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Questions on solvability of exterior boundary value problems with fractional boundary conditions

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Abstract. In this paper we study questions on solvability of some boundary value problems for the Laplace equation with boundary integro-differential operators in the exterior of a unit ball. We study properties of the given integral - differential operators of fractional order in a class of functions which are harmonic outside a ball. We prove theorems about existence and uniqueness of a solution of the problem. We construct explicit form of the solution of the problem in integral form, by solving the Dirichlet problem.

Keywords: regular harmonic function, Riemann–Liouville operator, exterior boundary value problem, Dirichlet problem.

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1 Introduction

Let D be a bounded domain in the space R^n, $n \geq 3$, with sufficiently smooth boundary S.

It is known (see e.g. [4]) that any function $u(x)$, which belongs to the class $C^2(D)$ and satisfies the Laplace equation

$$\Delta u(x) = 0, \qquad x \in D,$$

is called harmonic function in the domain D.

Laplace's equation is the most simple example of elliptic partial differential equations. The general theory of solutions to Laplace's equation is known as potential theory. The solutions of Laplace's equation are the harmonic functions, which are important in many fields of science, notably the fields of electromagnetism, robotic technique, astronomy, and fluid dynamics, because they can be used to accurately describe the behavior of electric, gravitational, and fluid potentials (see [5, 10, 12, 21, 24]). In the study of heat conduction, the Laplace equation is the steady-state heat equation. The Laplace operator has a great importance in quantum physics, in particular in the study of the Schrödinger equation.

The present work is devoted to the study of a exterior boundary value problem for the Laplace equation with a boundary operator of fractional order.

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Recently, interest in the study of various boundary value problems for elliptic equations is renewed [7,13,14,16,17,19,25–28,30].

Boundary value problem with boundary operators of fractional order appear in the problem of diffraction of waves and in the processes of electromagnetic waves. Details about this can be seen in [1,31,32].

In the study of boundary value problems for the Laplace equation on infinite domains additionally regular solutions are required. Namely, a function $u(x)$, harmonic in the domain $D_1 = R^n \backslash D$, is called regular harmonic (at infinity) if the following condition holds as $|x| \to \infty$:

$$|u(x)| \leq C|x|^{-(n-2)}, \qquad n \geq 3, \tag{1.1}$$

where $C = \text{const}$.

Note that if for the function $u(x)$ the estimation (1.1) holds, then for any multi-index $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ with $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_n$ the following estimation holds (see [33, p. 373]:

$$\frac{\partial^{|\beta|} u(x)}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} ... \partial x_n^{\beta_n}} = O\left(|x|^{-(n+|\beta|-2)}\right), \qquad n \geq 3. \tag{1.2}$$

Let $\Omega = \{x \in R^n : |x| < 1\}$, $n \geq 3$, be a unit ball, $\partial\Omega = \{x \in R^n : |x| = 1\}$ be a unit sphere. We denote by $\Omega_+ = R^n \backslash \Omega$ the exterior of the unit ball.

Assume that $u(x)$ is the regular harmonic function in the domain Ω_+ $r = |x|$, ($|x| = \sqrt{x_1^2 + \cdots + x_n^2}$ is the norm in R^n), $\theta = \frac{x}{|x|}$ and $0 < \alpha \leq 1$.

For the formulation of the problem, we need to define the fractional differentiation operator. In the class of functions, harmonic in the domain Ω_+, we define integral-differentiation operators of the fractional order. For a positive number α a fractional integration operator in Riemann–Liouville sense of α order is the expression [15]:

$$I_-^\alpha[u](x) = \frac{1}{\Gamma(\alpha)} \int_r^\infty (\tau - r)^{\alpha-1} u(\tau\theta) d\tau,$$

and the expression:

$$\begin{aligned} D_-^\alpha[u](x) &= \left(-\frac{d}{dr}\right)[I_-^\alpha[u]](x) \\ &\equiv \frac{1}{\Gamma(1-\alpha)} \left(-\frac{d}{dr}\right) \int_r^\infty (\tau - r)^{-\alpha} u(\tau\theta) d\tau, \qquad 0 < \alpha \leq 1, \end{aligned} \tag{1.3}$$

is called a fractional differentiation operator in Riemann–Liouville sense of α order, where $\frac{d}{dr}$ denotes differentiation operator of the form

$$\frac{d}{dr} = \sum_{j=1}^n \frac{x_j}{|x|} \frac{\partial}{\partial x_j}.$$

Furthermore, we will suppose, that $I_-^0[u](x) = u(x)$. Then

$$D_-^1[u](x) = \left(-\frac{d}{dr}\right) I_-^0[u](x) = -\frac{du}{dr}(x),$$

therefore, when $\alpha = 1$ the operator (1.3) coincides with derivative by direction of the vector $r = |x|$.

Introduce the additional notation

$$B_{-}^{\alpha}[u](x) = r^{\alpha} D_{-}^{\alpha}[u](x),$$
$$B_{-}^{-\alpha}[u](x) = I_{-}^{\alpha}\left[r^{-\alpha} u\right](x).$$

It is easy to show that the operators B_{-}^{α} and B_{-}^{β} commute for $\alpha,\beta \in (0,1]$. Similarly, we can show that the operators $B_{-}^{-\alpha}$ and $B_{-}^{-\beta}$ also commute.

Hence, we see that in the general case:

$$B_{-}^{\alpha}\left[B_{-}^{\beta}[u]\right](x) \neq B_{-}^{\alpha+\beta}[u](x),$$
$$B_{-}^{-\alpha}\left[B_{-}^{-\beta}[u]\right](x) \neq B_{-}^{-(\alpha+\beta)}[u](x).$$

Let now $\vec{\alpha}=(\alpha_1,\ldots,\alpha_m)$ and $0<\alpha_j \le 1,\ j=1,\ldots,m$. Consider more general operators:

$$\begin{aligned} B_{-}^{\vec{\alpha}}[u](x) &= B_{-}^{\alpha_1}[B_{-}^{\alpha_2}\cdots B_{-}^{\alpha_m}[u]](x) \\ &= B_{-}^{\alpha_m}\left[B_{-}^{\alpha_{m-1}}\cdots B_{-}^{\alpha_1}[u]\right](x), \\ B_{-}^{-\vec{\alpha}}[u](x) &= B_{-}^{-\alpha_1}\left[B_{-}^{-\alpha_2}\cdots B_{-}^{-\alpha_m}[u]\right](x) \\ &= B_{-}^{-\alpha_m}\left[B_{-}^{-\alpha_{m-1}}\cdots B_{-}^{-\alpha_1}[u]\right](x). \end{aligned}$$

Note that properties and applications of similar operators in the class of functions, harmonic in the ball Ω, were studied in [14,28]. Moreover, note that in [6] in the class of functions, harmonic in the ball, properties and applications of operators of the following form were studied:

$$\begin{aligned} &\delta_{c_1} = r\frac{d}{dr} + c_1, \\ &\delta_{\vec{c}}^{m} = \left(r\frac{d}{dr} + c_1\right)\cdots\left(r\frac{d}{dr} + c_m\right), \qquad \vec{c} = (c_1,\ldots,c_m). \end{aligned}$$

2 Formulation and solution of boundary value problems

Now let us consider formulation and solution of some exterior boundary value problems, including the operators $B_{-}^{\vec{\alpha}}$ and $B_{-}^{-\vec{\alpha}}$ on the boundary.

Problem 2.1. *Find a function $u(x)$, harmonic in the domain Ω_+, for which the function $B_{-}^{\alpha}[u](x)$ is continuous in $\Omega_+ \cup \partial\Omega$, satisfying the equality*

$$B_{-}^{\alpha}[u](x) = f(x), \qquad x \in \partial\Omega,$$

and the condition (1.1).

Problem 2.2. *Find a function $u(x)$, harmonic in the domain Ω_+, for which the function $B_{-}^{\vec{\alpha}}[u](x)$ is continuous in $\Omega_+ \cup \partial\Omega$, satisfying the equality*

$$B_{-}^{\vec{\alpha}}[u](x) = f(x), \qquad x \in \partial\Omega,$$

and the condition (1.1).

Papers [2, 3, 9, 11, 18, 20, 23, 29] are dedicated to the study of boundary problems with boundary operators of integer order in the infinite domains. And boundary value problems with boundary operators of the fractional order for elliptic equations were studied in [7, 13, 14, 16, 17, 19, 25–28, 30].

Let $v(x)$ be a regular solution of the Dirichlet problem in the domain Ω_+, i.e.

$$\begin{cases} \Delta v(x) = 0, & x \in \Omega_+, \\ v(x) = f(x), & x \in \partial\Omega, \\ |v(x)| \le C|x|^{-(n-2)}, & |x| \to \infty. \end{cases} \tag{2.1}$$

It is well known (see [8, p. 73]) that if $f(x) \in C(\partial\Omega)$, then the solution of the problem (2.1) exists, unique and can be represented as:

$$v(x) = \frac{1}{\omega_n} \int\limits_{\partial\Omega} \frac{|x|^2 - 1}{|x-y|^{n-1}} f(y) dS_y,$$

where $\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the area of the unit sphere in R^n.

Since when $\alpha = 1$ we have the equality:

$$B^1_-[u](x)\Big|_{\partial\Omega} = -r\frac{\partial u(x)}{\partial r}\Big|_{\partial\Omega} = \frac{\partial u(x)}{\partial \nu}\Big|_{\partial\Omega},$$

where ν is a vector of normal to the sphere $\partial\Omega$, then the Problem 2.1 coincides with the exterior Neumann problem for the Laplace equation. It is known that (see e.g. [23]), for any $f(x) \in C(\partial\Omega)$ the solution of the exterior Neumann problem exists, unique and can be represented as

$$u(x) = \int\limits_r^\infty \frac{v(tx)}{t} dt,$$

where $v(x)$ is a solution of the Dirichlet problem (2.1).

Analogously, the Problem 2.2 when $\alpha_j = 1, j = 1, \dots, m$ coincides with external problem with the boundary operator of the form:

$$(-1)^m \underbrace{r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\dots\left(r\frac{\partial}{\partial r}\right)\right)}_{m} = (-1)^m \left(r\frac{\partial}{\partial r}\right)^m = (-1)^m \left(\frac{\partial}{\partial \nu}\right)^m.$$

Let us formulate the main propositions concerning to the Problems 2.1 and 2.2.

Theorem 2.3. *Let $0 < \alpha \le 1$. Then for any $f(x) \in C(\partial\Omega)$ a solution of the Problem 2.1 exists, unique and can be represented as:*

$$u(x) = B_-^{-\alpha}[v](x), \tag{2.2}$$

where $v(x)$ is a solution of the Dirichlet problem (2.1).

Theorem 2.4. *Let $0 < \alpha_j \le 1, j = 1, 2, \dots, n$. Then for any $f(x) \in C(\partial\Omega)$ a solution of the Problem 2.2 exists, unique and can be represented as:*

$$u(x) = B_-^{-\vec{\alpha}}[v](x), \tag{2.3}$$

where $v(x)$ is a solution of the Dirichlet problem (2.1).

Hence, statement of Theorem 2.3 implies that the Problem 2.1 for any $0 < \alpha \le 1$ behaves as a solution of the exterior Neumann problem.

3 Properties of the operators $B_{-}^{\vec{\alpha}}$ and $B_{-}^{-\vec{\alpha}}$

In this section we investigate some properties of the operator $B_{-}^{\vec{\alpha}}$ and $B_{-}^{-\vec{\alpha}}$.

Lemma 3.1. *Let $0 < \alpha < 1$ and $u(x)$ is a regular harmonic function in the domain Ω_+. Then the following inequalities hold:*

$$\left|B_{-}^{-\alpha}u(x)\right| \le C|x|^{2-n}, \qquad |B_{-}^{\alpha}u(x)| \le C|x|^{2-n}, \qquad |x| \to \infty, \tag{3.1}$$

where C is a some constant.

Proof. Let $u(x)$ be the regular harmonic function in the domain Ω_+. Then

$$\begin{aligned}|B_{-}^{-\alpha}u(x)| &= \left|\frac{1}{\Gamma(\alpha)}\int\limits_1^\infty (s-1)^{\alpha-1}u(sx)ds\right| \\ &\le Cr^{2-n}\int\limits_1^\infty (s-1)^{\alpha-1}s^{2-n}ds \\ &= Cr^{2-n}B(\alpha, n-2-\alpha) = C_1|x|^{2-n}.\end{aligned}$$

Here $B(\alpha,\beta)$ is the Euler beta function. Furthermore, we represent the function

$$B_{-}^{\alpha}u(x) \equiv r^{\alpha}D_{-}^{\alpha}u(x)$$

in the form:

$$\begin{aligned}r^{\alpha}D_{-}^{\alpha}u(x) &= \frac{r^{\alpha}}{\Gamma(1-\alpha)}\frac{d}{dr}\int\limits_r^\infty (\tau-r)^{-\alpha}u(\tau\theta)d\tau \\ &= \frac{r^{\alpha}}{\Gamma(1-\alpha)}\frac{d}{dr}\left(r^{1-\alpha}\int\limits_1^\infty (s-1)^{-\alpha}u(sx)ds\right) \\ &= \frac{1}{\Gamma(1-\alpha)}\int\limits_1^\infty (s-1)^{-\alpha}r\frac{d}{dr}u(sx)ds \\ &\quad+\frac{1-\alpha}{\Gamma(1-\alpha)}\int\limits_1^\infty (s-1)^{-\alpha}u(sx)ds.\end{aligned}$$

Since

$$r\frac{d}{dr}u(x) = \sum_{i=1}^n x_i\frac{\partial u(x)}{\partial x_i},$$

then

$$\left|r\frac{d}{dr}u(x)\right| = \sum_{i=1}^n \left|x_i\frac{\partial u(x)}{\partial x_i}\right| \le C|x|^{2-n}, \qquad |x| \to \infty.$$

Consequently

$$\begin{aligned}|r^{\alpha}D_{-}^{\alpha}u(x)| &\le C_1|x|^{2-n}\int\limits_1^\infty (s-1)^{\alpha-1}s^{2-n}ds \\ &= C_2|x|^{2-n}, \qquad |x| \to \infty.\end{aligned}$$

Lemma 3.1 is proved. $\square$

Remark 3.2. From now on we will consider regular harmonic functions in Ω_+. Therefore all further investigated integrals converge.

Lemma 3.3. *Let $u(x)$ be a regular harmonic function in the domain Ω_+, then the functions $B_-^{\alpha}[u](x)$ and $B_-^{-\alpha}[u](x)$ are also harmonic in Ω_+.*

Proof. Let $u(x)$ be a regular harmonic function in the domain Ω_+. We represent $B_-^{\alpha}[u](x)$ in the form:

$$B_-^{\alpha}[u](x) = -\frac{1}{\Gamma(1-\alpha)}\left(r\frac{d}{dr}+1-\alpha\right)\int_1^{\infty}(s-1)^{-\alpha}u(sx)ds$$
$$= -\frac{1}{\Gamma(1-\alpha)}\int_1^{\infty}(s-1)^{-\alpha}\delta_{1-\alpha}[u](sx)ds.$$

Formally applying the operator Laplace Δ to the function $B_-^{\alpha}[u](x)$, we get

$$\Delta B_-^{\alpha}[u](x) = -\frac{1}{\Gamma(1-\alpha)}\Delta\left[\int_1^{\infty}(s-1)^{-\alpha}\delta_{1-\alpha}[u](sx)ds\right]$$
$$= -\frac{1}{\Gamma(1-\alpha)}\int_1^{\infty}(s-1)^{-\alpha}\delta_{3-\alpha}[\Delta[u]](sx)ds = 0.$$

Now we show harmony of the function $B_-^{-\alpha}[u](x)$. By direct calculation we find, that in the domain Ω_+ the following equality holds:

$$\Delta B_-^{-\alpha}[u](x) = \int_1^{\infty}\frac{(s-1)^{\alpha-1}}{\Gamma(\alpha)}s^{-\alpha}\Delta u(sx)ds = 0, \qquad x\in\Omega_+.$$

Consequently, functions $B_-^{\alpha}[u](x)$ and $B_-^{-\alpha}[u](x)$ are harmonic in Ω_+. Further, since the function $u(x)$ is regular at infinity, then the condition (1.1) holds for this function. Then as in the Lemma 3.1 for the functions $B_-^{\alpha}[u](x)$ and $B_-^{-\alpha}[u](x)$ are regulars at infinity. Lemma 3.3 is proved. □

Lemma 3.4. *Let $u(x)$ be a regular harmonic function in the domain Ω_+, then the functions $B_-^{\vec{\alpha}}[u](x)$ and $B_-^{-\vec{\alpha}}[u](x)$ are also harmonic in Ω_+.*

Proof. Let a function $u(x)$ be regular harmonic in the domain Ω_+. Then as in Lemma 3.3, the function $B_-^{\vec{\alpha}}[u](x)$ can be represented as:

$$B_-^{\vec{\alpha}}[u](x) = -\int_1^{\infty}\frac{ds_1}{\Gamma(1-\alpha_1)}\cdots\int_1^{\infty}\frac{(s-1)^{-\vec{\alpha}}}{\Gamma(1-\alpha_m)}\delta_{1-\alpha_1}[\cdots\delta_{1-\alpha_m}[u]](sx)ds.$$

where $(s-1)^{-\vec{\alpha}} = (s_1-1)^{-\alpha_1}\cdots(s_m-1)^{-\alpha_m}$, $sx = s_1\cdots s_m x$.

Applying Laplace operator to the function $B_-^{\vec{\alpha}}[u](x)$, we have

$$\Delta B_-^{\vec{\alpha}}[u](x) = \int_1^{\infty}\frac{-ds_1}{\Gamma(1-\alpha_1)}\cdots\int_1^{\infty}\frac{(s-1)^{-\vec{\alpha}}}{\Gamma(1-\alpha_m)}\delta_{3-\alpha_1}[\cdots\delta_{3-\alpha_m}[\Delta u]](sx)ds = 0.$$

Further, now we show that the function $B_-^{-\vec{\alpha}}[u](x)$ is harmonic. We represent the function $B_-^{-\vec{\alpha}}[u](x)$ as:

$$B_-^{-\vec{\alpha}}[u](x) = \int_1^\infty \frac{ds_1}{\Gamma(\alpha_1)} \int_1^\infty \frac{ds_2}{\Gamma(\alpha_2)} \cdots \int_1^\infty (s-1)^{\vec{\alpha}-1} \frac{s^{-\vec{\alpha}}}{\Gamma(\alpha_m)} u(sx) ds_m,$$

where $(s-1)^{\vec{\alpha}-1} = (s-1)^{\alpha_1-1} \cdots (s-1)^{\alpha_m-1}$, $s^{-\vec{\alpha}} = s_1^{-\alpha_1} \cdots s_m^{-\alpha_m}$, $sx = s_1 \cdots s_m x$.

Applying the Laplace operator to the function $B_-^{-\vec{\alpha}}[u](x)$, we get

$$\Delta B_-^{-\vec{\alpha}}[u](x) = \int_1^\infty \frac{ds_1}{\Gamma(\alpha_1)} \cdots \int_1^\infty \frac{(s-1)^{\alpha-1}}{\Gamma(\alpha_m)} s^{-\alpha} \Delta u(sx) ds_m = 0, \qquad x \in \Omega.$$

Regularity of the functions $B_-^{\vec{\alpha}}[u](x)$ and $B_-^{-\vec{\alpha}}[u](x)$ at infinity can be checked as in the case of Lemma 3.3. Lemma 3.4 is proved. □

Lemma 3.5. *Let $u(x)$ be a regular harmonic function in the domain Ω_+ and $0 < \alpha \leq 1$. Then, for any $x \in \Omega_+$ the following equality holds:*

$$u(x) = \frac{1}{\Gamma(\alpha)} \int_1^\infty (s-1)^{\alpha-1} s^{-\alpha} B_-^{\alpha}[u](sx) ds. \tag{3.2}$$

Proof. Let $x \in \Omega$ and $t \in [1, \infty)$. Consider the function:

$$\Im_t[u](x) = \frac{1}{\Gamma(\alpha)} \int_t^\infty (\tau - t)^{\alpha-1} \tau^{-\alpha} B_-^{\alpha}[u](\tau x) d\tau.$$

Since $u(x)$ is the regular harmonic function, then it satisfies the estimate (1.1). Then, by the assertion of Lemma 3.1 for the function $B_-^{\alpha}[u](x)$ satisfies the estimate (3.1). Therefore for all $t \geq 1$ the integral $\Im_t[u](x)$ exists.

We represent $\Im_t[u](x)$ as:

$$\Im_t[u](x) = \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \left\{ \int_t^\infty \frac{(\tau - t)^{\alpha}}{\alpha} \tau^{-\alpha} B_-^{\alpha}[u](\tau x) d\tau \right\}.$$

Further, using definition of the operator B_-^{α}, we get

$$\begin{aligned}
\Im_t[u](x) &= -\frac{d}{dt} \left\{ \int_t^\infty \frac{(\tau - t)^{\alpha}}{\alpha \Gamma(\alpha)} \tau^{-\alpha} \tau^{\alpha} \frac{d}{d\tau} \int_\tau^\infty \frac{(\xi - \tau)^{-\alpha}}{\Gamma(1-\alpha)} u(\xi x) d\xi d\tau \right\} \\
&= -\frac{d}{dt} \left\{ \int_t^\infty \frac{(\tau - t)^{\alpha}}{\Gamma(\alpha+1)} \frac{d}{d\tau} \int_\tau^\infty \frac{(\xi - \tau)^{-\alpha}}{\Gamma(1-\alpha)} u(\xi x) d\xi d\tau \right\} \\
&= -\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{d}{dt} \left\{ \frac{(\tau - t)^{\alpha}}{\alpha} \int_\tau^\infty (\xi - \tau)^{-\alpha} u(\xi x) d\xi \Bigg|_{\tau=t}^{\tau=\infty} \right. \\
&\quad \left. + \int_t^\infty (\tau - t)^{\alpha-1} \int_\tau^\infty (\xi - \tau)^{-\alpha} u(\xi x) d\xi d\tau \right\}
\end{aligned}$$

$$= -\frac{d}{dt}\left\{\int\limits_t^\infty \frac{(\tau-t)^{\alpha-1}}{\Gamma(\alpha)}\int\limits_\tau^\infty \frac{(\xi-\tau)^{-\alpha}}{\Gamma(1-\alpha)}u(\xi x)d\xi d\tau\right\}$$
$$= -\frac{d}{dt}\left\{\int\limits_t^\infty u(\xi x)\int\limits_t^\xi \frac{(\tau-t)^{\alpha-1}(\xi-\tau)^{-\alpha}}{\Gamma(\alpha)\Gamma(1-\alpha)}d\tau d\xi\right\}.$$

It is easy to show that

$$\int\limits_t^\xi (\tau-t)^{\alpha-1}(\xi-\tau)^{-\alpha}d\tau = \Gamma(\alpha)\Gamma(1-\alpha).$$

Then

$$\Im_t[u](x) = -\frac{d}{dt}\int\limits_t^\infty u(\xi x)d\xi = u(tx).$$

If now we put $t=1$, then we get the equality (3.2). Lemma 3.5 is proved. $\square$

Lemma 3.6. *Let $\vec{\alpha} = (\alpha_1, \dots, \alpha_m)$, $0 < \alpha_j \le 1, j = 1, \dots, m$ and $u(x)$ be harmonic function in the domain Ω. Then for any $x \in \Omega$ the following equality holds:*

$$u(x) = \int\limits_1^\infty \frac{ds_1}{\Gamma(\alpha_1)}\cdots\int\limits_1^\infty \frac{(s-1)^{\vec{\alpha}-1}}{\Gamma(\alpha_m)}s^{-\vec{\alpha}}B_-^{\vec{\alpha}}[u](sx)ds_m. \tag{3.3}$$

Proof. Let $x \in \Omega$ and $t_j \in [1, \infty)$, $j = \overline{1, m}$. Denote

$$(s-t)^{\alpha-1} = (s_1-t_1)^{\alpha_1-1}\cdots(s_m-t_m)^{\alpha_m-1}.$$

Consider the function:

$$I_t[u](x) = \int\limits_{t_1}^\infty \frac{ds_1}{\Gamma(\alpha_1)}\cdots\int\limits_{t_m}^\infty \frac{(s-t)^{\vec{\alpha}-1}}{\Gamma(\alpha_m)}s^{-\vec{\alpha}}B_-^{\vec{\alpha}}[u](sx)ds_m.$$

Denote

$$I_{t_m}[u](x) = \frac{1}{\Gamma(\alpha)}\int\limits_{t_m}^\infty (s_m-t_m)^{\alpha_m-1}s_m^{-\alpha_m}B_-^{\alpha_m}[v](sx)ds_m,$$

where

$$v(x) = B_-^{\alpha_{m-1}}\left[\cdots B_-^{\alpha_1}[u]\right](x).$$

By using results of Lemma 3.5, we obtain:

$$I_{t_m}[u](x) = \frac{1}{\Gamma(\alpha)}\int\limits_{t_m}^\infty (s_m-t_m)^{\alpha_m-1}s_m^{-\alpha_m}B_-^{\alpha_m}[v](sx)ds_m = v(tx).$$

Further, repeating this process by all t_j, $j = 1, \dots, m-1$, we have

$$I_t[u](x) = u(tx),$$

where $tx = t_1\cdots t_m x$.

If now we put $t_1 = 1, t_2 = 1, \dots, t_m = 1$, then we get the equality (3.3). Lemma 3.6 is proved. $\square$

Lemma 3.7. *If function $u(x)$ is harmonic in the domain Ω, then the following equalities hold:*

$$B_-^{-\alpha}[B_-^{\alpha}[u]](x) = u(x), \qquad B_-^{\alpha}\left[B_-^{-\alpha}[u]\right](x) = u(x). \tag{3.4}$$

Proof. Let us prove the first equality of Lemma 3.7. We apply operator $B_-^{-\alpha}$ to the function $B_-^{\alpha}[u](x)$. By definition of the operator $B_-^{-\alpha}[u](x)$ and according to Lemma 3.5, we have

$$B_-^{-\alpha}[B_-^{\alpha}[u]](x) = \frac{1}{\Gamma(\alpha)} \int_1^{\infty} (s-1)^{\alpha-1} s^{-\alpha} B_-^{\alpha}[u](sx)ds = u(x).$$

To prove the second equality of Lemma 3.7 we apply the operator $B_-^{\alpha}[u](x) = r^{\alpha} D_-^{\alpha}[u](x)$ to the function $B_-^{-\alpha}[u](x)$. Then we get

$$\begin{aligned} B_-^{\alpha}\left[B_-^{-\alpha}[u]\right](x) &= \frac{1}{\Gamma(\alpha)} B_-^{\alpha}\left[\int_1^{\infty} (s-1)^{\alpha-1} s^{-\alpha} u(sx)ds\right] \\ &= \frac{-r^{\alpha}}{\Gamma(\alpha)} \frac{d}{dr} \int_r^{\infty} \frac{(\tau-r)^{-\alpha}}{\Gamma(1-\alpha)} \left[\int_1^{\infty} (s-1)^{\alpha-1} s^{-\alpha} u(s\tau\theta)ds\right] d\tau. \end{aligned}$$

Since $u(x)$ is the regular harmonic function, then it satisfies the estimate (1.1). Therefore, each of the considered integrals exist and by Fubini's theorem, we can change the order of integration. Then

$$B_-^{\alpha}\left[B_-^{-\alpha}[u]\right](x) = -\int_1^{\infty} \frac{(s-1)^{\alpha-1}}{\Gamma(\alpha)} s^{-\alpha} \left[r^{\alpha} \frac{d}{dr} \int_r^{\infty} \frac{(\tau-r)^{-\alpha}}{\Gamma(1-\alpha)} u(s\tau\theta)d\tau\right] ds.$$

Further, it is easily seen correctness of the following equalities:

$$\begin{aligned} r^{\alpha} \frac{d}{dr} \int_r^{\infty} \frac{(\tau-r)^{-\alpha}}{\Gamma(1-\alpha)} u(s\tau\theta)d\tau \underset{s\tau=t}{=} & \frac{r^{\alpha}}{\Gamma(1-\alpha)} \frac{d}{dr} \int_{rs}^{\infty} \left(\frac{t}{s} - r\right)^{-\alpha} u(t\theta)\frac{dt}{s} \\ = & \frac{r^{\alpha} s^{\alpha-1}}{\Gamma(1-\alpha)} \frac{d}{dr} \int_{rs}^{\infty} (t-rs)^{-\alpha} u(t\theta)dt \\ = & \frac{(rs)^{\alpha}}{\Gamma(1-\alpha)} \frac{d}{d(rs)} \int_{rs}^{\infty} (t-rs)^{-\alpha} u(t\theta)dt \\ = & B_-^{\alpha}[u](sx), \end{aligned}$$

taking into account $\theta = \frac{x}{|x|} = \frac{sx}{|sx|}$. Therefore, we have

$$B_-^{\alpha}\left[B_-^{-\alpha}[u]\right](x) = \frac{1}{\Gamma(\alpha)} \int_1^{\infty} (s-1)^{\alpha-1} s^{-\alpha} B_-^{\alpha}[u](sx)ds.$$

Consequently, the second equality of Lemma 3.7 is proved. □

Lemma 3.8. *Let a function $u(x)$ be harmonic in the domain Ω. Then the following equalities hold:*

$$B_-^{-\vec{\alpha}}\left[B_-^{\vec{\alpha}}[u]\right](x) = u(x), \qquad B_-^{\vec{\alpha}}\left[B_-^{-\vec{\alpha}}[u]\right](x) = u(x). \tag{3.5}$$

Proof. Let us prove the first equality. To the function $B_-^{\vec{\alpha}}[u](x)$ we apply the operator $B_-^{-\vec{\alpha}}$. Then by definition of the operator $B_-^{-\vec{\alpha}}$ and according to Lemma 3.6, we get

$$B_-^{-\vec{\alpha}}\left[B_-^{\vec{\alpha}}[u]\right](x)=\int_1^\infty\frac{ds_1}{\Gamma(\alpha_1)}\cdots\int_1^\infty\frac{(s-1)^{\vec{\alpha}-1}}{\Gamma(\alpha_m)}s^{-\vec{\alpha}}B_-^{\vec{\alpha}}[u](sx)ds_m=u(x).$$

To prove the second equality of Lemma 3.8 we apply the operator $B_-^{\alpha_1}[u](x)=r^{\alpha_1}D_-^{\alpha_1}[u](x)$ to the function $B_-^{\vec{\alpha}}[u](x)$. Then

$$\begin{aligned}B_-^{\alpha_1}\left[B_-^{-\vec{\alpha}}[u]\right](x)&=B_-^{\alpha_1}\left[\int_1^\infty\frac{ds_1}{\Gamma(\alpha_1)}\cdots\int_1^\infty\frac{(s-1)^{\vec{\alpha}-1}}{\Gamma(\alpha_m)}s^{-\vec{\alpha}}u(sx)ds_m\right]\\&=-r^{\alpha_1}\frac{d}{dr}\int_r^\infty\frac{(\tau-r)^{-\alpha_1}}{\Gamma(1-\alpha_1)}\left[\int_1^\infty\frac{ds_1}{\Gamma(\alpha_1)}\cdots\int_1^\infty\frac{(s-1)^{\vec{\alpha}-1}}{\Gamma(\alpha_m)}s^{-\vec{\alpha}}u(s\tau\theta)ds_m\right]d\tau\\&=-\int_1^\infty\frac{ds_1}{\Gamma(\alpha_1)}\cdots\int_1^\infty\frac{(s-1)^{\vec{\alpha}-1}}{\Gamma(\alpha_m)}s^{-\vec{\alpha}}\left[r^{\alpha_1}\frac{d}{dr}\int_r^\infty\frac{(\tau-r)^{-\alpha_1}}{\Gamma(1-\alpha_1)}u(s\tau\theta)d\tau\right]ds_m.\end{aligned}$$

It is easy to show implementation of the following equality:

$$\begin{aligned}r^{\alpha_1}\frac{d}{dr}\int_r^\infty\frac{(\tau-r)^{-\alpha_1}}{\Gamma(1-\alpha_1)}u(s\tau\theta)d\tau\underset{s\tau=t}{=}&\frac{r^{\alpha_1}}{\Gamma(1-\alpha_1)}\frac{d}{dr}\int_{rs}^\infty\left(\frac{t}{s}-r\right)^{-\alpha_1}u(t\theta)\frac{dt}{s}\\=&\frac{r^{\alpha_1}s^{\alpha_1-1}}{\Gamma(1-\alpha_1)}\frac{d}{dr}\int_{rs}^\infty(t-rs)^{-\alpha_1}u(t\theta)dt\\=&\frac{(rs)^{\alpha_1}}{\Gamma(1-\alpha_1)}\frac{d}{d(rs)}\int_{rs}^\infty(t-rs)^{-\alpha_1}u(t\theta)dt\\=&B_-^{\alpha_1}[u](sx),\end{aligned}$$

where $\theta=\frac{x}{|x|}=\frac{sx}{|sx|}$. Then

$$B_-^{\alpha_1}\left[B_-^{-\vec{\alpha}}[u]\right](x)=\frac{1}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_m)}\int_1^\infty ds_1\cdots\int_1^\infty(s-1)^{\vec{\alpha}-1}s^{-\vec{\alpha}}B_-^{\alpha_1}[u](sx)ds_m.$$

Consequently, taking into account definition of the operator

$$B_-^{\vec{\alpha}}[u](x)=B_-^{\alpha_1}[B_-^{\alpha_2}\cdots B_-^{\alpha_m}[u]](x),$$

we can write that

$$B_-^{\vec{\alpha}}\left[B_-^{-\vec{\alpha}}[u]\right](x)=\int_1^\infty\frac{ds_1}{\Gamma(\alpha_1)}\cdots\int_1^\infty\frac{(s-1)^{\vec{\alpha}-1}}{\Gamma(\alpha_m)}s^{-\vec{\alpha}}B_-^{\vec{\alpha}}[u](sx)ds_m=u(x).$$

The second equality of Lemma 3.8 is proved. □

Therefore, Lemma 3.8 yields that $B_-^{\vec{\alpha}}$ and $B_-^{-\vec{\alpha}}$ are inverse on functions, which are harmonic in Ω_+.

4 Proofs of the main propositions

Proof of Theorem 2.3. Let a solution of the Problem 2.1 exist and be equal to $u(x)$. We apply the operator B^{α}_{-} to the function $u(x)$ and denote it by $B^{\alpha}_{-}[u](x) = v(x)$. By assumption $B^{\alpha}_{-}[u](x) \in C(\Omega_{+} \cup \partial\Omega)$, then $v(x) \in C(\Omega_{+} \cup \partial\Omega)$. Since $u(x)$ is a harmonic function in Ω_{+}, regular at infinity, then due to Lemma 3.3 the function $v(x)$ is also harmonic in the domain Ω_{+} and regular at infinity. Moreover,

$$v(x)|_{\partial\Omega} = B^{\alpha}_{-}[u](x)|_{\partial\Omega} = f(x).$$

We apply the operator $B^{-\alpha}_{-}$ to the equality $B^{\alpha}_{-}[u](x) = v(x)$. Since the integral of the form

$$\int\limits_{1}^{\infty} (\tau-1)^{\alpha-1}\tau^{-\alpha} v(\tau x)\, d\tau$$

has week singularity when $\alpha \in (0,1]$, $\tau = 1$ and $\tau = \infty$, then it is a continuous function by $x \in \Omega_{+} \cup \partial\Omega$, where $v(x) \in C(\Omega_{+} \cup \partial\Omega)$ is continuous. Thus, the operator $B^{-\alpha}_{-}$ can be applied to functions from $C(\Omega_{+} \cup \partial\Omega)$. Due to the first equality (3.4) we get (2.2). Moreover, due to Lemma 3.3, the function $B^{-\alpha}_{-}[v](x)$ is regular at infinity. Therefore, the function $v(x)$ is a solution of the Dirichlet problem (2.1). Moreover, if $f(x) \in C(\partial\Omega)$, then solution of the problem exists, unique and $v(x) \in C(\Omega_{+} \cup \partial\Omega)$. Let, on the contrary, function $v(x)$ be a solution of the Dirichlet problem (2.1) with boundary value $f(x) \in C(\partial\Omega)$. Then $v(x) \in C(\Omega_{+} \cup \partial\Omega)$. Consider the function $u(x) = B^{-\alpha}_{-}[v](x)$. Due to the second equality (3.4), we have

$$B^{\alpha}_{-}[u](x) = B^{\alpha}_{-}\left[B^{-\alpha}_{-}[v]\right](x) = v(x).$$

It means that the function $u(x)$ is harmonic in Ω, regular at infinity and

$$B^{\alpha}_{-}[u](x)|_{\partial\Omega} = v(x)|_{\partial\Omega} = f(x).$$

Theorem 2.3 is proved. □

Proof of Theorem 2.4. Let a solution of the Problem 2.2 exist and be $u(x)$. Apply the differential operator $B^{\vec{\alpha}}_{-}$ to the function $u(x)$ and denote it by

$$B^{\vec{\alpha}}_{-}[u](x) = v(x).$$

By assumption $B^{\vec{\alpha}}_{-}[u](x) \in C(\Omega_{+} \cup \partial\Omega)$, and, therefore $v(x) \in C(\Omega_{+} \cup \partial\Omega)$. Since $u(x)$ is a harmonic function in the domain Ω_{+}, regular at infinity, then due to the Lemma 3.4 the function $v(x)$ is also harmonic outside the ball, regular at infinity and

$$v(x)|_{\partial\Omega} = B^{\vec{\alpha}}_{-}[u](x)|_{\partial\Omega} = f(x).$$

Therefore, $v(x)$ is a solution of the problem (2.1). Moreover, if $f(x) \in C(\partial\Omega)$, then solution of the problem exists, unique and $v(x) \in C(\Omega_{+} \cup \partial\Omega)$. We apply the integral operator $B^{-\vec{\alpha}}_{-}$ to the function $B^{\vec{\alpha}}_{-}[u](x) = v(x)$. Since the integral of the form

$$\int\limits_{1}^{\infty} ds_1 \ldots \int\limits_{1}^{\infty} (s-1)^{\vec{\alpha}-1} s^{-\vec{\alpha}} v(sx) ds_m$$

has week singularity at $s_j = 1$ and $s_j = \infty, j = 1, \ldots, m$, when $\alpha \in (0, 1]$, then it is a continuous function by $x \in \Omega_+ \cup \partial\Omega$ where $v(x) \in C(\Omega_+ \cup \partial\Omega)$ is continuous. Thus, the operator $B_-^{-\vec{\alpha}}$ can be applied to functions from $C(\Omega_+ \cup \partial\Omega)$. Due to the first equality of (3.5), we obtain (2.3). On the contrary, let function $v(x)$ be a solution of the problem (2.1) when $f(x) \in C(\partial\Omega)$. Then $v(x) \in C(\Omega_+ \cup \partial\Omega)$. Consider the function

$$u(x) = B_-^{-\vec{\alpha}}[v](x).$$

Due to (3.5) we have $B_-^{\vec{\alpha}}[u](x) = B_-^{\vec{\alpha}}\left[B_-^{-\vec{\alpha}}[v]\right](x) = v(x)$. Hence, $u(x)$ is harmonic in the domain Ω_+ and $B_-^{\vec{\alpha}}[u](x)|_{\partial\Omega} = v(x)|_{\partial\Omega} = f(x)$. Theorem 2.4 is proved. □

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