right]^{m/2} d\theta \\ &\leq C\|u_\lambda\|_\infty^{m+1}(\lambda\exp(-k\sqrt{\lambda}))^{m/2}\int_{u_\lambda(t)/\|u_\lambda\|_\infty}^1 (1-s^2)^{m/2} ds \\ &\leq C\|u_\lambda\|_\infty^{m+1}(\lambda\exp(-k\sqrt{\lambda}))^{m/2}\int_{u_\lambda(t)/\|u_\lambda\|_\infty}^1 (1-s)^{m/2} ds \\ &\leq C\xi_\lambda(\|u_\lambda\|_\infty - u_\lambda(t))^{1+m/2}. \end{aligned}$$

By this, (3.4), (3.8) and (4.1), for $-1/2 < t < 0$,

$$\begin{aligned} \frac{1}{2}B_\lambda(t) &= \int_t^0 u_\lambda'(s)^{m+1} ds \geq \int_t^0 A_\lambda(u_\lambda(s))^{m/2} u_\lambda'(s) ds \qquad (4.6)\\ &= \int_{u_\lambda(t)}^{\|u_\lambda\|_\infty} \left[\lambda(\|u_\lambda\|_\infty^2 - \theta^2) - \frac{2}{p+1}(\|u_\lambda\|_\infty^{p+1} - \theta^{p+1})\right]^{m/2} d\theta \\ &= \int_{u_\lambda(t)}^{\|u_\lambda\|_\infty} A_{0,\lambda}(\theta)^{m/2} d\theta + \int_{u_\lambda(t)}^{\|u_\lambda\|_\infty} \left(A_\lambda(\theta)^{m/2} - A_{0,\lambda}(\theta)^{m/2}\right) d\theta \\ &\geq \|u_\lambda\|_\infty^{1+m(p+1)/2} \int_{u_\lambda(t)/\|u_\lambda\|_\infty}^1 \left[1 - x^2 - \frac{2}{p+1}(1-x^{p+1})\right]^{m/2} dx \\ &\quad - C\xi_\lambda(\|u_\lambda\|_\infty - u_\lambda(t))^{1+m/2} \\ &= \frac{1}{2}D_\lambda(u_\lambda(t))\|u_\lambda\|_\infty^{1+m(p+1)/2} - C\xi_\lambda(\|u_\lambda\|_\infty - u_\lambda(t))^{1+m/2}. \end{aligned}$$

Thus the proof is complete. ∎

We next calculate the estimate of $B_\lambda(t)$ from above.

Lemma 4.2. *For $-1/2 < t < 0$ and $\lambda \gg 1$*

$$B(t) \leq D_\lambda(u_\lambda(t))\|u_\lambda\|_\infty^{m(p+1)/2+1} + C\|u_\lambda\|_\infty^{m^2(p+1)/4}(\|u_\lambda\|_\infty - u_\lambda(t))^{m/2+1} + C\xi_\lambda(\|u_\lambda\|_\infty - u_\lambda(t))^{1+m/2}. \tag{4.7}$$

Proof. We first show that for $\lambda \gg 1$,

$$u'_\lambda\left(-\frac{1}{2}\right)^2 = \frac{p-1}{p+1}\|u_\lambda\|_\infty^{p+1}(1+o(1)). \tag{4.8}$$

To do this, we assume that there exists a subsequence of $\{\lambda\}$, which is denoted by $\{\lambda\}$ again, such that as $\lambda \to \infty$

$$\frac{u'_\lambda(-\frac{1}{2})^2}{\|u_\lambda\|_\infty^{p+1}} \to \infty. \tag{4.9}$$

By (1.1), (2.3) and (2.4), $u''_\lambda(t) < 0$ for $t \in I$. Therefore, $u'_\lambda(-\frac{1}{2}) = \max_{t\in\bar{I}}|u'_\lambda(t)|$. Then by (2.5), (3.1) and (4.9), we have

$$\begin{aligned}\frac{1}{2}u'_\lambda\left(-\frac{1}{2}\right)^2(1-o(1)) &= \int_{-1/2}^0 u'_\lambda(s)^{m+1}ds \leq u'_\lambda\left(-\frac{1}{2}\right)^m\int_{-1/2}^0 u'_\lambda(s)ds \\ &= u'_\lambda\left(-\frac{1}{2}\right)^m\|u_\lambda\|_\infty.\end{aligned} \tag{4.10}$$

This implies that

$$u'_\lambda\left(-\frac{1}{2}\right) \leq C\|u_\lambda\|_\infty^{1/(2-m)}. \tag{4.11}$$

Since $m < 2p/(p+1)$, as $\lambda \to \infty$

$$\frac{u'_\lambda(-\frac{1}{2})^2}{\|u_\lambda\|_\infty^{p+1}} \leq C\|u_\lambda\|_\infty^{2/(2-m)-(p+1)} = C\|u_\lambda\|_\infty^{-(2p-(p+1)m)/(2-m)} \to 0. \tag{4.12}$$

This contradicts to (4.9). Therefore, for $\lambda \gg 1$,

$$\frac{u'_\lambda(-\frac{1}{2})^2}{\|u_\lambda\|_\infty^{p+1}} \leq C. \tag{4.13}$$

By this, we obtain

$$\begin{aligned}\int_{-1/2}^0 u'_\lambda(s)^{m+1}ds &\leq u'_\lambda\left(-\frac{1}{2}\right)^m\|u_\lambda\|_\infty \leq C\|u_\lambda\|_\infty^{m(p+1)/2+1} \\ &= o(\|u_\lambda\|_\infty^{p+1}).\end{aligned} \tag{4.14}$$

By this, (2.5) and (3.1), we easily obtain (4.8). Then by (4.8), we obtain

$$\begin{aligned} B_\lambda(t) &= 2\int_t^0 u_\lambda'(s)^{m+1}ds \le 2u_\lambda'(t)^m \int_t^0 u_\lambda'(s)ds \\ &\le Cu_\lambda'\left(-\frac{1}{2}\right)^m (\|u_\lambda\|_\infty - u_\lambda(t)) \le C\|u_\lambda\|_\infty^{m(p+1)/2}(\|u_\lambda\|_\infty - u_\lambda(t)). \end{aligned}$$

By this, (3.8), (4.4) and (4.6), for $\lambda \gg 1$,

$$\begin{aligned} \frac{1}{2}B_\lambda(t) &= \int_t^0 u_\lambda'(s)^{m+1}ds = \int_t^0 (A_\lambda(u_\lambda(s)) + B_\lambda(s))^{m/2}u_\lambda'(s)ds \\ &\le \int_t^0 A_\lambda(u_\lambda(s))^{m/2}u_\lambda'(s)ds + C^{m/2}\int_t^0 B_\lambda(s)^{m/2}u_\lambda'(s)ds \\ &\le \int_t^0 A_\lambda(u_\lambda(s))^{m/2}u_\lambda'(s)ds + C\|u_\lambda\|_\infty^{m^2(p+1)/4}(\|u_\lambda\|_\infty - u_\lambda(t))^{m/2+1} \\ &= \frac{1}{2}D_\lambda(u_\lambda(t))\|u_\lambda\|_\infty^{m(p+1)/2+1} + C\xi_\lambda(\|u_\lambda\|_\infty - u_\lambda(t))^{m/2+1} \\ &\quad + C\|u_\lambda\|_\infty^{m^2(p+1)/4}(\|u_\lambda\|_\infty - u_\lambda(t))^{m/2+1}. \end{aligned} \tag{4.15}$$

Thus the proof is complete. ∎

Now we are in the position to prove Lemmas 3.2 and 3.3.

Proof of Lemma 3.2. Let $0 < \delta \ll 1$ be fixed. By (3.6) and Taylor expansion, we see that for $1 - \delta \le x \le 1$,

$$C^{-1}(1-x)^2 \le S_\lambda(x) \le C(1-x)^2. \tag{4.16}$$

Furthermore, if we choose $C \gg 1$, then for $0 \le x \le 1$

$$S_\lambda(x) \le C(1-x)^2. \tag{4.17}$$

By this and (4.1), for $0 \le s \le 1$,

$$D_\lambda(\|u_\lambda\|_\infty s) = 2\int_s^1 S_\lambda(x)^{m/2}dx \le C\int_s^1 (1-x)^m dx \le C(1-s)^{m+1}. \tag{4.18}$$

By (3.10) and Lemma 4.2,

$$\begin{aligned} |II| &\le 2\int_{-1/2}^0 \frac{B_\lambda(t)(\|u_\lambda\|_\infty^q - u_\lambda(t)^q)u_\lambda'(t)}{2A_\lambda(u_\lambda(t))^{3/2}}dt \\ &\le Y_1 + Y_2 \\ &= \int_{-1/2}^0 \frac{D_\lambda(u_\lambda(t))\|u_\lambda\|_\infty^{m(p+1)/2+1}(\|u_\lambda\|_\infty^q - u_\lambda(t)^q)u_\lambda'(t)}{A_{0,\lambda}(u_\lambda(t))^{3/2}}dt \\ &\quad + \int_{-1/2}^0 \frac{C(\|u_\lambda\|_\infty^{m^2(p+1)/4} + \xi_\lambda)(\|u_\lambda\|_\infty - u_\lambda(t))^{m/2+1}(\|u_\lambda\|_\infty^q - u_\lambda(t)^q)u_\lambda'(t)}{A_{0,\lambda}(u_\lambda(t))^{3/2}}dt. \end{aligned} \tag{4.19}$$

Then by (3.7), (4.2), (4.19) and putting $s = u_\lambda(t)/\|u_\lambda\|_\infty$, we obtain

$$\begin{aligned} Y_1 &= \|u_\lambda\|_\infty^{q-(p-1)/2-(2p-(p+1)m)/2} \int_0^1 \frac{2\left(\int_s^1 S_\lambda(x)^{m/2}dx\right)(1-s^q)}{S_\lambda(s)^{3/2}} ds \qquad (4.20) \\ &= C_2 \|u_\lambda\|_\infty^{q-(p-1)/2-(2p-(p+1)m)/2}, \\ Y_2 &\le C\|u_\lambda\|_\infty^{q-(p-1)/2-(m+2)(2p-(p+1)m)/4} \int_0^1 \frac{(1-s)^{m/2+1}(1-s^q)}{S_\lambda(s)^{3/2}} ds. \qquad (4.21) \end{aligned}$$

Note that by (4.16) and (4.18), the right hand side of (4.20) and (4.21) are integrable. By (4.19)–(4.21), we obtain Lemma 3.2. Thus the proof is complete. ∎

Proof of Lemma 3.3. By (3.10), we have

$$\begin{aligned} |II| &\ge 2\int_{-1/2}^0 \frac{B_\lambda(t)(\|u_\lambda\|_\infty^q - u_\lambda(t)^q)u_\lambda'(t)}{2\sqrt{A_\lambda(u_\lambda(t))}(A_\lambda(u_\lambda(t)) + B_\lambda(t))} dt \qquad (4.22) \\ &= \int_{-1/2}^0 \frac{B_\lambda(t)(\|u_\lambda\|_\infty^q - u_\lambda(t)^q)u_\lambda'(t)}{\sqrt{A_\lambda(u_\lambda(t))}A_\lambda(u_\lambda(t))} dt \\ &+ \left(\int_{-1/2}^0 \frac{B_\lambda(t)(\|u_\lambda\|_\infty^q - u_\lambda(t)^q)u_\lambda'(t)}{\sqrt{A_\lambda(u_\lambda(t))}(A_\lambda(u_\lambda(t)) + B_\lambda(t))} dt - \int_{-1/2}^0 \frac{B_\lambda(t)(\|u_\lambda\|_\infty^q - u_\lambda(t)^q)u_\lambda'(t)}{\sqrt{A_\lambda(u_\lambda(t))}A_\lambda(u_\lambda(t))} dt\right) \\ &= Z_1 - Z_2. \end{aligned}$$

We first calculate Z_1. By (2.5), (3.2) and (3.3),

$$\begin{aligned} \frac{1}{A_\lambda(u_\lambda(t))^{3/2}} &= \frac{1}{A_{0,\lambda}(u_\lambda(t))^{3/2}} + \left(\frac{1}{A_\lambda(u_\lambda(t))^{3/2}} - \frac{1}{A_{0,\lambda}(u_\lambda(t))^{3/2}}\right) \qquad (4.23) \\ &= \frac{1}{A_{0,\lambda}(u_\lambda(t))^{3/2}} - \frac{A_\lambda(u_\lambda(t))^3 - A_{0,\lambda}(u_\lambda(t))^3}{A_{0,\lambda}(u_\lambda(t))^{3/2}A_\lambda(u_\lambda(t))^{3/2}(A_{0,\lambda}(u_\lambda(t))^{3/2} + A_\lambda(u_\lambda(t))^{3/2})} \\ &\ge \frac{1}{A_{0,\lambda}(u_\lambda(t))^{3/2}} - \frac{3(A_\lambda(u_\lambda(t)) - A_{0,\lambda}(u_\lambda(t)))}{A_{0,\lambda}(u_\lambda(t))^{3/2}A_\lambda(u_\lambda(t))} \\ &= \frac{1}{A_{0,\lambda}(u_\lambda(t))^{3/2}} - \frac{3\eta_\lambda(\|u_\lambda\|_\infty^2 - u_\lambda(t)^2)}{A_{0,\lambda}(u_\lambda(t))^{3/2}A_\lambda(u_\lambda(t))}. \end{aligned}$$

By this, (4.22) and Lemmas 4.1 and 4.2, we obtain

$$\begin{aligned} Z_1 &\ge \int_{-1/2}^0 \frac{D_\lambda(u_\lambda(t))\|u_\lambda\|_\infty^{m(p+1)/2+1}(\|u_\lambda\|_\infty^q - u_\lambda(t)^q)u_\lambda'(t)}{A_{0,\lambda}(u_\lambda(t))^{3/2}} dt \qquad (4.24) \\ &\quad - \int_{-1/2}^0 \frac{C\xi_\lambda(\|u_\lambda\|_\infty - u_\lambda(t))^{m/2+1}(\|u_\lambda\|_\infty^q - u_\lambda(t)^q)u_\lambda'(t)}{A_{0,\lambda}(u_\lambda(t))^{3/2}} dt \\ &\quad - \int_{-1/2}^0 \frac{3\eta_\lambda(\|u_\lambda\|_\infty^2 - u_\lambda(t)^2)D_\lambda(u_\lambda(t))\|u_\lambda\|_\infty^{m(p+1)/2+1}(\|u_\lambda\|_\infty^q - u_\lambda(t)^q)u_\lambda'(t)}{A_{0,\lambda}(u_\lambda(t))^{3/2}A_\lambda(u_\lambda(t))} dt \\ &\quad - \int_{-1/2}^0 \frac{3\eta_\lambda\|u_\lambda\|_\infty^{m^2(p+1)/4}L_\lambda(u_\lambda(t))(\|u_\lambda\|_\infty^q - u_\lambda(t)^q)u_\lambda'(t)}{A_{0,\lambda}(u_\lambda(t))^{3/2}A_\lambda(u_\lambda(t))} dt \\ &= W_1 - W_2 - W_3 - W_4, \end{aligned}$$

where

$$L_\lambda(u_\lambda(t)) := (\|u_\lambda\|_\infty^2 - u_\lambda(t)^2)(\|u_\lambda\|_\infty - u_\lambda(t))^{m/2+1}.$$

It is clear that $W_1 = Y_1$ in (4.20). By (4.16),

$$\begin{aligned} W_2 &\le C\xi_\lambda \|u_\lambda\|_\infty^{q+2+m/2-3(p+1)/2} \int_0^1 \frac{(1-s)^{m/2+1}(1-s^q)}{S_\lambda(s)^{3/2}} ds \\ &\le C\xi_\lambda \|u_\lambda\|_\infty^{q+2+m/2-3(p+1)/2} \\ &= o(\|u_\lambda\|_\infty^{q-(p-1)/2-(2p-(p+1)m)}). \end{aligned} \tag{4.25}$$

Next, we calculate W_3. We fix $0 < \delta \ll 1$. By (4.18) and putting $s = u_\lambda(t)/\|u_\lambda\|_\infty$,

$$\begin{aligned} W_3 &\le C\eta_\lambda \|u_\lambda\|_\infty^{q+3/2-5p/2+m(p+1)/2} \int_0^1 \frac{(1-s^2)(1-s^q)D_\lambda(\|u_\lambda\|_\infty s)}{S_\lambda(s)^{3/2} R_\lambda(s)} ds \\ &= W_{3,1} + W_{3,2} = C\eta_\lambda \|u_\lambda\|_\infty^{q+3/2-5p/2+m(p+1)/2} \left(\int_{1-\delta}^1 + \int_0^{1-\delta} \right). \end{aligned} \tag{4.26}$$

For $1 - \delta \le s \le 1$, by Taylor expansion and Hölder's inequality,

$$R_\lambda(s) \ge \frac{2\eta_\lambda}{\|u_\lambda\|_\infty^{p-1}}(1-s) + C(p-1)(1-s)^2 \ge C\left(\frac{2\eta_\lambda}{\|u_\lambda\|_\infty^{p-1}}\right)^{2/3}(1-s)^{4/3}. \tag{4.27}$$

By this, (2.5), (4.16) and (4.26),

$$\begin{aligned} W_{3,1} &\le C\eta_\lambda \|u_\lambda\|_\infty^{q+3/2-5p/2+m(p+1)/2} \left(\frac{2\eta_\lambda}{\|u_\lambda\|_\infty^{p-1}}\right)^{-2/3} \int_{1-\delta}^1 \frac{(1-s^2)(1-s^q)(1-s)^{m+1}}{(1-s)^3(1-s)^{4/3}} ds \\ &\le C\eta_\lambda^{1/3} \|u_\lambda\|_\infty^{q+3/2-5p/2+m(p+1)/2+2(p-1)/3} \\ &= o(\|u_\lambda\|_\infty^{q-(p-1)/2-(2p-(p+1)m)}). \end{aligned} \tag{4.28}$$

By the same argument as that just above, we obtain

$$W_{3,2} \le C\eta_\lambda \|u_\lambda\|_\infty^{q+3/2-5p/2+m(p+1)/2} = o(\|u_\lambda\|_\infty^{q-(p-1)/2-(2p-(p+1)m)}), \tag{4.29}$$

$$\begin{aligned} W_4 &\le C\eta_\lambda^{1/3} \|u_\lambda\|_\infty^{q+4+m^2(p+1)/4+m/2-5(p+1)/2+2(p-1)/3} \\ &= o(\|u_\lambda\|_\infty^{q-(p-1)/2-(2p-(p+1)m)}). \end{aligned} \tag{4.30}$$

By (4.20), (4.24), (4.25), (4.26) and (4.28)–(4.30), we obtain the estimate of Z_1. Now we calculate Z_2. To do this, we consider the case where $1 < m < 2$ and $m = 1$ separately.

Case 1. Let $1 < m < 2$. By (3.7), Lemma 4.2 and putting $s := u_\lambda(t)/\|u_\lambda\|_\infty$, we obtain

$$Z_2 = \int_{-1/2}^0 \frac{B_\lambda(t)^2(\|u_\lambda\|_\infty^q - u_\lambda(t)^q)u_\lambda'(t)}{A_\lambda(u_\lambda(t))^{3/2}(A_\lambda(u_\lambda(t)) + B_\lambda(t))} dt \tag{4.31}$$

$$
\begin{aligned}
&\leq\ C\int_{-1/2}^{0}\frac{D_\lambda(u_\lambda(t))^2\|u_\lambda\|_\infty^{mp+m+2}(\|u_\lambda\|_\infty^q-u_\lambda(t)^q)u_\lambda'(t)}{A_0(u_\lambda(t)^{5/2}}dt\\
&\quad +C\int_{-1/2}^{0}\frac{\|u_\lambda\|_\infty^{m^2(p+1)/2}(\|u_\lambda\|_\infty-u_\lambda(t))^{m+2}(\|u_\lambda\|_\infty^q-u_\lambda(t)^q)u_\lambda'(t)}{A_0(u_\lambda(t)^{5/2}}dt\\
&\leq\ C\|u_\lambda\|_\infty^{q+1-5(p+1)/2}\\
&\quad\times\int_{1-\delta}^{1}\frac{[D_\lambda(\|u_\lambda\|_\infty s)^2\|u_\lambda\|_\infty^{mp+m+2}+\|u_\lambda\|_\infty^{m^2(p+1)/2+m+2}(1-s)^{m+2}](1-s^q)}{S_\lambda(s)^{5/2}}ds\\
&\quad + C\|u_\lambda\|_\infty^{q+1-5(p+1)/2}\\
&\quad\times\int_{0}^{1-\delta}\frac{[D_\lambda(\|u_\lambda\|_\infty s)^2\|u_\lambda\|_\infty^{mp+m+2}+\|u_\lambda\|_\infty^{m^2(p+1)/2+m+2}(1-s)^{m+2}](1-s^q)}{S_\lambda(s)^{5/2}}ds\\
&\leq\ C\|u_\lambda\|_\infty^{q-(p-1)/2-(2p-(p+1)m)}.
\end{aligned}
$$

Note that by (4.16) and (4.18), all the definite integrals in the right hand side of (4.31) are integrable.

Case 2. Let $m=1$. By (3.8),

$$
\begin{aligned}
\frac{1}{2}B_\lambda(t)\ &=\ \int_t^0 u_\lambda'(s)^2ds=\int_t^0(A_\lambda(u_\lambda(s))+B_\lambda(s))^{1/2}u_\lambda'(s)ds\\
&\leq\ \int_t^0 A_\lambda(u_\lambda(s))^{1/2}u_\lambda'(s)ds+\int_t^0 B_\lambda(s)^{1/2}u_\lambda'(s)ds\\
&\leq\ \int_t^0 A_\lambda(u_\lambda(s))^{1/2}u_\lambda'(s)ds+B_\lambda(t)^{1/2}(\|u_\lambda\|_\infty-u_\lambda(t)).
\end{aligned}
\tag{4.32}
$$

By this, we obtain

$$
B_\lambda(t)^{1/2}\leq C(\|u_\lambda\|_\infty-u_\lambda(t))+\left(2\int_t^0 A_\lambda(u_\lambda(s))^{1/2}u_\lambda'(s)ds\right)^{1/2}
\tag{4.33}
$$

This along with (4.6) implies

$$
B_\lambda(t)^2\ \leq\ CD_\lambda(u_\lambda(t))^2\|u_\lambda\|_\infty^{p+3}+C\xi_\lambda^2(\|u_\lambda\|_\infty-u_\lambda(t))^3+C(\|u_\lambda\|_\infty-u_\lambda(t))^4.
$$

By (4.2), (4.27) and putting $s:=u_\lambda(t)/\|u_\lambda\|_\infty$, we obtain

$$
\begin{aligned}
Z_2\ &\leq\ \int_{-1/2}^{0}\frac{B_\lambda(t)^2(\|u_\lambda\|_\infty^q-u_\lambda(t)^q)u_\lambda'(t)}{A_{0,\lambda}(u_\lambda(t))^{5/2}}dt\\
&\leq\ C\int_{-1/2}^{0}\frac{(D_\lambda(u_\lambda(t))^2\|u_\lambda\|_\infty^{p+3}(\|u_\lambda\|_\infty^q-u_\lambda(t)^q)u_\lambda'(t)}{A_{0,\lambda}(u_\lambda(t))^{5/2}}dt\\
&\quad +C\xi_\lambda^2\int_{-1/2}^{0}\frac{(\|u_\lambda\|_\infty-u_\lambda(t))^3(\|u_\lambda\|_\infty^q-u_\lambda(t)^q)u_\lambda'(t)}{A_{0,\lambda}(u_\lambda(t))^{3/2}A_\lambda(u_\lambda(t))}dt\\
&\quad +C\int_{-1/2}^{0}\frac{(\|u_\lambda\|_\infty-u_\lambda(t))^4(\|u_\lambda\|_\infty^q-u_\lambda(t)^q)u_\lambda'(t)}{A_{0,\lambda}(u_\lambda(t))^{5/2}}dt
\end{aligned}
\tag{4.34}
$$

$$
\begin{aligned}
&\le C\|u_\lambda\|_\infty^{q+1-5(p+1)/2+p+3}\int_0^1 \frac{(1-s^q)D_\lambda(\|u_\lambda\|_\infty s)^2}{S_\lambda(s)^{5/2}}ds\\
&\quad + Ce^{-k\sqrt{\lambda}/3}\|u_\lambda\|_\infty^{p+q+4-5(p+1)/2-2(p-1)/3}\int_0^1 \frac{(1-s)^3(1-s^q)}{S_\lambda(s)^{3/2}(1-s)^{4/3}}ds\\
&\quad + C\|u_\lambda\|_\infty^{q+1-5(p+1)/2+4}\int_0^1 \frac{(1-s^q)(1-s)^4}{S_\lambda(s)^{5/2}}ds\\
&\le C\|u_\lambda\|_\infty^{q-(p-1)/2-p+1}.
\end{aligned}
$$

Note that by (4.16) and (4.18), all the integrals in the right hand side of (4.34) is integrable, and (4.34) coincides with (4.31) with $m = 1$. Thus the proof is complete. ∎

5 Appendix

Proof of Lemma 3.1. By (3.7), (3.9) and putting $s = u_\lambda(t)/\|u_\lambda\|_\infty$, we obtain

$$
\begin{aligned}
I &= 2\int_0^1 \frac{\|u_\lambda\|_\infty^{q+1}(1-s^q)}{\sqrt{\|u_\lambda\|_\infty^{p+1}R_\lambda(s)}}ds \qquad (5.1)\\
&= 2\|u_\lambda\|_\infty^{q-(p-1)/2}\int_0^1 \frac{1-s^q}{\sqrt{S_\lambda(s)}}ds + 2\|u_\lambda\|_\infty^{q-(p-1)/2}\left[\int_0^1 \frac{1-s^q}{\sqrt{R_\lambda(s)}}ds - \int_0^1 \frac{1-s^q}{\sqrt{S_\lambda(s)}}ds\right]\\
&:= \|u_\lambda\|_\infty^{q-(p-1)/2}(C(q)+2X),
\end{aligned}
$$

where

$$
\begin{aligned}
X &:= \int_0^1 \left(\frac{1-s^q}{\sqrt{R_\lambda(s)}} - \frac{1-s^q}{\sqrt{S_\lambda(s)}}\right)ds \qquad (5.2)\\
&= \left(1-\frac{\lambda}{\|u_\lambda\|_\infty^{p-1}}\right)\int_0^1 \frac{(1-s^2)(1-s^q)}{\sqrt{R_\lambda(s)}\sqrt{S_\lambda(s)}(\sqrt{R_\lambda(s)}+\sqrt{S_\lambda(s)})}ds\\
&= -(X_1+X_2)\\
&:= -\frac{\eta_\lambda}{\|u_\lambda\|_\infty^{p-1}}\int_{1-\delta}^1 \frac{(1-s^2)(1-s^q)}{\sqrt{R_\lambda(s)}\sqrt{S_\lambda(s)}(\sqrt{R_\lambda(s)}+\sqrt{S_\lambda(s)})}ds\\
&\quad - \frac{\eta_\lambda}{\|u_\lambda\|_\infty^{p-1}}\int_0^{1-\delta} \frac{(1-s^2)(1-s^q)}{\sqrt{R_\lambda(s)}\sqrt{S_\lambda(s)}(\sqrt{R_\lambda(s)}+\sqrt{S_\lambda(s)})}ds.
\end{aligned}
$$

Here, $0 < \delta \ll 1$ is a constant. Then we apply the argument in [11, Lemma 3.2] to our case and obtain that for $\lambda \gg 1$,

$$
0 \le X_1 \le C\sqrt{\lambda}e^{-k\sqrt{\lambda}}, \qquad (5.3)
$$

$$
0 \le X_2 \le Ce^{-k\sqrt{\lambda}}. \qquad (5.4)
$$

By (5.1)–(5.4), we obtain Lemma 3.1. Thus the proof is complete. ∎

Acknowledgements

The author thanks the referee for the following helpful suggestions that improved the manuscript.

(i) Let $f(\beta, x, y) = |x|^{p-1}x - \beta|y|^m$, where $\beta \geq 0$ is a parameter. Our nonlinear term is given by $f = f(1, x, y)$. In the referee's report of the first version of this paper, it is suggested that it is interesting to analyze the transitions between the asymptotic formulas from the case when $\beta = 0$ to the case $\beta > 0$. This analysis gives us the good information how the asymptotic behavior of $\lambda_q(\alpha)$ depends on the damped term. By following the argument in Section 3 carefully, we find that the asymptotic formula corresponding to (1.13) is obtained by C_2 replaced with βC_2. Therefore, if $\beta \to 0$, then the formula (1.13) for the case $f(\beta, x, y)$ converges to (1.6), which is the asymptotic formula for the case $f(0, x, y)$, namely, the case without damped term. In this way, by introducing a new parameter β, we understand well how the asymptotic behavior of $\lambda_q(\alpha)$ depends on the damped term.

(ii) It is also suggested in the referee's report that it seems interesting to consider the case $\beta = -1$. To treat this case, more precise observation than above (i) will be necessary. From this point we might go on to an extension of the results here to the case $\beta = -1$.

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