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LIAPUNOV FUNCTIONALS AND PERIODICITY IN A SYSTEM OF NONLINEAR INTEGRAL EQUATIONS

Bo Zhang

Department of Mathematics and Computer Science
Fayetteville State University
Fayetteville, NC 28301-4298 USA
e-mail: bzhang@uncfsu.edu

Honoring the Career of John Graef on the Occasion of His Sixty-Seventh Birthday

Abstract

In this paper, we construct a Liapunov functional for a system of nonlinear integral equations. From that Liapunov functional we are able to establish the existence of periodic solutions to the system by applying some well-known fixed point theorems for the sum of a nonlinear contraction mapping and compact operator.

Key words and phrases: Fixed point, Periodicity, System of integral equations.
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1 Introduction

This paper is concerned with the existence of periodic solutions to the system of nonlinear integral equations

$$x(t) = h(t, x(t)) - \int_{-\infty}^{t} D(t,s)g(s,x(s))ds \tag{1.1}$$

where $x(t) \in R^n$, $h : R \times R^n \to R^n$, $D : R \times R \to R^{n \times n}$, $g : R \times R^n \to R^n$ are continuous, and $R = (-\infty, \infty)$.

The existence of periodic solutions of (1.1) or its differential form has been the subject of extensive investigations for many years. Our interest here centers on the use of Liapunov's direct method and fixed point theorems of continuation type, which are nonlinear alternatives of Leray-Schauder degree theory, to derive the existence of periodic solutions. Continuation theorems, such as Schaefer's [26] fixed point theorem without actually calculating degree, require less restrictive growth conditions on the functions involved. For the historical background and discussion of applications, we refer the reader to, for example, the work of Burton [4], Burton, Eloe, and Islam [10], Graef and Kong [12], Miller [24], O'Regan [25], Zeidler [28] and Zhang [29].

It is well-known that Liapunov's direct method has been used very effectively for differential equations. The method has not, however, been used with much success on integral equations until recently. The reason for this lies in the fact that it is very difficult to relate the derivative of a scalar function to the unknown non-differentiable solution of an integral equation. In the present paper, we construct a Liapunov functional for (1.1). From that Liapunov functional we are able to establish an *a priori* bound for all possible T-periodic solutions of a companion system of (1.1), and then, to prove the existence of a T-periodic solution to (1.1). As in the case for differential equations, once the Liapunov function is constructed, we can take full advantage of its simplicity in qualitative analysis. A good summary for recent development of the subject may be found in Burton [8].

A continuous function $x : R \to R^n$ is called a solution of (1.1) on R if it satisfies (1.1) on R. If $x(t)$ is specified to be a certain initial function on an initial interval, say

$$x(t) = \phi(t) \quad \text{for} \quad -\infty < t \leq 0,$$

we are then looking for a solution of

$$\begin{aligned} x(t) &= h(t, x(t)) - \int_{-\infty}^{0} D(t,s)g(s,\phi(s))ds - \int_0^t D(t,s)g(s,x(s))ds \\ &=: \tilde{h}(t,x(t)) - \int_0^t D(t,s)g(s,x(s))ds. \end{aligned} \tag{1.2}$$

Note that the initial function ϕ is absorbed into the forcing function term $\tilde{h}(t, x(t))$.

There is substantial literature on equations (1.1) and (1.2). Much of the literature can be traced back to the pioneering work of Levin and Nohel ([18]-[21]) in the study of asymptotic behavior of solutions of the scalar integral equation

$$x(t) = a(t) - \int_0^t D(t,s)g(x(s))ds \tag{1.3}$$

and the integro-differential equation

$$x'(t) = a(t) - \int_0^t D(t,s)g(x(s))ds. \tag{1.4}$$

These equations arise in problems related to evolutionary processes in biology, in nuclear reactors, and in control theory (see Corduneanu [11], Burton [7], Levin and Nohel [20], Kolmanovskii and Myshkis [16]). It is often required that $a \in C(R^+, R)$ and

$$D(t,s) \geq 0, \quad D_s(t,s) \geq 0, \quad \text{and} \quad D_{st}(t,s) \leq 0, \tag{1.5}$$

with $G(x) = \int_0^x g(s)ds \to \infty$ as $|x| \to \infty$. When $D(t,s) = D(t-s)$ is of convolution type, (1.5) represents for $t \in R^+ = [0,\infty)$

$$D(t) \geq 0, \quad D'(t) \leq 0, \quad \text{and} \quad D''(t) \geq 0. \tag{1.6}$$

Levin's work was based on (1.6) and the construction of a Liapunov functional for (1.4). The method was further extended into a long line of investigation drawing together such different notions of positivity as Liapunov functions, completely monotonic functions, and kernels of positive type (see Corduneanu [11], Gripenberg et al [14], Levin and Nohel [21], MacCamy and Wong [23]). In a series of papers ([4]-[6]), Burton obtains results on boundedness and periodicity of solutions for a scalar equation in the form of (1.1) without asking the growth condition on g. Liapunov functionals play an essential role in his proofs.

Many investigators mentioned above frequently use the fact that (1.4) can be put into the form of (1.3) by integration to study the existence and qualitative behavior of solutions by applying fixed point theorems. We now consider the right-hand side of (1.1) as a mapping $Fx = Bx + Ax$ on a convex subset M of the Banach space $(P_T, \|\cdot\|)$ of continuous T-periodic functions $\phi : R \to R^n$ with the supremum norm, where $(Bx)(t) = h(t, x(t))$ and Ax represents the integral term, with a view of showing that B is a nonlinear contraction and A is compact. Observe that F in general is a non-self map; that is, F may not necessarily map M into M. This presents a significant challenge to investigators. A modern approach to such a problem is to use topological degree theory or transversality method to prove the existence of fixed points. The later requires the construction of a homotopy U_λ and uses conditions on U_λ which may be less general, but more easily established in application (see Burton and Kirk [9], Liu and Liu [22], Granas and Dugundji [13], Wu, Xia, and Zhang [27]). The following formulation is from O'Regan [25].

Theorem 1.1 *Let U be an open set in a closed, convex set C of a Banach space $(E, \|\cdot\|)$ with $0 \in U$. Suppose that $F : \overline{U} \to C$ is given by $F = F_1 + F_2$ and $F(\overline{U})$ is a bounded set in C. In addition, assume that $F_1 : \overline{U} \to C$ is continuous and completely continuous and for $F_2 : \overline{U} \to C$, there exists a continuous, nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ satisfying $\phi(t) < t$ for $t > 0$ such that $\|F_2(x) - F_2(y)\| \leq \phi(\|x-y\|)$ for all $x, y \in \overline{U}$. Then either*
(A1) *F has a fixed point in $\overline{U}$, or*
(A2) *there is a point $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda F(u)$.*

We will apply Theorem 1.1 to show that $F = B + A$ has a fixed point in M which is a T-periodic solution of (1.1). This will be done in Section 2. Our proofs are by Liapunov functionals V. It is to be noted that the technique used allows us to prove that V is bounded without ever asking a growth condition on g that makes the derivative of V negative in any region. In Section 3, we discuss some special cases of (1.1) with illustrative examples to show application of the main result.

For $x \in R^n$, $|x|$ denotes the Euclidean norm of x. Let $C(X,Y)$ denote the space of continuous functions $\phi : X \to Y$. For an $n \times n$ matrix $B = (b_{ij})_{n\times n}$, we denote the norm of B by $\|B\| = \sup\{|Bx| : |x| \le 1\}$. If B is symmetric, we use the convention for self-adjoint positive operators to write $B \ge 0$ whenever B is positive semi-definite. Similarly, if B is negative semi-definite, then $B \le 0$. Also, if $B \ge 0$, we denote its square root by $\sqrt{B}$. Concerning the terminology of a completely continuous mapping, we use the usual convention to mean the following: Let E be a Banach space and $P : X \subseteq E \to E$. If $P(Y)$ is relatively compact for every bounded set $Y \subseteq X$, we say that P is completely continuous. In particular, X need not be bounded (see Agarwal, Meehan and O'Regan [1, p. 56]).

2 The Main Result

In this section, we will apply Theorem 1.1 with $F_1 = A,\ F_2 = B,\ U = M,$ and $C = E = P_T$ to show that the mapping $F = B + A$ has a fixed point in $M = \{x \in P_T : \|x\| < \mu\}$ for some constant $\mu > 0$, where

$$(Bx)(t) = h(t, x(t)) \ \text{ and } \ (Ax)(t) = -\int_{-\infty}^{t} D(t,s)g(s,x(s))ds \tag{2.1}$$

for any $x \in P_T$.

To show that F has a fixed point in M, we must prove that the alternative (A2) does not hold and the homotopy $U_\lambda(x) = \lambda F(x)$ is fixed point free on ∂M for $\lambda = 1$. This can be achieved by establishing the existence of an *a priori* bound for all possible fixed points of $\lambda(B + A)$ for $0 < \lambda \le 1$. To accomplish this, we assume that

(H_1) there exists a constant $T > 0$ such that $D(t+T, s+T) = D(t,s),\ h(t+T, x) = h(t,x),\ g(t+T, x) = g(t,x)$ for all $t, s \in R$ and all $x \in R^n$,

(H_2) there exist constants $K > 0$ and $\eta > 0$ such that

$$g^T(t,x)[x - \lambda h(t,x)] \ge \eta|g(t,x)| \quad \text{for all } |x| \ge K,\ t \ge 0, \text{ and } \lambda \in (0,1] \tag{2.2}$$

where g^T is the transpose of g,

(H_3) $|h(t,x) - h(t,y)| \le \psi(|x-y|)$ for all $t \in R$ and $x, y \in R^n$, where ψ is continuous, nondecreasing with $\psi(r) < r$ for all $r > 0$ and

$$\lim_{r\to\infty}(r - \psi(r)) = \infty, \tag{2.3}$$

(H$_4$) $D_s(t,s) \geq 0$ and $D_{st}(t,s) \leq 0$ with $D_s(t,s)$ and $D_{st}(t,s)$ continuous in the matrix norm for all $t \geq s \geq 0$,

(H$_5$) $\displaystyle\int_{-\infty}^{t} \Big(\|D(t,s)\| + \|D_s(t,s)\|\Big)ds$ and $\displaystyle\int_{-\infty}^{t} \Big(\|D_s(t,s)\| + \|D_{st}(t,s)\|\Big)(t-s)^2 ds$ are continuous in t with $\lim\limits_{s\to-\infty}(t-s)\|D(t,s)\| = 0$ for fixed t,

(H$_6$) there exists a function $Q \in C([0,T], R^+)$ with $Q(0) = 0$ such that

$$\int_{-\infty}^{t_1} \|D(t_1,s) - D(t_2,s)\|ds \leq Q(|t_2 - t_1|) \text{ for } 0 \leq t_1 \leq t_2 \leq T.$$

Remark 2.1 *We observe that all of these conditions on $D(t,s)$ can be verified if*

$$D(t,s) = [(t-s)+1]^{-k}\widetilde{D}$$

where $\widetilde{D} = (d_{ij})_{n\times n}$ is a positive definite matrix and $k > 2$. Also, (H$_1$) implies that $\int_{-\infty}^{t} D(t,s)g(s,x(s))ds$ is T-periodic whenever $x \in P_T$. We also see that some of these conditions are interconnected. For example, (H$_3$) nearly implies (H$_2$) if $x^T g(t,x) \geq \gamma|x||g(t,x)|$ for all $|x| \geq K$ and a constant $\gamma > 0$. It is also to be noted that $h(t,x)$ satisfying (H$_3$) is a nonlinear contraction in the sense of Boyd and Wong [3].

We now prove the following theorem by constructing a Liapunov functional which has its roots in the work of Burton [4], Kemp [15], and Zhang [30]. The result here generalizes a theorem of Burton [4] for scalar equations.

Theorem 2.1 *If (H$_1$)-(H$_6$) hold, then (1.1) has a T-periodic solution.*

Proof. Let A and B be defined in (2.1) for each $x \in P_T$. A change of variable shows that if $\phi \in P_T$, then $(A\phi)(t+T) = (A\phi)(t)$. Thus, $A, B : P_T \to P_T$ are well defined. By (H$_3$), B satisfies the conditions for F_2 in Theorem 1.1. To establish that $A : \overline{M} \to P_T$ is continuous and completely continuous, we need several steps which follow. Let us first show that there exists a constant $\mu > 0$ such that $\|x\| < \mu$ whenever $x \in P_T$ and $x = \lambda(Bx + Ax)$ for $\lambda \in (0,1]$. Suppose now that $x \in P_T$ satisfying

$$x(t) = \lambda\left[h(t,x(t)) - \int_{-\infty}^{t} D(t,s)g(s,x(s))ds\right] \tag{2.4}$$

and define

$$V(t,x(\cdot)) = \lambda^2 \int_{-\infty}^{t} \left(\int_s^t g(v,x(v))dv\right)^T D_s(t,s)\left(\int_s^t g(v,x(v))dv\right)ds.$$

Then $V(t,x(\cdot))$ is T-periodic and

$$\begin{aligned} V'(t,x(\cdot)) &= \lambda^2\int_{-\infty}^t\left(\int_s^t g(v,x(v))dv\right)^T D_{st}(t,s)\left(\int_s^t g(v,x(v))dv\right)ds \\ &\quad + 2\lambda^2 g^T(t,x(t))\int_{-\infty}^t D_s(t,s)\left(\int_s^t g(v,x(v))dv\right)ds. \end{aligned}$$

If we integrate the last term by parts, we have

$$2\lambda^2 g^T(t,x(t))\left[D(t,s)\int_s^t g(v,x(v))dv\Big|_{s=-\infty}^{s=t}+\int_{-\infty}^t D(t,s)g(s,x(s))ds\right].$$

The first term vanishes at both limits by (H_5); the first term of V' is not positive since $D_{st}(t,s)\le 0$, and if we use (2.4) on the last term, then we obtain

$$\begin{aligned} V'(t,x(\cdot)) &\le 2\lambda^2 g^T(t,x(t))\int_{-\infty}^t D(t,s)g(s,x(s))ds \\ &= 2\lambda g^T(t,x(t))[-x(t)+\lambda h(t,x(t))]. \end{aligned} \tag{2.5}$$

By (H_2), we see that if $|x(t)|\ge K$, then

$$V'(t,x(\cdot)) \le -\lambda\eta|g(t,x(t))|. \tag{2.6}$$

It is clear that $V'(t,x(\cdot))$ is bounded above for $0\le|x(t)|\le K$ since $g(t,x)$ and $h(t,x)$ are bounded for x bounded, and hence, there exists a constant $L>0$ depending on K such that

$$V'(t,x(\cdot)) \le -\lambda\eta|g(t,x(t))|+\lambda\eta L \tag{2.7}$$

for all $t\in R$. By the Schwarz inequality, we have

$$\begin{aligned} &\lambda^2\left|\int_{-\infty}^t D_s(t,s)\int_s^t g(v,x(v))dvds\right|^2 \\ =\ &\lambda^2\left|\int_{-\infty}^t\sqrt{D_s(t,s)}\left[\sqrt{D_s(t,s)}\int_s^t g(v,x(v))dv\right]ds\right|^2 \\ \le\ &\lambda^2\int_{-\infty}^t\|\sqrt{D_s(t,s)}\|^2ds\int_{-\infty}^t\left|\sqrt{D_s(t,s)}\int_s^t g(v,x(v))dv\right|^2ds \\ =\ &\lambda^2\int_{-\infty}^t\|\sqrt{D_s(t,s)}\|^2ds\int_{-\infty}^t\left(\int_s^t g(v,x(v))dv\right)^T D_s(t,s)\left(\int_s^t g(v,x(v))dv\right)ds \\ \le\ &\int_{-\infty}^t\|D_s(t,s)\|ds\,V(t,x(\cdot))\le JV(t,x(\cdot)) \end{aligned} \tag{2.8}$$

where

$$J = \sup_{0\le t\le T} \int_{-\infty}^{t} \|D_s(t,s)\|ds.$$

We have just integrated the left side by parts, obtaining

$$\left|\int_{-\infty}^{t} D(t,s)g(s,x(s))ds\right|^2 = \left|\int_{-\infty}^{t} D_s(t,s)\int_s^t g(v,x(v))dvds\right|^2$$

so that by (2.4) we now have

$$|x(t) - \lambda h(t,x(t))|^2 \le JV(t,x(\cdot)). \tag{2.9}$$

Since V is T-periodic, there exists a sequence $\{t_n\} \uparrow \infty$ with $V(t_n,x(\cdot)) \ge V(s,x(\cdot))$ for $s \le t_n$. Thus,

$$\begin{aligned} 0 &\le V(t_n,x(\cdot)) - V(s,x(\cdot)) \\ &\le -\lambda\eta \int_s^{t_n} |g(v,x(v))|dv + \lambda\eta L(t_n - s), \end{aligned}$$

and so

$$\int_s^{t_n} |g(v,x(v))|dv \le L(t_n - s).$$

Thus,

$$\begin{aligned} V(t_n,x(\cdot)) &\le \lambda^2 \int_{-\infty}^{t_n} \|D_s(t_n,s)\| \left|\int_s^{t_n} g(v,x(v))dv\right|^2 ds \\ &\le \lambda^2 \int_{-\infty}^{t_n} \|D_s(t_n,s)\| L^2(t_n-s)^2 ds \le \gamma L^2 \end{aligned}$$

where

$$\gamma = \sup_{0\le t\le T} \int_{-\infty}^{t} \|D_s(t,s)\|(t-s)^2 ds.$$

This implies that $V(t,x(\cdot)) \le \gamma L^2$ for all $t \in R$, and therefore by (2.9) we obtain

$$|x(t) - \lambda h(t,x(t))|^2 \le JV(t,x(\cdot)) \le \gamma J L^2. \tag{2.10}$$

Observe that

$$\begin{aligned} |x(t) - h(t,x(t))| &\ge |x(t)| - |h(t,x(t)) - h(t,0)| - |h(t,0)| \\ &\ge |x(t)| - \psi(|x(t)|) - h^* \end{aligned}$$

where $h^* = \sup\{|h(t,0)| : 0 \le t \le T\}$ and ψ is defined in (H_3). By (2.3), there exists a constant $\mu > K$ such that $r \ge \mu$ implies

$$r - \psi(r) - h^* > \sqrt{\gamma J L^2}. \tag{2.11}$$

We now claim that $|x(t)| < \mu$ for all $t \in [0, T]$. If for some $t^* \in [0, T]$ such that $|x(t^*)| \geq \mu$, then by (2.10) and (2.11), we have

$$\begin{aligned} \gamma J L^2 &< (|x(t^*)| - \psi(|x(t^*)|) - h^*)^2 \\ &\leq |x(t^*) - h(t^*, x(t^*))|^2 \leq JV(t^*, x(t^*)) \leq \gamma J L^2, \end{aligned}$$

a contradiction, and thus, $\|x\| < \mu$ whenever x is a solution of (2.4) for $0 < \lambda \leq 1$. We now define

$$M = \{x \in P_T : \|x\| < \mu\}.$$

It is clear that M is an open subset of P_T. By the argument above, if $x = \lambda(Bx + Ax)$ for $0 < \lambda \leq 1$, then $\|x\| < \mu$. This implies $x \in M$, and therefore, (A2) of Theorem 1.1 fails to hold.

Next, we show that $A : \overline{M} \to P_T$ is continuous and $A\overline{M}$ is contained in a compact subset of P_T. Let $\phi_1, \phi_2 \in \overline{M}$. Then for all $t \in [0, T]$, we have

$$|A\phi_1(t) - A\phi_2(t)| \leq \int_{-\infty}^{t} \|D(t, s)\| |g(s, \phi_1(s)) - g(s, \phi_2(s))| ds. \tag{2.12}$$

Since g is uniformly continuous on $\{(t, x) : 0 \leq t \leq T, |x| \leq \mu\}$, then for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|\phi_1 - \phi_2\| < \delta$ implies $|g(s, \phi_1(s)) - g(s, \phi_2(s))| < \varepsilon$ for all $s \in [0, T]$. It then follows from (2.12) that $\|A\phi_1 - A\phi_2\| \leq J^*\varepsilon$, where

$$J^* = \sup_{0 \leq t \leq T} \int_{-\infty}^{t} \|D(t, s)\| ds.$$

Thus, A is continuous on $\overline{M}$. Now if $\phi \in \overline{M}$ and $0 \leq t_1 \leq t_2 \leq T$, then

$$\begin{aligned} & |A\phi(t_1) - A\phi(t_2)| \\ = & \left| \int_{-\infty}^{t_1} D(t_1, s) g(s, \phi(s)) ds - \int_{-\infty}^{t_2} D(t_2, s) g(s, \phi(s)) ds \right| \\ \leq & \int_{-\infty}^{t_1} \|D(t_1, s) - D(t_2, s)\| |g(s, \phi(s))| ds + \int_{t_1}^{t_2} \|D(t_2, s)\| |g(s, \phi(s))| ds \\ \leq & \; g^* Q(|t_2 - t_1|) + g^* D^* |t_2 - t_1| \end{aligned}$$

where $D^* = \sup\{\|D(t, s)\| : 0 \leq s \leq t \leq T\}$ and $g^* = \sup\{|g(t, x)| : 0 \leq t \leq T, |x| \leq \mu\}$. Here we have used ($H_6$) in the last inequality. This implies that $A\overline{M}$ is equicontinuous. The uniform boundedness of $A\overline{M}$ follows from the following inequality

$$|A\phi(t)| \leq \int_{-\infty}^{t} \|D(t, s)\| |g(s, \phi(s))| ds \leq g^* J^*$$

for all $\phi \in \overline{M}$. So, by the Ascoli-Arzela Theorem, $A\overline{M}$ lies in a compact subset of P_T, and therefore, A is completely continuous. Moreover, for each $x \in \overline{M}$, we have

$$\begin{aligned} |(Bx)(t)| &= |h(t, x(t))| \leq \psi(\|x\|) + |h(t, 0)| \\ &\leq \|x\| + h^* \leq \mu + h^*. \end{aligned}$$

This implies that $B\overline{M}$ is bounded, and hence, $F = B + A$ is bounded on $\overline{M}$. By Theorem 1.1, F has a fixed point $x^* \in \overline{M}$. In this case, $x^* \in M$ and x^* is a T-periodic solution of (1.1). The proof is complete.

Remark 2.2 *It is clear from (2.11) that Theorem 2.1 remains true if the condition* $\lim_{r\to\infty}(r - \psi(r)) = \infty$ *in (2.3) is replaced by*

$$\liminf_{r\to\infty} (r - \psi(r)) > \ell \tag{2.13}$$

where $\ell = h^* + L\sqrt{\gamma J}$. *The constant* L *can be expressed as a function of* K *and* η. *In fact, letting*

$$g_K = \sup\{|g(t,x)| : 0 \le t \le T, |x| \le K\}$$

we see from (2.5) that if $|x(t)| \le K$, *then*

$$\begin{aligned} V'(t,x(\cdot)) &\le 2\lambda \left|g^T(t,x(t))\right| \Big[|x(t)| + |h(t,x(t)) - h(t,0)| + |h(t,0)|\Big] \\ &\le 2\lambda g_K[|x(t)| + \psi(|x(t)|) + h^*] \\ &\le 2\lambda g_K[K + \psi(K) + h^*] \\ &\le -\lambda\eta|g(t,x(t))| + \lambda\eta g_K[1 + 2(K + \psi(K) + h^*)/\eta] \end{aligned}$$

Combining this with (2.6), we arrive at (2.7) with $L = g_K[1 + 2(K + \psi(K) + h^*)/\eta]$.

Remark 2.3 *We observe that if* (H_1)-(H_6) *hold for a different norm of* R^n, *say* $|\cdot|_*$, *then Theorem 2.1 is still true. In this case, we choose* $\|D(t,s)\|_*$ *to be the norm induced by* $|\cdot|_*$ *or any matrix norm that is compatible with* $|\cdot|_*$.

3 Special Equations and Examples

In this section, we discuss some special cases of (1.1) with examples and remarks concerning conditions (H_1)-(H_6). These special equations not only have deep roots in application, but possess rich properties that provide much needed insight for investigation of highly nonlinear equations. We first consider the system

$$x(t) = a(t) - \int_{-\infty}^{t} D(t,s)g(x(s))ds \tag{3.1}$$

where $x(t) \in R^n$, $a : R \to R^n$, $D : R \times R \to R^{n\times n}$, $g : R^n \to R^n$ are continuous and assume there exists a constant $K > 0$ such that

$$g(x) = (x_1^m, x_2^m, \cdots, x_n^m)^T \tag{3.2}$$

whenever $|x| \ge K$. Here $x = (x_1, x_2, \cdots, x_n)^T \in R^n$ and m is an odd positive integer.

We now apply the generalized arithmetic-mean and geometric-mean inequality (see Beckenbach and Bellman [2])

$$a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n} \le \Big(\frac{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n}{p_1 + p_2 + \cdots + p_n}\Big)^{(p_1+p_2+\cdots+p_n)} \tag{3.3}$$

where $a_k \ge 0, p_k > 0$ for $k = 1, 2, \cdots, n$, to prove the following lemma.

Lemma 3.1 *Let $x = (x_1, x_2, \cdots, x_n)^T \in R^n$ and $\alpha > 0$. Then*

$$\left(\sum_{i=1}^n |x_i|^\alpha\right)\left(\sum_{j=1}^n |x_j|\right) \le n\left(\sum_{i=1}^n |x_i|^{\alpha+1}\right). \tag{3.4}$$

Proof. Using inequality (3.3) and the convexity of $r^{\alpha+1}$ for $r \ge 0$, we obtain

$$|x_i|^\alpha |x_j| \le \Big(\frac{\alpha|x_i| + |x_j|}{\alpha+1}\Big)^{\alpha+1} \le \frac{\alpha}{\alpha+1}|x_i|^{\alpha+1} + \frac{1}{\alpha+1}|x_j|^{\alpha+1},$$

and thus,

$$\begin{aligned}\left(\sum_{i=1}^n |x_i|^\alpha\right)\left(\sum_{j=1}^n |x_j|\right) &\le \sum_{j=1}^n \left[\sum_{i=1}^n \left(\frac{\alpha}{\alpha+1}|x_i|^{\alpha+1} + \frac{1}{\alpha+1}|x_j|^{\alpha+1}\right)\right] \\ &= \frac{n\alpha}{\alpha+1}\sum_{i=1}^n |x_i|^{\alpha+1} + \frac{n}{\alpha+1}\sum_{j=1}^n |x_j|^{\alpha+1}.\end{aligned}$$

This yields (3.4).

Theorem 3.1 *Suppose that $a \in P_T$ and $D(t+T, s+T) = D(t,s)$ for all $t, s \in R$. If (H$_4$)-(H$_6$) hold, then (3.1) has a T-periodic solution .*

Proof. We first observe that (3.1) is in the form of (1.1) with $h(t,x) = a(t)$. Thus, (H$_1$) and (H$_3$) are satisfied. To verify (H$_2$), we start with the inequality

$$\begin{aligned} g^T(x)\Big[x - \lambda a(t)\Big] &\ge g^T(x)x - \|a\||g(x)| \\ &= g^T(x)x - \|a\|\sqrt{\sum_{j=1}^n |x_j|^{2m}} \\ &\ge g^T(x)x - \|a\|\sum_{j=1}^n |x_j|^m \end{aligned} \tag{3.5}$$

for $|x| \ge K$ and $\lambda \in (0, 1]$, and apply (3.4) to obtain

$$g^T(x)\Big[x - \lambda a(t)\Big] \ge \sum_{j=1}^n |x_j|^{m+1} - \|a\|\sum_{j=1}^n |x_j|^m$$

$$
\begin{aligned}
&\geq \frac{1}{n}\left(\sum_{i=1}^{n}|x_i|^m\right)\left(\sum_{j=1}^{n}|x_j|\right)-\|a\|\sum_{j=1}^{n}|x_j|^m \\
&= \left(\sum_{j=1}^{n}|x_j|^m\right)\left[\frac{1}{n}\sum_{j=1}^{n}|x_j|-\|a\|\right] \\
&\geq \left(\sum_{j=1}^{n}|x_j|^m\right)\left[\frac{1}{n}|x|-\|a\|\right] \geq |g(x)|\left[\frac{1}{n}|x|-\|a\|\right].
\end{aligned}
$$

Letting $K > n\|a\|$, we see (H_2) is satisfied. Thus, by Theorem 2.1, (3.1) has a T-periodic solution. This completes the proof.

Remark 3.1 *Theorem 3.1 is still valid if the function g in (3.2) is replaced by*

$$g(x) = \left(x_1^{1/m}, x_2^{1/m}, \cdots, x_n^{1/m}\right)^T \tag{3.6}$$

where m is an odd positive integer, or by the bounded function

$$g(x) = \left(\frac{x_1}{1+|x|}, \frac{x_2}{1+|x|}, \cdots, \frac{x_n}{1+|x|}\right)^T \tag{3.7}$$

for $x = (x_1, x_2, \cdots, x_n)^T \in R^n$ and $|x| > K$.

Next, we consider the system with a nonlinear contraction term

$$x(t) = h(t, x(t)) - \int_{-\infty}^{t} D(t,s)g(s, x(s))ds \tag{3.8}$$

where $x(t) \in R^n$, $h : R \times R^n \to R^n$, $D : R \times R \to R^{n\times n}$, $g : R \times R^n \to R^n$ are continuous, and $D(t+T, s+T) = D(t,s)$ for all $t, s \in R$.

We now let

$$h(t, x) = \tilde{\alpha}(x) + a(t) \tag{3.9}$$

for all $x \in R^n$ and $t \in R$, and for $x = (x_1, x_2, \cdots, x_n)^T \in R^n$

$$g(t, x) = b(t)\left(x_1^3, x_1^3, \cdots, x_n^3\right)^T \tag{3.10}$$

whenever $|x| \geq K$ for some positive constant K, where $a : R \to R^n$, $b : R \to R^+$ are continuous and T-periodic, and

$$\tilde{\alpha}(x) = \left(\alpha(x_1), \alpha(x_2), \cdots, \alpha(x_n)\right)^T \tag{3.11}$$

with

$$\alpha(u) = u - \frac{u^3}{4(1+u^2)} \quad \text{for all } u \in R. \tag{3.12}$$

It will be easier for us to verify (H$_1$)-(H$_6$) for (3.8) if we endow R^n with the norm

$$|x|_* = \max_{1\le j\le n} |x_j|, \quad x = (x_1, x_2, \cdots, x_n)^T \in R^n$$

and use the matrix norm $\|\cdot\|_*$ induced by $|\cdot|_*$. Thus, if $A = (a_{ij})_{n\times n}$, then

$$\|A\|_* = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$$

(see Lancaster and Tismenetsky [17, p. 365]).

Theorem 3.2 *Under (3.9)-(3.12), if (H$_4$)-(H$_6$) hold with respect to the norm $|\cdot|_*$, then (3.8) has a T-periodic solution.*

Proof. We first observe that for the function α defined in (3.12), it is a straightforward calculation to obtain

$$|\alpha(u) - \alpha(v)| \le \psi(|u - v|) \tag{3.13}$$

for all $u, v \in R$, where

$$\psi(r) = \begin{cases} \frac{61}{64}\, r, & r \ge \sqrt{3} \\ (1 - \frac{1}{64}r^2)\, r, & 0 \le r < \sqrt{3}. \end{cases} \tag{3.14}$$

It is clear that ψ is continuous, increasing with $\psi(r) < r$ for all $r > 0$ and

$$\lim_{r\to\infty} (r - \psi(r)) = \infty.$$

We now proceed to verify that (H$_2$) and (H$_3$) hold. For any $x = (x_1, x_2, \cdots, x_n)^T$, $y = (y_1, y_2, \cdots, y_n)^T \in R^n$, by the definition of $h(t, x)$ in (3.9), we have

$$\begin{aligned} |h(t,x) - h(t,y)|_* &= |\tilde{\alpha}(x) - \tilde{\alpha}(y)|_* \\ &= \max_{1\le j\le n} |\alpha(x_j) - \alpha(y_j)| \\ &\le \max_{1\le j\le n} \psi(|x_j - y_j|) \le \psi(|x - y|_*). \end{aligned} \tag{3.15}$$

Thus, $h(t, x)$ satisfies (H$_3$). To show (H$_2$) holds, we first observe that by (3.10), there exists $K_1 > 0$ such that $|x|_* \ge K_1$ implies

$$g(t, x) = b(t) \left(x_1^3, x_1^3, \cdots, x_n^3\right)^T. \tag{3.16}$$

This implies $|g(t,x)|_* = b(t)|x|_*^3$ for all $|x|_* \geq K_1$ so that

$$\begin{aligned}
& g^T(t,x)[x-\lambda h(t,x)] \\
= \ & (1-\lambda)g^T(t,x)x + \lambda b(t)\sum_{j=1}^{n}\frac{x_j^6}{4(1+x_j^2)} - \lambda g^T(t,x)a(t) \\
\geq \ & (1-\lambda)g^T(t,x)x + \lambda|g(t,x)|_*\left[\frac{|x|_*^3}{4(1+|x|_*^2)} - n\|a\|\right] \\
\geq \ & (1-\lambda)b(t)\frac{1}{n}\left(\sum_{i=1}^{n}|x_i|^3\right)\left(\sum_{j=1}^{n}|x_j|\right) + \lambda|g(t,x)|_*\left[\frac{1}{4}|x|_* - (1+n\|a\|)\right] \\
\geq \ & (1-\lambda)\frac{1}{n}|g(t,x)|_*|x|_* + \lambda|g(t,x)|_*\left[\frac{1}{4}|x|_* - (1+n\|a\|)\right] \\
\geq \ & |g(t,x)|_*\left[\min\{\frac{1}{n},\frac{1}{4}\}|x|_* - (1+n\|a\|)\right].
\end{aligned}$$

Here we have used inequality (3.4). Now there exists $K_2 > 0$ such that

$$g^T(t,x)[x-\lambda h(t,x)] \geq |g(t,x)|_*$$

for all $|x|_* \geq K_2$, $t \in R$, and $\lambda \in (0,1]$. Thus, (H_2) holds, and by Theorem 2.1, (3.8) has a T-periodic solution.

Remark 3.2 *We again point out that (H_2) is a quite mild condition which allows $g(t,x)$ to be highly nonlinear and nearly independent of $h(t,x)$. We also observe that $g(t,x)$ in (3.10) can take the form of (3.6) or (3.7), and Theorem 3.2 is still valid.*

Remark 3.3 For $h(t,x)$ defined in (3.9), the mapping $(Bx)(t) = h(t,x(t))$ for $x \in P_T$ is almost a contraction, but fails near $x = 0$. Thus, we don't expect to find a constant $0 < \alpha < 1$ satisfying $\|Bx - By\| \leq \alpha\|x-y\|$ for all $x, y \in P_T$.

Finally, we wish to point out that the sign condition (H_4) is essential for the existence of periodic solutions of (1.1). For example, the scalar equation

$$x(t) = 1 + \sin t + \int_{-\infty}^{t} e^{-(t-s)}x(s)ds \tag{3.17}$$

does not have a periodic solution. In fact, all solutions of (3.17) are unbounded. It is clear that (H_4) is not satisfied with $D(t,s) = -e^{-(t-s)}$.

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