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Sturm comparison theorems for some elliptic type equations via Picone-type inequalities

Aydın Tiryaki[1], **Sinem Şahiner**[✉ 1] and **Emine Mısırlı**[2]

[1]İzmir University, Üçkuyular, İzmir 35350, Turkey
[2]Ege University, Bornova, İzmir 35100, Turkey

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Abstract. Sturm theorems have appeared as one of the fundamental subjects in qualitative theory to determine properties of the solutions of differential equations. Motivated by some recent developments for half-linear type elliptic equations, we obtain Picone-type inequalities for two pairs of elliptic type equations with damping and external terms in order to establish Sturmian comparison theorems. Some oscillation results are given as applications.

Keywords: elliptic equations, half-linear equations, oscillation, Picone's inequality, Sturm theory.

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1 Introduction

Differential equations are widely used to construct mathematical models for various types of problems. Therefore, we want to examine the solutions of the differential equations. General solutions of most differential equations cannot be obtained by elementary methods analytically, that is, the solution cannot be expressed by a formula. For this reason, the qualitative approach helps us to describe some properties of the solutions without finding the analytical solution.

The determination of the qualitative character of half linear equations is a current interest; oscillatory behavior of solutions has been studied by many authors. Sturm comparison theorems and Picone identities or Picone-type inequalities play an important role in determining the oscillatory behavior of half linear elliptic equations. There are many papers (and books) dealing with Picone identities and Picone-type inequalities for certain type differential equations. For example see [5, 11, 12, 16, 18, 19, 23, 28–30, 35–37].

After the pioneering work of Sturm [27] in 1836, Sturmian comparison theorems have been derived for differential equations of various types, especially via Picone type inequalities. We refer to Kreith [21, 22], Swanson [29] for Picone identities and Sturmian comparison theorems for linear elliptic equations, and to Allegretto [3], Allegretto and Huang [5, 6], Bognár and

✉Corresponding author. Email: sinem.uremen@izmir.edu.tr

Došlý [9], Dunninger [13], Jaroš et al. [16,17,19,20], Kusano et al. [23], Yoshida [36,38], Tiryaki [31], Tiryaki and Sahiner [33] for Picone identities, Sturmian comparison and/or oscillation theorems for half linear elliptic equations.

It is known that, there are several results dealing with the solutions of linear equations. Thus, comparing the behavior of solutions of nonlinear equations with linear equations would help us to learn more about nonlinear equations. For instance in [17], Jaroš et al. considered the linear elliptic operator

$$\bar{\ell}(u) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + c(x)u \tag{1.1}$$

with the nonlinear elliptic operators

$$\bar{L}(v) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(A_{ij}(x)\frac{\partial v}{\partial x_j}\right) + C(x)|v|^{\beta-1}v \tag{1.2}$$

and

$$\tilde{L}(v) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(A_{ij}(x)\frac{\partial v}{\partial x_j}\right) + C(x)|v|^{\beta-1}v + D(x)|v|^{\gamma-1}v, \tag{1.3}$$

where β and γ are positive constants with $\beta > 1$ and $0 < \gamma < 1$, $(a_{ij}(x))$ and $(A_{ij}(x))$ are matrices. They derived Sturm comparison theorems on the basis of the Picone-type inequalities for the pairs of $\bar{\ell}(u) = 0$, $\bar{L}(v) = f(x)$ and $\bar{\ell}(u) = 0$, $\tilde{L}(v) = 0$, and they gave oscillation theorems for the equations $\bar{L}(v) = f(x)$ and $\tilde{L}(v) = 0$.

Recently, motivated by this paper, Şahiner et al. [33] obtained some new results related to Sturmian comparison theory for a damped linear elliptic equation and a forced nonlinear elliptic equation with a damping term. They considered the damped linear elliptic operator

$$\ell^*(u) = \nabla \cdot (a(x)\nabla u) + 2b(x) \cdot \nabla u + c(x)u \tag{1.4}$$

with a forced nonlinear elliptic operator with damping term of the form

$$\begin{aligned} P^*(v) &= \nabla \cdot (A(x)|\nabla v|^{\alpha-1}\nabla v) + (\alpha+1)|\nabla v|^{\alpha-1}B(x) \cdot \nabla v + g(x,v), \\ g(x,v) &= C(x)|v|^{\alpha-1}v + \sum_{i=1}^{\ell} D_i(x)|v|^{\beta_i-1}v + \sum_{j=1}^{m} E_j(x)|v|^{\gamma_j-1}v, \end{aligned} \tag{1.5}$$

where $|\cdot|$ denotes the Euclidean length, and " $\cdot$ "denotes the scalar product. It is assumed that $0 < \gamma_j < \alpha < \beta_i$, $(i = 1,2,\ldots,\ell;\ j = 1,2,\ldots,m)$.

Here the following question arises: is it possible to extend the Sturm comparison results in [33] such that equations (1.4) and (1.5) are replaced with equations with matrix coefficients? The objective of this paper is to give an affirmative answer to this question.

We organize this paper as follows: in Section 2, we establish Picone-type inequalities for a pair of $\{\ell^*, P^*\}$ with matrix coefficients. In Section 3, we present Sturmian comparison theorems, and Section 4 is left for applications.

2 Picone-type inequalities

In this section we establish Picone-type inequalities and some Sturmian comparsion results for a pair of differential equations. In this respect, we examine the following damped operators:

$$\ell(u) := \sum_{k=1}^{m} \nabla \cdot (a_k(x)\nabla u) + 2b(x) \cdot \nabla u + c(x)u \tag{2.1}$$

and

$$P(v) := \sum_{k=1}^{m} \nabla \cdot \left(A_k(x) \left| \sqrt{A_k(x)} \nabla v \right|^{\alpha-1} \nabla v \right) + (\alpha+1) \left| \sqrt{A_k(x)} \nabla v \right|^{\alpha-1} B(x) \cdot \nabla v + g(x,v), \quad (2.2)$$

where

$$g(x,v) = C(x)|v|^{\alpha-1}v + \sum_{i=1}^{\ell} D_i(x)|v|^{\beta_i-1}v + \sum_{j=1}^{m} E_j(x)|v|^{\gamma_j-1}v.$$

It is noted that $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n})^T$ and the operator norm $\|A(x)\|_2$ of an $n \times n$ matrix function $A(x)$ is defined by

$$\|A(x)\|_2 = \sup\{|A(x)\xi|;\ \ \xi \in R^n,\ \ |\xi| < 1\}.$$

For a real, symmetric positive definite matrix $A(x)$, there exists a unique symmetric positive semidefinite matrix $\sqrt{A_k(x)}$ satisfying $(\sqrt{A_k(x)})^2 = A(x)$ and $(A_k(x))^{-1}$ is the inverse of the matrix $A_k(x)$.

It is known that

$$\|A(x)\|_2 = \sqrt{\lambda_{\max}(A^T(x)A(x))}$$

where the superscript T denotes the transpose and $\lambda_{\max}(A^T(x)A(x))$ denotes the eigenvalue of $A^T(x)A(x)$.

It is assumed that $\beta_i > \alpha > \gamma_j > 0$, $(i = 1,2,\dots,\ell;\, j = 1,2,\dots,m)$. When $m = 1$ and $A_1(x)$ is the identity matrix I_n, the principal part of (2.2) reduced to the p-Laplacian $\nabla \cdot (|\nabla v|^{p-2}\nabla v)$, $p = \alpha + 1$. We know that a variety of physical phenomena are modeled by equations involving the p-Laplacian [2, 7, 8, 24–26]. We refer the reader to Diaz [10] for detailed references on physical background of the p-Laplacian.

In this section, we first establish Picone-type inequalities for a pair of differential equations $\ell(u) = 0$ and $P(v) = 0$ and then for $\ell(u) = 0$ and $P(v) = f(x)$, where the operators ℓ and P are defined by (2.1) and (2.2), respectively.

Let G be a bounded domain in R^n with piecewise smooth boundary ∂G. We assume that: matrices $a_k(x), A_k(x) \in C(\bar{G}, R^{n\times n})$, $(k = 1,2,\dots,m)$ are symmetric and positive semidefinite in G; $b(x), B(x) \in C(\bar{G}, R^n)$; $c(x), C(x) \in C(\bar{G}, R)$; $D_i(x)$, $(i = 1,2,\dots,\ell)$, $E_j(x) \in C(\bar{G}, R^+ \bigcup\{0\})$, $(j = 1,2,\dots,m)$ and $f(x) \in C(\bar{G}, R)$.

The domain $D_\ell(G)$ of $\ell(u)$ is defined to be set of all functions u of class $C^1(\bar{G}, R)$ with the property that $a_k(x)\nabla u \in C^1(G, R^n) \cap C(\bar{G}, R^n)$. The domain $D_P(G)$ of P is defined to be the set of all functions v of class $C^1(\bar{G}, R)$ with the property that $A_k(x)|\sqrt{A_k(x)}\nabla v|^{\alpha-1}\nabla v \in C^1(G, R^n) \cap C(\bar{G}, R^n)$.

Let $N = \min\{\ell, m\}$,

$$C_1(x) = C(x) + \sum_{i=1}^{N} H_1(\beta_i, \alpha, \gamma_i; D_i(x), E_i(x)), \quad (2.3)$$

where

$$H_1(\beta, \alpha, \gamma; D(x), E(x)) = \frac{\beta-\gamma}{\alpha-\gamma}\left(\frac{\beta-\alpha}{\alpha-\gamma}\right)^{\frac{\alpha-\beta}{\beta-\gamma}} (D(x))^{\frac{\alpha-\gamma}{\beta-\gamma}} (E(x))^{\frac{\beta-\alpha}{\beta-\gamma}},$$

and

$$C_2(x) = C(x) + \sum_{i=1}^{N} H_2(\beta_i, \alpha, D_i(x), f(x)), \tag{2.4}$$

where

$$H_2(\beta,\ \alpha,\ D(x),\ f(x)) = \left(\frac{\beta}{\alpha}\right)\left(\frac{\beta-\alpha}{\alpha}\right)^{\frac{\alpha-\beta}{\beta}} (D(x))^{\frac{\alpha}{\beta}} |f(x)|^{\frac{\beta-\alpha}{\beta}}.$$

We need the following lemma in order to give the proof of our results.

Lemma 2.1 ([23])**.** *The inequality*

$$|\xi|^{\alpha+1} + \alpha|\eta|^{\alpha+1} - (\alpha+1)|\eta|^{\alpha-1}\xi\cdot\eta \geq 0$$

is valid for any $\xi \in R^n$ and $\eta \in R^n$, where the equality holds if and only if $\xi = \eta$.

Theorem 2.2 (Picone-type inequality for $\ell(u) = 0$ and $P(v) = 0$)**.** *If $u \in D_\ell(G)$ and $v \in D_P(G)$, $v \neq 0$ in G, then for any $u \in C^1(G, R)$ the following Picone-type inequality holds:*

$$\begin{aligned}
&\sum_{k=1}^{m} \nabla\cdot\left(\frac{u}{\varphi(v)}\left[\varphi(v)a_k(x)\nabla u - \varphi(u)A_k(x)\left|\sqrt{A_k(x)}\nabla v\right|^{\alpha-1}\nabla v\right]\right)\\
&\quad\geq -\sum_{k=1}^{m}\left|\sqrt{A_k(x)}\nabla u - u\left(\sqrt{A_k(x)}\right)^{-1}B(x)\right|^{\alpha+1}\\
&\qquad + (\nabla u)^T\cdot\left(\sum_{k=1}^{m} a_k(x)\right)(\nabla u) - |b(x)||\nabla u|^2\\
&\qquad - (|b(x)| + c(x))u^2 + C_1(x)|u|^{\alpha+1} - \frac{u}{\varphi(v)}(\varphi(v)\ell(u) - \varphi(u)P(v))\\
&\qquad + \sum_{k=1}^{m}\left[\left|\sqrt{A_k(x)}\nabla u - u\left(\sqrt{A_k(x)}\right)^{-1}B(x)\right|^{\alpha+1} + \alpha\left|\frac{u}{v}\sqrt{A_k(x)}\nabla v\right|^{\alpha+1}\right.\\
&\qquad\quad \left. - (\alpha+1)\left(\sqrt{A_k(x)}\nabla u - u\left(\sqrt{A_k(x)}\right)^{-1}B(x)\right)\cdot\Phi\left(\frac{u}{v}\sqrt{A_k(x)}\nabla v\right)\right],
\end{aligned} \tag{2.5}$$

where $\varphi(s) = |s|^{\alpha-1}s$, $s \in R$, $\Phi(\xi) = |\xi|^{\alpha-1}\xi$, $\xi \in R^n$ and $C_1(x)$ is defined by (2.3).

Proof. We easily see that

$$u\ell(u) = u\sum_{k=1}^{m}\nabla\cdot(a_k(x)\nabla u) + 2ub(x)\cdot\nabla u + c(x)u^2$$

or

$$\sum_{k=1}^{m}\nabla\cdot(ua_k(x)\nabla u) = (\nabla u)^T\cdot\left(\sum_{k=1}^{m}a_k(x)\right)(\nabla u) - 2ub(x)\cdot\nabla u - c(x)u^2. \tag{2.6}$$

Using Young's inequality, we have,

$$2ub(x)\cdot\nabla u \leq |b(x)|(u^2 + |\nabla u|^2). \tag{2.7}$$

Using (2.6) and (2.7), we obtain the following inequality:

$$\begin{aligned}
&\sum_{k=1}^{m}\nabla\cdot\left(\frac{u}{\varphi(v)}[\varphi(v)a_k(x)\nabla u]\right)\\
&\quad\geq (\nabla u)^T\cdot\left(\sum_{k=1}^{m}a_k(x)\right)(\nabla u) - |b(x)|\nabla u|^2 - (|b(x)| + c(x))u^2
\end{aligned} \tag{2.8}$$

On the other hand, we observe that the following identity holds:

$$
\begin{aligned}
&-\sum_{k=1}^{m} \nabla \cdot \left(u\varphi(u) \frac{A_k(x)|\sqrt{A_k(x)}\nabla v|^{\alpha-1}\nabla v}{\varphi(v)} \right) \\
&\quad = -\sum_{k=1}^{m} \left| \sqrt{A_k(x)}\nabla u - u\left(\sqrt{A_k(x)}\right)^{-1} B(x) \right|^{\alpha+1} \\
&\qquad + \sum_{k=1}^{m} \left[\left| \sqrt{A_k(x)}\nabla u - u\left(\sqrt{A_k(x)}\right)^{-1} B(x) \right|^{\alpha+1} + \alpha \left| \frac{u}{v}\sqrt{A_k(x)}\nabla v \right|^{\alpha+1} \right. \\
&\qquad\qquad \left. -(\alpha+1)\left(\sqrt{A_k(x)}\nabla u - u\left(\sqrt{A_k(x)}\right)^{-1} B(x) \right) \cdot \Phi\left(\frac{u}{v}\sqrt{A_k(x)}\nabla v \right) \right] \\
&\qquad + \frac{u\varphi(u)}{\varphi(v)} g(x,v) - \frac{u\varphi(u)}{\varphi(v)} P(v)
\end{aligned}
\tag{2.9}
$$

and that

$$
\frac{u\varphi(u)}{\varphi(v)} g(x,v) = C(x)|u|^{\alpha+1} + |u|^{\alpha+1} \left(\sum_{i=1}^{\ell} D_i(x)|v|^{\beta_i-\alpha} + \sum_{j=1}^{m} E_j(x)|v|^{\gamma_j-\alpha} \right).
$$

By Young's inequality,

$$
\begin{aligned}
\sum_{i=1}^{\ell} D_i(x)|v|^{\beta_i-\alpha} + \sum_{j=1}^{m} E_j(x)|v|^{\gamma_j-\alpha} &\geq \sum_{i=1}^{N} \left(D_i(x)|v|^{\beta_i-\alpha} + E_i(x)|v|^{\gamma_i-\alpha} \right) \\
&\geq \sum_{i=1}^{N} H_1(\beta_i, \alpha, \gamma_i; D_i(x), E_i(x)),
\end{aligned}
\tag{2.10}
$$

which yields

$$
\begin{aligned}
\frac{u\varphi(u)}{\varphi(v)} g(x,v) &\geq \left(C(x) + \sum_{i=1}^{N} H_1(\beta_i, \alpha, \gamma_i; D_i(x), E_i(x)) \right) |u|^{\alpha+1} \\
&= C_1(x)|u|^{\alpha+1}.
\end{aligned}
\tag{2.11}
$$

Combining (2.8), (2.9) and (2.11) we get the desired inequality (2.5). □

Theorem 2.3 (Picone-type inequality for $P(v) = 0$)**.** *If $v \in D_P(G)$, $v \neq 0$ in G, then for any $u \in C^1(G,R)$, the following Picone-type inequality holds:*

$$
\begin{aligned}
&-\sum_{k=1}^{m} \nabla \cdot \left(\frac{u\varphi(u)}{\varphi(v)} A_k(x) \left| \sqrt{A_k(x)}\nabla v \right|^{\alpha-1} \nabla v \right) \\
&\quad \geq -\sum_{k=1}^{m} \left| \sqrt{A_k(x)}\nabla u - u\left(\sqrt{A_k(x)}\right)^{-1} B(x) \right|^{\alpha+1} + C_1(x)|u|^{\alpha+1} \\
&\qquad + \sum_{k=1}^{m} \left[\left| \sqrt{A_k(x)}\nabla u - u\left(\sqrt{A_k(x)}\right)^{-1} B(x) \right|^{\alpha+1} + \alpha \left| \frac{u}{v}\sqrt{A_k(x)}\nabla v \right|^{\alpha+1} \right. \\
&\qquad\qquad \left. -(\alpha+1)\left(\sqrt{A_k(x)}\nabla u - u\left(\sqrt{A_k(x)}\right)^{-1} B(x)(A_k(x))^{-1} \right) \cdot \Phi\left(\frac{u}{v}\sqrt{A_k(x)}\nabla v \right) \right] \\
&\qquad - \frac{u\varphi(u)}{\varphi(v)} P(v),
\end{aligned}
\tag{2.12}
$$

where $\varphi(s) = |s|^{\alpha-1}s$, $s \in R$, $\Phi(\xi) = |\xi|^{\alpha-1}\xi$, $\xi \in R^n$ and $C_1(x)$ is defined by (2.3).

Proof. Combining (2.9) with (2.11) yields the desired inequality (2.12). □

Theorem 2.4 (Picone-type inequality for $\ell(u) = 0$ and $P(v) = f(x)$). *If $u \in D_\ell(G)$ and $v \in D_P(G)$, $v \neq 0$ in G and $vf(x) \leq 0$ in G, then for any $u \in C^1(G, R)$ the following Picone-type inequality holds:*

$$\begin{aligned}
&\sum_{k=1}^{m} \nabla \cdot \left(\frac{u}{\varphi(v)} \left[\varphi(v) a_k(x) \nabla u - \varphi(u) A_k(x) \left| \sqrt{A_k(x)} \nabla v \right|^{\alpha-1} \nabla v \right] \right) \\
&\quad \geq -\sum_{k=1}^{m} \left| \sqrt{A_k(x)} \nabla u - u \left(\sqrt{A_k(x)} \right)^{-1} B(x) \right|^{\alpha+1} \\
&\qquad + (\nabla u)^T \cdot \left(\sum_{k=1}^{m} a_k(x) \right) (\nabla u) - |b(x)| |\nabla u|^2 \\
&\qquad - (|b(x)| + c(x)) u^2 + C_2(x) |u|^{\alpha+1} - \frac{u\varphi(u)}{\varphi(v)} [P(v) - f(x)] \\
&\qquad + \sum_{k=1}^{m} \Bigg[\left| \sqrt{A_k(x)} \nabla u - u \left(\sqrt{A_k(x)} \right)^{-1} B(x) \right|^{\alpha+1} + \alpha \left| \frac{u}{v} \sqrt{A_k(x)} \nabla v \right|^{\alpha+1} \\
&\qquad\qquad - (\alpha+1) \left(\sqrt{A_k(x)} \nabla u - u \left(\sqrt{A_k(x)} \right)^{-1} B(x) \right) \cdot \Phi\left(\frac{u}{v} \sqrt{A_k(x)} \nabla v \right) \Bigg],
\end{aligned} \tag{2.13}$$

where $\varphi(s) = |s|^{\alpha-1}s$, $s \in R$, $\Phi(\xi) = |\xi|^{\alpha-1}\xi$, $\xi \in R^n$ and $C_2(x)$ is defined with (2.4).

Proof. We easily obtain that:

$$\begin{aligned}
&-\sum_{k=1}^{m} \nabla \cdot \left(u\varphi(u) \frac{A_k(x) |\sqrt{A_k(x)} \nabla v|^{\alpha-1} \nabla v}{\varphi(v)} \right) \\
&\quad = -\sum_{k=1}^{m} \left| \sqrt{A_k(x)} \nabla u - u \left(\sqrt{A_k(x)} \right)^{-1} B(x) \right|^{\alpha+1} \\
&\qquad + \frac{u\varphi(u)}{\varphi(v)} (g(x,v) - f(x)) - \frac{u\varphi(u)}{\varphi(v)} [P(v) - f(x)] \\
&\qquad + \sum_{k=1}^{m} \Bigg[\left| \sqrt{A_k(x)} \nabla u - u \left(\sqrt{A_k(x)} \right)^{-1} B(x) \right|^{\alpha+1} + \alpha \left| \frac{u}{v} \sqrt{A_k(x)} \nabla v \right|^{\alpha+1} \\
&\qquad\qquad - (\alpha+1) \left(\sqrt{A_k(x)} \nabla u - u \left(\sqrt{A_k(x)} \right)^{-1} B(x) \right) \cdot \Phi\left(\frac{u}{v} \sqrt{A_k(x)} \nabla v \right) \Bigg],
\end{aligned} \tag{2.14}$$

and it is clear that

$$\begin{aligned}
&\frac{u\varphi(u)}{\varphi(v)} \left(C(x)|v|^{\alpha-1}v + \sum_{i=1}^{\ell} D_i(x)|v|^{\beta_i-1} + \sum_{j=1}^{m} E_j(x)|v|^{\gamma_j-1} - f(x) \right) \\
&\quad \geq \left(C(x) + \sum_{i=1}^{\ell} D_i(x)|v|^{\beta_i-\alpha}v - \frac{f(x)}{|v|^{\alpha-1}v} \right) |u|^{\alpha+1}.
\end{aligned} \tag{2.15}$$

It can be shown that by using $vf(x) \leq 0$ and Young's inequality,

$$\begin{aligned}
C(x) + \sum_{i=1}^{\ell} D_i(x)|v|^{\beta_i-\alpha}v - \frac{f(x)}{|v|^{\alpha-1}v} &= C(x) + \sum_{i=1}^{\ell} D_i(x)|v|^{\beta_i-\alpha}v + \frac{|f(x)|}{|v|^{\alpha}} \\
&\geq C(x) + \sum_{i=1}^{N} H_2(\beta_i, \alpha; D_i(x), f(x)).
\end{aligned}$$

Finally,

$$\frac{u\varphi(u)}{\varphi(v)}(g(x,v)-f(x)) \geq C_2(x)|u|^{\alpha+1}. \tag{2.16}$$

Combining (2.8), (2.14) and (2.16), we get the desired inequality (2.13). □

Theorem 2.5 (Picone-type inequality for $P(v) = f(x)$). *If $v \in D_P(G)$, $v \neq 0$ in G and $vf(x) \leq 0$ in G, then for any $u \in C^1(G,R)$ the following Picone-type inequality holds:*

$$\begin{aligned}
&-\sum_{k=1}^{m} \nabla\cdot\left(\frac{u\varphi(u)}{\varphi(v)}A_k(x)\left|\sqrt{A_k(x)}\nabla v\right|^{\alpha-1}\nabla v\right)\\
&\quad\geq \sum_{k=1}^{m}\Bigg[\left|\sqrt{A_k(x)}\nabla u - u\left(\sqrt{A_k(x)}\right)^{-1}B(x)\right|^{\alpha+1} + \alpha\left|\frac{u}{v}\sqrt{A_k(x)}\nabla v\right|^{\alpha+1}\\
&\qquad - (\alpha+1)\left(\sqrt{A_k(x)}\nabla u - u\left(\sqrt{A_k(x)}\right)^{-1}B(x)(A_k(x))^{-1}\right)\cdot\Phi\left(\frac{u}{v}\sqrt{A_k(x)}\nabla v\right)\Bigg]\\
&\qquad - \sum_{k=1}^{m}\left|\sqrt{A_k(x)}\nabla u - u\left(\sqrt{A_k(x)}\right)^{-1}B(x)\right|^{\alpha+1} + C_2(x)|u|^{\alpha+1}\\
&\qquad - \frac{u\varphi(u)}{\varphi(v)}[P(v)-f(x)],
\end{aligned} \tag{2.17}$$

where $\varphi(s) = |s|^{\alpha-1}s$, $s \in R$, $\Phi(\xi) = |\xi|^{\alpha-1}\xi$, $\xi \in R^n$ and $C_2(x)$ is defined with (2.4).

Proof. Combining (2.14) with (2.16) yields the desired inequality (2.17). □

By using the ideas in [41], the condition on $f(x)$ can be removed if we impose another condition on v, as $|v| \geq k_0$. The proofs of the following theorems are similar to that of Theorems 2.2–2.5 and Lemma 1 in [41], and hence are omitted.

Theorem 2.6 (Picone-type inequality for $\ell(u) = 0$ and $P(v) = f(x)$). *If $u \in D_\ell(G)$ of $\ell(u) = 0$, $v \in D_{P_\alpha}(G)$ and $|v| \geq k_0$ then the following Picone-type inequality holds for any $u \in C^1(G,R)$:*

$$\begin{aligned}
&\sum_{k=1}^{m}\nabla\cdot\left(\frac{u}{\varphi(v)}\left[\varphi(v)a_k(x)\nabla u - \varphi(u)A_k(x)\left|\sqrt{A_k(x)}\nabla v\right|^{\alpha-1}\nabla v\right]\right)\\
&\quad\geq -\sum_{k=1}^{m}\left|\sqrt{A_k(x)}\nabla u - u\left(\sqrt{A_k(x)}\right)^{-1}B(x)\right|^{\alpha+1} + (\nabla u)^T\cdot\left(\sum_{k=1}^{m}a_k(x)\right)(\nabla u) - |b(x)||\nabla u|^2\\
&\quad - (|b(x)| + c(x))u^2 + (C_1(x) - k_0^{-\alpha})|u|^{\alpha+1}\\
&\quad + \sum_{k=1}^{m}\Bigg[\left|\sqrt{A_k(x)}\nabla u - u\left(\sqrt{A_k(x)}\right)^{-1}B(x)\right|^{\alpha+1} + \alpha\left|\frac{u}{v}\sqrt{A_k(x)}\nabla v\right|^{\alpha+1}\\
&\qquad - (\alpha+1)\left(\sqrt{A_k(x)}\nabla u - u\left(\sqrt{A_k(x)}\right)^{-1}B(x)\right)\cdot\Phi\left(\frac{u}{v}\sqrt{A_k(x)}\nabla v\right)\Bigg]\\
&\quad - \frac{u\varphi(u)}{\varphi(v)}[P(v)-f(x)],
\end{aligned} \tag{2.18}$$

where $\varphi(s) = |s|^{\alpha-1}s$, $s \in R$, $\Phi(\xi) = |\xi|^{\alpha-1}\xi$, $\xi \in R^n$ and $C_1(x)$ is defined with (2.3).

Theorem 2.7 (Picone-type inequality for $P(v) = f(x)$)**.** *If $v \in D_{P_\alpha}(G)$ and $|v| \geq k_0$ then the following Picone-type inequality holds for any $u \in C^1(G, R)$:*

$$\begin{aligned}
&-\sum_{k=1}^{m} \nabla \cdot \Big(\frac{u\varphi(u)}{\varphi(v)} A_k(x)\Big|\sqrt{A_k(x)}\nabla v\Big|^{\alpha-1}\nabla v\Big) \\
&\geq \sum_{k=1}^{m}\Bigg[\Big|\sqrt{A_k(x)}\nabla u - u\Big(\sqrt{A_k(x)}\Big)^{-1}B(x)\Big|^{\alpha+1} + \alpha\Big|\frac{u}{v}\sqrt{A_k(x)}\nabla v\Big|^{\alpha+1} \\
&\qquad - (\alpha+1)\Big(\sqrt{A_k(x)}\nabla u - u\Big(\sqrt{A_k(x)}\Big)^{-1}B(x)(A_k(x))^{-1}\Big)\cdot\Phi\Big(\frac{u}{v}\sqrt{A_k(x)}\nabla v\Big)\Bigg] \\
&\quad - \sum_{k=1}^{m}\Big|\sqrt{A_k(x)}\nabla u - u\Big(\sqrt{A_k(x)}\Big)^{-1}B(x)\Big|^{\alpha+1} + (C_1(x) - k_0^{-\alpha})|u|^{\alpha+1} \\
&\quad - \frac{u\varphi(u)}{\varphi(v)}[P(v) - f(x)], \qquad (2.19)
\end{aligned}$$

where $\varphi(s) = |s|^{\alpha-1}s$, $s \in R$, $\Phi(\xi) = |\xi|^{\alpha-1}\xi$, $\xi \in R^n$ and $C_1(x)$ is defined with (2.3).

3 Sturmian comparison theorems

In this section we establish some Sturmian comparison results on the basis of the Picone-type inequalities obtained in Section 2. Let us begin with the differential equations $\ell(u) = 0$ and $P(v) = 0$ which contain damping terms.

Theorem 3.1. *Assume $\sum_{k=1}^{m}\sqrt{A_k(x)}$ is positive definite in G. If there is a nontrivial function $u \in C^1(\bar{G}, R)$ such that $u = 0$ on ∂G and*

$$M[u] := \int_G \sum_{k=1}^{m}\Big\{\Big|\sqrt{A_k(x)}\nabla u - u\Big(\sqrt{A_k(x)}\Big)^{-1}B(x)\Big|^{\alpha+1} - C_1(x)|u|^{\alpha+1}\Big\}\,dx \leq 0, \qquad (3.1)$$

then every solution $v \in D_P(G)$ of $P(v) = 0$ vanishes at some point of $\bar{G}$. Furthermore, if $\partial G \in C^1$, then every solution $v \in D_P(G)$ of $P(v) = 0$ has one of the following properties:

(1) v has a zero in G, or

(2) $u = k_0 e^{\alpha(x)} v$, where $k_0 \neq 0$ is a constant and $\nabla\alpha(x) = \sum_{k=1}^{m}(A_k(x))^{-1}B(x)$.

Proof. (The first statement) Suppose to the contrary that there exists a solution $v \in D_P(G)$ of $P(v) = 0$ and $v \neq 0$ on $\bar{G}$. Then the inequality (2.12) of Theorem 2.4 holds. Integrating (2.12) over G and then using divergence theorem, we get

$$\begin{aligned}
M[u] \geq \int_G \sum_{k=1}^{m}\Bigg[&\Big|\sqrt{A_k(x)}\nabla u - u\Big(\sqrt{A_k(x)}\Big)^{-1}B(x)\Big|^{\alpha+1} + \alpha\Big|\frac{u}{v}\sqrt{A_k(x)}\nabla v\Big|^{\alpha+1} \\
&- (\alpha+1)\Big(\sqrt{A_k(x)}\nabla u - u\Big(\sqrt{A_k(x)}\Big)^{-1}B(x)\Big)\cdot\Phi\Big(\frac{u}{v}\sqrt{A_k(x)}\nabla v\Big)\Bigg]. \qquad (3.2)
\end{aligned}$$

Since $u = 0$ on ∂G and $v \neq 0$ on $\bar{G}$, we observe that $u \neq k_0 e^{\alpha(x)} v$ and hence $\nabla\left(\frac{u}{v}\right) - B(x)(A_k(x))^{-1}\frac{u}{v} \neq 0$. We see that

$$\begin{aligned}
\int_G \sum_{k=1}^{m}\Bigg[&\Big|\sqrt{A_k(x)}\nabla u - u\Big(\sqrt{A_k(x)}\Big)^{-1}B(x)\Big|^{\alpha+1} + \alpha\Big|\frac{u}{v}\sqrt{A_k(x)}\nabla v\Big|^{\alpha+1} \\
&- (\alpha+1)\Big(\nabla u - u\Big(\sqrt{A_k(x)}\Big)^{-1}B(x)\Big)\cdot\Phi\Big(\frac{u}{v}\sqrt{A_k(x)}\nabla v\Big)\Bigg]dx > 0, \qquad (3.3)
\end{aligned}$$

which together with (3.2) implies that $M[u] > 0$. This contradicts the hypothesis $M[u] \leq 0$. The proof of the first statement (1) is complete.

(The second statement) Next we consider the case where $\partial G \in C^1$. Let $v \in D_P(G)$ be a solution of $P(v) = 0$ and $v \neq 0$ on G. Since $\partial G \in C^1$, $u \in C^1(\bar{G}, R)$ and $u = 0$ on ∂G, we find that u belongs to the Sobolev space $W_0^{1,\alpha+1}(G)$ which is the closure in the norm

$$\|w\| := \left(\int_G \left[|w|^{\alpha+1} + \sum_{k=1}^{m} \left| \frac{\partial w}{\partial x_i} \right|^{\alpha+1} \right] dx \right)^{\frac{1}{\alpha+1}} \tag{3.4}$$

of the class $C_0^\infty(G)$ of infinitely differentiable functions with compact supports in G, [1, 14]. Let u_j be a sequence of functions in $C_0^\infty(G)$ converging to u in the norm (3.4). Integrating (2.12) with $u = u_j$ over G and then applying the divergence theorem, we observe that

$$\begin{aligned} M[u_j] \geq \int_G \sum_{k=1}^{m} \Bigg[\Big| \sqrt{A_k(x)} \nabla u_j - u_j \Big(\sqrt{A_k(x)} \Big)^{-1} B(x) \Big|^{\alpha+1} + \alpha \Big| \frac{u_j}{v} \sqrt{A_k(x)} \nabla v \Big|^{\alpha+1} \\ -(\alpha+1) \Big(\sqrt{A_k(x)} \nabla u_j - u_j \Big(\sqrt{A_k(x)} \Big)^{-1} B(x) \Big) \cdot \Phi\Big(\frac{u_j}{v} \sqrt{A_k(x)} \nabla v \Big) \Bigg] dx \\ \geq 0. \end{aligned} \tag{3.5}$$

We first claim that $\lim_{j\to\infty} M[u_j] = M[u] = 0$. Since $C(x)$, $D_i(x)$, $(i = 1, 2, \ldots, \ell)$, and $f(x)$ are bounded on $\bar{G}$, there exists a constant $K_1 > 0$ such that

$$|C_1(x)| \leq K_1.$$

It is easy to see that

$$\begin{aligned} |M[u_j] - M[u]| \leq + \int_G \sum_{k=1}^{m} \Bigg\{ \Big| \Big| \sqrt{A_k(x)} \nabla u_j - u_j \Big(\sqrt{A_k(x)} \Big)^{-1} B(x) \Big|^{\alpha+1} \\ - \Big| \sqrt{A_k(x)} \nabla u - u \Big(\sqrt{A_k(x)} \Big)^{-1} B(x) \Big|^{\alpha+1} \Big| \Bigg\} dx \\ + K_1 \int_G \Big| |u_j|^{\alpha+1} - |u|^{\alpha+1} \Big| dx \end{aligned} \tag{3.6}$$

It follows from the mean value theorem that

$$\begin{aligned} &\Big| \Big| \sqrt{A_k(x)} \nabla u_j - u_j \Big(\sqrt{A_k(x)} \Big)^{-1} B(x) \Big|^{\alpha+1} - \Big| \sqrt{A_k(x)} \nabla u - u \Big(\sqrt{A_k(x)} \Big)^{-1} B(x) \Big|^{\alpha+1} \Big| \\ &\quad \leq (\alpha+1) \Big(\Big| \sqrt{A_k(x)} \nabla u_j - u_j \Big(\sqrt{A_k(x)} \Big)^{-1} B(x) \Big| + \Big| \sqrt{A_k(x)} \nabla u - u \Big(\sqrt{A_k(x)} \Big)^{-1} B(x) \Big| \Big)^{\alpha} \\ &\quad \times \Big(\Big| \sqrt{A_k(x)} \nabla (u_j - u) + \Big(\sqrt{A_k(x)} \Big)^{-1} B(x)(u_j - u) \Big| \Big) \\ &\quad \leq (\alpha+1) \Big(\Big\| \sqrt{A_k(x)} \Big\|_2 \big(|\nabla u_j| + |\nabla u| \big) \\ &\qquad + |B(x)| \Big\| \Big(\sqrt{A_k(x)} \Big)^{-1} \Big\|_2 |u_j| + |B(x)| \Big\| \Big(\sqrt{A_k(x)} \Big)^{-1} \Big\|_2 |u| \Big)^{\alpha} \\ &\quad \times \Big(\Big\| \sqrt{A_k(x)} \Big\|_2 |\nabla (u_j - u)| + \Big\| \Big(\sqrt{A_k(x)} \Big)^{-1} \Big\|_2 |u_j - u| |B(x)| \Big). \end{aligned}$$

Since also $B(x)$ is bounded on $\bar{G}$, there is a constant K_3 such that $|B(x)| \|(A_k(x))^{-1}\| \leq K_3$ on $\bar{G}$. Thus,

$$\left| \left| \sqrt{A_k(x)} \nabla u_j - u_j \left(\sqrt{A_k(x)} \right)^{-1} B(x) \right|^{\alpha+1} - \left| \sqrt{A_k(x)} \nabla u - u \left(\sqrt{A_k(x)} \right)^{-1} B(x) \right|^{\alpha+1} \right|$$
$$\leq (\alpha+1) \Big(K_2(|\nabla u_j| + |\nabla u|) + K_3(|u_j| + |u|) \Big)^{\alpha} \Big(K_2 |\nabla(u_j - u)| + K_3 |u_j - u| \Big), \quad (3.7)$$

where $K_2 = \max_{x \in \bar{G}} \| \sqrt{A_k(x)} \|_2$.

Let us take $N_k = \max\{1, K_2, K_3\}$. From the above inequality we have

$$\left| \left| \sqrt{A_k(x)} \nabla u_j - u_j \left(\sqrt{A_k(x)} \right)^{-1} B(x) \right|^{\alpha+1} - \left| \sqrt{A_k(x)} \nabla u - u \left(\sqrt{A_k(x)} \right)^{-1} B(x) \right|^{\alpha+1} \right|$$
$$\leq (\alpha+1) N_k^{\alpha+1} \Big(|\nabla u_j| + |\nabla u| + |u_j| + |u| \Big)^{\alpha} \Big(|\nabla(u_j - u) + |u_j - u|| \Big). \quad (3.8)$$

Using (3.8) and applying Hölder's inequality, we find that

$$\int_G \left| \sum_{k=1}^m \left| \sqrt{A_k(x)} \nabla u_j - u_j \left(\sqrt{A_k(x)} \right)^{-1} B(x) \right|^{\alpha+1} - \left| \sqrt{A_k(x)} \nabla u - u \left(\sqrt{A_k(x)} \right)^{-1} B(x) \right|^{\alpha+1} \right| dx$$
$$\leq (\alpha+1) N_k^{\alpha+1} \left(\int_G (|\nabla u_j| + |\nabla u| + |u_j| + |u|)^{\alpha+1} dx \right)^{\frac{\alpha}{\alpha+1}}$$
$$\times \left(\int_G \Big(|\nabla(u_j - u)|^{\alpha+1} + |u_j - u| \Big)^{\alpha+1} dx \right)^{\frac{1}{\alpha+1}}$$
$$\leq (\alpha+1) N_k^{\alpha+1} n^{\alpha} \left(\|u_j\| + \|u\| \right)^{\alpha} \|u_j - u\|. \quad (3.9)$$

Similarly we obtain

$$\int_G \Big| |u_j|^{\alpha+1} - |u|^{\alpha+1} \Big| dx \leq (\alpha+1) \left(\|u_j\| + \|u\| \right)^{\alpha} \|u_j - u\|. \quad (3.10)$$

Combining (3.6), (3.9) and (3.10), we have

$$|M[u_j] - M[u]| \leq (\alpha+1) K_4 \left(\|u_j\| + \|u\| \right)^{\alpha} \|u_j - u\|$$

for some positive constant K_4 depending on N_k, α, n and m, from which it follows that $\lim_{j \to \infty} M[u_j] = M[u]$. We see from (3.1) that $M[u] \geq 0$, which together with (3.2) implies $M[u] = 0$.

Let $\mathcal{B}$ be an arbitrary ball with $\bar{\mathcal{B}} \subset G$ and define

$$Q_{\mathcal{B}}[w] := \int_G \sum_{k=1}^m \left[\left| \sqrt{A_k(x)} \nabla w - w \left(\sqrt{A_k(x)} \right)^{-1} B(x) \right|^{\alpha+1} + \alpha \left| \frac{w}{v} \sqrt{A_k(x)} \nabla v \right|^{\alpha+1} \right.$$
$$\left. - (\alpha+1) \Big(\nabla w - w \Big(\sqrt{A_k(x)} \Big)^{-1} B(x) \Big) \cdot \Phi \Big(\frac{w}{v} \sqrt{A_k(x)} \nabla v \Big) \right] \quad (3.11)$$

for $w \in C^1(G;\ R)$. It is easily verified that

$$0 \leq Q_{\mathcal{B}}[u_j] \leq Q_G[u_j] \leq M[u_j], \quad (3.12)$$

where $Q_G[u_j]$ denotes the right hand side of (3.11) with $w = u_j$ and with $\mathcal{B}$ replaced by G. By a simple computation,

$$|Q_{\mathcal{B}}(u_j) - Q_{\mathcal{B}}(u)| \leq K_5 (\|u_j\|_{\mathcal{B}} + \|u\|_{\mathcal{B}})^{\alpha} \|u_j - u\|_{\mathcal{B}}$$

$$+K_6(\|u_j\|_{\mathcal{B}})^{\alpha}\|u_j-u\|_{\mathcal{B}}+K_7\|\varphi(u_j)-\varphi(u)\|_{L^q_{(\mathcal{B})}}\|u\|_{\mathcal{B}}, \quad (3.13)$$

where $q=\frac{\alpha+1}{\alpha}$, the constants K_5, K_6 and K_7 are independent of k, and the subscript $\mathcal{B}$ indicates the integrals involved in the norm (3.4) are to be taken over B instead of G. It is known that Nemitski operator $\varphi : L^{\alpha+1}(G) \to L^q(G)$ is continuous [6] and it is clear that $\|u_j-u\|_{\mathcal{B}} \to 0$ as $\|u_j-u\|_G \to 0$. Therefore, letting $j\to\infty$ in (3.12), we find that $Q_{\mathcal{B}}[u]=0$. In view of (3.11), we obtain

$$\Bigg[\Big|\sqrt{A_k(x)}\nabla u-u\Big(\sqrt{A_k(x)}\Big)^{-1}B(x)\Big|^{\alpha+1}+\alpha\Big|\frac{u}{v}\sqrt{A_k(x)}\nabla v\Big|^{\alpha+1} \\ -(\alpha+1)\Big(\nabla u-u\Big(\sqrt{A_k(x)}\Big)^{-1}B(x)\Big)\cdot\Phi\Big(\frac{u}{v}\sqrt{A_k(x)}\nabla v\Big)\Bigg]\equiv 0 \quad \text{in } \mathcal{B}, \quad (3.14)$$

from which Lemma 2.1 implies that

$$\sqrt{A_k(x)}\nabla u-u\Big(\sqrt{A_k(x)}\Big)^{-1}B(x)\equiv\frac{u}{v}\sqrt{A_k(x)}\nabla v \quad \text{or} \quad \nabla\left(\frac{u}{v}\right)-B(x)(A_k(x))^{-1}\frac{u}{v}\equiv 0 \quad \text{in } \mathcal{B}.$$

Hence, we observe that $u/v=k_0e^{\alpha(x)}$ in $\mathcal{B}$ for some constant k_0 and some continuous function $\alpha(x)$ satisfying $\alpha(x)=\sum_{k=1}^{m}(\sqrt{A_k(x)})^{-1}B(x)$. Since $\mathcal{B}$ is an arbitrary ball with $\bar{B}\subset G$, we conclude that $u/v=k_0e^{\alpha(x)}$ in G , where $k_0\neq 0$ in the view of the hypothesis that u is nontrivial and therefore v is a function such that $u=k_0e^{\alpha(x)}v$ in G. This completes the proof of the second statement. □

Remark 3.2. If we omit the damping term, that is $B(x)\equiv 0$ in $M[u]$ in Theorem 3.1 (with $D_i(x)\equiv 0$, $(i=1,2,\ldots,\ell)$, $E_j(x)\equiv 0$, $(j=1,2,\ldots,m)$, we obtain Theorem 2.4 given in [38]. If $B(x)\equiv 0$ in Theorem 3.1, the Theorem 4 given in [40] is observed. Furthermore, in this case we can derive the Wirtinger inequality as given by Corollary 3 in [40]. If we take $m=\alpha=1$ and $B(x)\equiv 0$ with $D_1\equiv C(x)$, $D_i\equiv E_j(x)\equiv 0$, $(i=2,\ldots,\ell)$, $(j=1,2,\ldots,m)$, we obtain the inequality (14) in [17] and (2.21) in [20]. Therefore, Theorem 3.1 is a partial extension of the theorems that are cited above.

Theorem 3.3 (Sturmian comparison theorem)**.** *Assume that $\sum_{k=1}^{m}\sqrt{A_k(x)}$ is positive definite in G. If there is a nontrivial solution $u\in D_\ell(G)$ of $\ell(u)=0$ such that $u=0$ on ∂G and*

$$V[u]:=\int_G\Bigg\{(\nabla u)^T\cdot\left(\sum_{k=1}^{m}a_k(x)\right)(\nabla u)-\sum_{k=1}^{m}\Big|\sqrt{A_k(x)}\nabla u-u\Big(\sqrt{A_k(x)}\Big)^{-1}B(x)\Big|^{\alpha+1} \\ -|b(x)||\nabla u|^2+C_1(x)|u|^{\alpha+1}-(|b(x)|+c(x))\,u^2\Bigg\}dx\geq 0,$$

then every solution $v\in D_P(G)$ of $P(v)=0$ in G must vanish at some point of $\bar{G}$. Furthermore, if $\partial G\in C^1$, then every solution $v\in D_P(G)$ of $P(v)=0$ in G has one of the following properties:

(1) v has a zero in G or

(2) $u=k_0e^{\alpha(x)}v$, where $k_0\neq 0$ is a constant and $\nabla\alpha(x)=\sum_{k=1}^{m}B(x)(A_k(x))^{-1}$.

Proof. This theorem can be proven by applying the same argument used in the proof of Theorem 3.1 via Picone inequality (2.5). But we prefer to give an alternative proof here. By using the definition of $M[u]$ and $V[u]$, we have

$$M[u]=-V[u]+\int_G\Bigg\{(\nabla u)^T\cdot\left(\sum_{k=1}^{m}a_k(x)\right)(\nabla u)-|b(x)||\nabla u|^2-(c(x)+|b(x)|)u^2\Bigg\}dx.$$

For the last integral over G, considering the integral of the inequality (2.8), by using the divergence theorem, and in view of the above inequality, implies that $M[u] \leq 0$. Then the conclusion of the theorem follows from Theorem 3.1. □

Remark 3.4. If we set $\alpha = 1$ in $P(v) = 0$ and take $b(x) \equiv 0$ in $\ell(u) = 0$, $V[u]$ in Theorem 3.3 becomes the following:

$$V[u] = \int_G \sum_{k=1}^{m} \Big\{ (\nabla u)^T \big(a_k(x) - A_k(x)\big)(\nabla u) + \Big(C_1(x) - c(x) - B(x)\Big(\sqrt{A_k(x)}\Big)^{-1} B^T(x) - \nabla \cdot B(x)\Big) u^2 \Big\} dx \geq 0.$$

It can be shown that $V[u] \geq 0$ for any $u \in C^1(\bar{G}, R)$ if $\sum_{k=1}^{m}(a_k(x) - A_k(x))$ is positive semidefinite in G and

$$C_1(x) \geq c(x) + \nabla \cdot B(x) + \sum_{k=1}^{m} B(x)(A_k(x))^{-1} B^T(x) \quad \text{in } G.$$

Remark 3.5. For a special case if we set $\alpha = 1$ in $P(v) = 0$ and take $b(x) \equiv 0$ and $B(x) \equiv 0$ in $\ell(u) = 0$ and $P(v) = 0$, respectively, that is for the following equations:

$$\sum_{k=1}^{m} \nabla \cdot (a_k(x) \nabla u) + c(x) u = 0 \tag{3.15}$$

and

$$\sum_{k=1}^{m} \nabla \cdot (A_k(x) \nabla v) + C(x) v + \sum_{i=1}^{\ell} D_i(x) |v|^{\beta_i - 1} v + \sum_{j=1}^{m} E_j(x) |v|^{\gamma_j - 1} v = 0, \tag{3.16}$$

$V[u]$ in Theorem 3.3 can be expressed as:

$$V[u] = \int_G \sum_{k=1}^{m} \Big\{ (\nabla u)^T \cdot \big(a_k(x) - A_k(x)\big)(\nabla u) + (C_1(x) - c(x)) u^2 \Big\} dx \geq 0. \tag{3.17}$$

For the equation (3.15) and (3.16) the following corollary can be given as a result of special case of Theorem 3.3.

Corollary 3.6. *Assume that $\sum_{k=1}^{m} \sqrt{A_k(x)}$ is positive definite in G, and furthermore assume that*

$$a_k(x) - A_k(x), \quad (k = 1, 2, \ldots, m) \quad \textit{are positive semidefinite in } G$$

$$C_1(x) \geq c(x) \quad \textit{in } G.$$

If there is a nontrivial solution u of (3.15) such that $u = 0$ on ∂G, then every solution v of (3.16) must vanish at some point of $\bar{G}$.

Note that Corollary 3.6 gives Corollary 1 in the case $\alpha = 1$ in [40]. We have used Picone-type inequalities that we obtained in Theorem 2.2 and Theorem 2.3 to establish Theorem 3.1 and Theorem 3.3 in Section 2. Inspired by Yoshida's paper [41], we obtained alternative Picone-type inequalities to establish the following theorems.

Theorem 3.7. *Let $k_0 > 0$ be a constant. Assume $\sum_{k=1}^{m} \sqrt{A_k(x)}$ is positive definite in G. If there is a nontrivial function $u \in C^1(\bar{G}, R)$ such that $u = 0$ on ∂G and*

$$\tilde{M}[u] := \int_G \sum_{k=1}^{m} \left\{ \left| \sqrt{A_k(x)}\nabla u - u\left(\sqrt{A_k(x)}\right)^{-1} B(x) \right|^{\alpha+1} - (C_2(x) - k_0^{-\alpha})|u|^{\alpha+1} \right\} dx \le 0, \quad (3.18)$$

then for every solution $v \in D_{P_\alpha(G)}$ of $P(v) = 0$, either v has a zero on $\bar{G}$ or $|v(x_0)| < k_0$ for some $x_0 \in G$.

Theorem 3.8. *Assume that $\sum_{k=1}^{m} \sqrt{A_k(x)}$ is positive definite in G. If there is a nontrivial solution $u \in D_\ell(G)$ of $\ell(u) = 0$ such that $u = 0$ on ∂G and*

$$\tilde{V}[u] := \int_G \left\{ (\nabla u)^T \cdot \left(\sum_{k=1}^{m} a_k(x) \right)(\nabla u) - \sum_{k=1}^{m} \left| \sqrt{A_k(x)}\nabla u - u\left(\sqrt{A_k(x)}\right)^{-1} B(x) \right|^{\alpha+1} \right.$$
$$\left. - |b(x)||\nabla u|^2 + (C_2(x) - k_0^{-\alpha})|u|^{\alpha+1} - (|b(x)| + c(x))\, u^2 \right\} dx \ge 0,$$

then every solution $v \in D_{P_\alpha}(G)$ of $P_\alpha(v) = 0$ in G must vanish at some point of $\bar{G}$ or $|v(x_0)| < k_0$ for $x_0 \in G$.

These theorems can be proven by using the same ideas in the proof of Theorems 3.1 and 3.3 and Theorem 1 in [41]; hence the proofs are omitted.

Recently there has been considerable interest in studying forced differential equations and their oscillations. Yoshida studied the forced oscillations of second order elliptic equations. For additional examples about oscillation of forced differential equations, the reader may refer to [15,17,20,33,39] and the references cited therein.

Now we continue to give Sturmian comparison result on the basis of the Picone-type inequality obtained in Theorems 2.4 and 2.5 for the differential equations $\ell(u) = 0$ and $P(v) = f(x)$ that contain damping and forcing terms.

Theorem 3.9. *Assume $\sum_{k=1}^{m} \sqrt{A_k(x)}$ is positive definite in G. If there is a nontrivial function $u \in C^1(\bar{G}, R)$ such that $u = 0$ on ∂G and*

$$\tilde{M}_G[u] := \int_G \sum_{k=1}^{m} \left\{ \left| \sqrt{A_k(x)}\nabla u - u\left(\sqrt{A_k(x)}\right)^{-1} B(x) \right|^{\alpha+1} - C_2(x)|u|^{\alpha+1} \right\} dx \le 0, \quad (3.19)$$

then every solution $v \in D_P(G)$ of $P(v) = f(x)$ satisfying $vf(x) \le 0$ in G must vanish at some point of $\bar{G}$. Furthermore, if $\partial G \in C^1$, then every solution $v \in D_P(G)$ of $P(v) = f(x)$ satisfying $vf(x) \le 0$ in G has one of the following properties:

(1) v has a zero in G or

(2) $u = k_0 e^{\alpha(x)} v$, where $k_0 \neq 0$ is a constant and $\nabla\alpha(x) = B(x)(A_k(x))^{-1}$.

Proof. Suppose, to the contrary that, there is a solution $v \in D_P(G)$ of $P(v) = f(x)$ satisfying $vf(x) \le 0$ and $v \neq 0$ on $\bar{G}$. Then the inequality (2.17) of Theorem 2.5 holds for the nontrivial function u. Integrating the inequality (2.17) over G and applying the same idea used in the proof of Theorem 3.1 we observe that $\tilde{M}_G[u] > 0$, which contradicts the hypothesis $\tilde{M}_G[u] \le 0$. This completes the proof of the first statement. In the case where $\partial G \in C^1$, let $v \in D_P(G)$ be a solution of $P(v) = f(x)$ such that $v \neq 0$ in G. By the same arguments as in the proof of Theorem 2.2, we obtain that $\tilde{M}_G[u] = 0$, which implies that u and v can be written in the form $u = k_0 e^{\alpha(x)} v$, where $k_0 \neq 0$ is a constant and $\nabla\alpha(x) = B(x)(A_k(x))^{-1}$. Thus, the proof of the theorem is complete. □

Remark 3.10. If we omit the damping term, that is $B(x) \equiv 0$ in $M[u]$ in Theorem 3.1, we obtain Theorem 8 given in [40] and the inequality (14) in [17] for $m = \alpha = 1$. Furthermore if we take $m = \alpha = 1$ with $D_1(x) \equiv C(x)$, $D_i(x) \equiv 0$, $(i = 2, \ldots, \ell)$ and $E_j(x) \equiv 0$, $(j = 1, 2, \ldots, m)$, we get the inequality (2.23) in [20]. Therefore this theorem is a partial extension of these theorems cited above.

Corollary 3.11. *Let $\sum_{k=1}^{m} \sqrt{A_k(x)}$ be positive definite in G. Assume that $f(x) \geq 0$ (or $f(x) \leq 0$) in G. If there is a nontrivial function $u \in C^1(G, R)$ such that $u = 0$ on ∂G and $\tilde{M}_G[u] \leq 0$, then $P(v) = f(x)$ has no negative (or positive) solution on $\bar{G}$.*

Proof. Suppose that $P(v) = f(x)$ has a negative (or positive) solution v on $\bar{G}$. Then $vf(x) \leq 0$ in G, and therefore it follows from Theorem 3.3 that v must vanish at some point of $\bar{G}$. This is a contradiction and the proof is complete. □

Theorem 3.12 (Sturmian comparison theorem)**.** *Assume that $\sum_{k=1}^{m} \sqrt{A_k(x)}$ is positive definite in G. If there is a nontrivial solution $u \in D_\ell(G)$ of $\ell(u) = 0$ such that $u = 0$ on ∂G and*

$$\tilde{V}_G[u] := \int_G \Bigg\{ (\nabla u)^T \cdot \left(\sum_{k=1}^{m} a_k(x) \right) (\nabla u) - \sum_{k=1}^{m} \left| \sqrt{A_k(x)} \nabla u - u \left(\sqrt{A_k(x)} \right)^{-1} B(x) \right|^{\alpha+1} - |b(x)| |\nabla u|^2 + C_2(x) |u|^{\alpha+1} - (|b(x)| + c(x))\, u^2 \Bigg\} dx \geq 0,$$

then every solution $v \in D_P(G)$ of $P(v) = f(x)$ satisfying $vf(x) \leq 0$ in G must vanish at some point of $\bar{G}$.

Proof. Let us apply the same argument that we used in the proof of Theorem 3.3. We get

$$\tilde{M}_G[u] = -\tilde{V}_G[u] + \int_G \left\{ (\nabla u)^T \cdot \left(\sum_{k=1}^{m} a_k(x) \right) (\nabla u) - |b(x)| |\nabla u|^2 - (c(x) + |b(x)|) u^2 \right\} dx. \quad (3.20)$$

If we integrate both sides of inequality (2.8) over G and use the divergence theorem, we get $\tilde{M}_G[u] \leq 0$. Thus Theorem 3.12 follows from Theorem 3.9. □

Remark 3.13. If we take $m = \alpha = 1$, $C(x) \equiv 0$, $D_1(x) \equiv C(x)$, $D_i(x) \equiv E_j(x) \equiv 0$, $(i = 2, \ldots, \ell;\ j = 1, 2, \ldots, m)$ in $\tilde{V}_G[u]$ in Theorem 3.12, we observe Theorem 2.6 in [20]. Furthermore, by omitting the damping terms, that is $b(x) \equiv B(x) \equiv 0$ and $m = \alpha = 1$, Theorem 3.12 becomes Theorem 4 in [17]. If we take $\alpha = 1$, $b(x) \equiv B(x) \equiv 0$ and $D_i(x) \equiv 0$, $(i = 1, \ldots, \ell)$ and $E_j(x) \equiv 0$, $(j = 1, 2, \ldots, m)$ in $\tilde{V}_G[u]$ in the Theorem 3.12, we see this theorem becomes Theorem 2.3 in the case $\alpha = 1$ in [38].

By using the Picone-type inequalities that are obtained in Theorems 2.6 and 2.7, and by using the same idea in [41], the following theorems can be established.

Theorem 3.14. *Let $k_0 > 0$ be a constant. Assume $\sum_{k=1}^{m} \sqrt{A_k(x)}$ is positive definite in G. If there is a nontrivial function $u \in C^1(\bar{G}, R)$ such that $u = 0$ on ∂G and*

$$\tilde{\tilde{M}}_G[u] := \int_G \sum_{k=1}^{m} \left\{ \left| \sqrt{A_k(x)} \nabla u - u \left(\sqrt{A_k(x)} \right)^{-1} B(x) \right|^{\alpha+1} - (C_1(x) - k_0^{-\alpha}) |u|^{\alpha+1} \right\} dx \leq 0,$$

then for every solution $v \in D_{P_\alpha(G)}$ of $P(v) = f(x)$, either v has a zero on $\bar{G}$ or $|v(x_0)| < k_0$ for some $x_0 \in G$.

Theorem 3.15 (Sturmian comparison theorem). *Assume that $\sum_{k=1}^{m}\sqrt{A_k(x)}$ is positive definite in G. If there is a nontrivial solution $u \in D_\ell(G)$ of $\ell(u)=0$ such that $u=0$ on ∂G and*

$$\tilde{\tilde{V}}_G[u] := \int_G \Big\{ (\nabla u)^T \cdot \Big(\sum_{k=1}^{m} a_k(x)\Big)(\nabla u) - \sum_{k=1}^{m}\Big|\sqrt{A_k(x)}\nabla u - u\Big(\sqrt{A_k(x)}\Big)^{-1}B(x)\Big|^{\alpha+1} - |b(x)||\nabla u|^2 + (C_1(x)-k_0^{-\alpha})|u|^{\alpha+1} - (|b(x)|+c(x))\,u^2\Big\}dx \geq 0,$$

then for every solution $v \in D_{P_\alpha(G)}$ of $P(v)=f(x)$, either v has a zero on $\bar{G}$ or $|v(x_0)|<k_0$ for some $x_0 \in G$.

The proofs of the Theorems 3.14 and 3.15 can be given by following the same steps in Theorems 3.9 and 3.12 respectively, hence omitted.

However Theorems 3.14 and 3.15 cannot guarantee a zero in $\bar{G}$, Theorems 3.9 and 3.12 guarantee a zero in $\bar{G}$. But considering the Theorems 3.14 and 3.15, can be used to obtain other results as in [41].

Theorem 3.16. *Let $\sum_{k=1}^{m}\sqrt{A_k(x)}$ be positive definite in G. Assume that G is divided into subdomains G_1 and G_2 by an $(n-1)$-dimensional piecewise smooth hypersurface in such a way that*

$$f(x) \geq 0 \;\text{ in }\; G_1 \;\text{ and }\; f(x) \leq 0 \;\text{ in }\; G_2 \tag{3.21}$$

If there are nontrivial functions $u_k \in C^1(\bar{G}_k, R)$ such that $u_k=0$, on ∂G_k $(k=1,2)$ and $\tilde{M}_G[u_k] \leq 0$ then every solution $v \in D_P(G)$ of $P(v)=f(x)$ has a zero on $\bar{G}$.

Proof. Suppose that $P(v)=f(x)$ has a solution $v \in D_{P_1}(G)$ with no zero on $\bar{G}$. Then either $v<0$ on $\bar{G}$ or $v>0$ on $\bar{G}$. If $v<0$ on $\bar{G}$, then $v<0$ on $\bar{G}_1$, so that $vf(x) \leq 0$ in $\bar{G}_1$. Using Corollary 3.11, we see that no solution of $P(v)=f(x)$ can be negative on $\bar{G}_1$. This contradiction shows that it is impossible to have $v<0$ on $\bar{G}$. In the case where $v>0$ on $\bar{G}$, a similar argument leads us to a contradiction and the proof is complete. □

4 Applications

In this section we will give an oscillation result for the equations $P(v)=0$ and $P(v)=f(x)$ in an unbounded domain $\Omega \subset R^n$ which contains $\{x \in R^n; |x| \geq r_0\}$ for some $r_0>0$ where

$$\Omega_r := \Omega \cap \{x \in R;\; |x|>r\}.$$

Definition 4.1. A bounded domain G with $\bar{G} \subset \Omega$ is said to be a nodal domain for $\ell(u)=0$ in G and $u=0$ on ∂G. The equation $\ell(u)=0$ is called nodally oscillatory in Ω if it has a nodal domain contained in Ω_r for any $r>0$.

It is assumed that matrices $(a_k(x)),(A_k(x)) \in C(\Omega, R^{n\times n})$, $(k=1,2,\ldots,m)$ are symmetric and positive definite in Ω, $b(x),B(x) \in C(\Omega,R^n)$; $c(x),C(x) \in C(\Omega,R)$; $D_i(x)$, $(i=1,2,\ldots,\ell)$, $E_j(x) \in C(\Omega, R^+\bigcup\{0\})$, $(j=1,2,\ldots,m)$ and $f(x) \in C(\Omega,R)$.

Theorem 4.2. *Let $b_i(x)\equiv 0$ in $\ell(u)=0$ and $\alpha=1$ in $P(v)=0$, and assume that*

$$(a_k(x)-A_k(x)) \text{ is positive semidefinite in } \Omega,$$

$$c(x)+\nabla\cdot B(x)+B(x)(A_k(x))^{-1}B^T(x) \leq C_1(x) \text{ in } \Omega.$$

Every solution $v \in D_P(\Omega)$ of

$$\sum_{k=1}^{m} \nabla \cdot \big(A_k(x)\nabla v\big) + 2B(x)\cdot\nabla v + C(x)v + \sum_{i=1}^{\ell} D_i(x)|v|^{\beta_i-1}v + \sum_{j=1}^{m} E_j(x)|v|^{\gamma_j-1}v = 0 \tag{4.1}$$

is oscillatory in Ω if $\ell(u) = 0$ is nodally oscillatory in Ω.

Proof. Since $\ell(u) = 0$ is nodally oscillatory in Ω, there exists a nodal domain $G \subset \Omega_r$ for any $r > 0$, and therefore there is a nontrivial solution u of $\ell(u) = 0$ in G such that $u = 0$ on ∂G. It follows from the hypotheses of the theorem that $V[u]$ defined in Theorem 3.3 is nonnegative. Theorem 3.3 implies that every solution $v \in D_P(\Omega)$ of (4.1) must vanish at some point of $\bar{G}$, that is, v has a zero in Ω_r for any $r > 0$. This implies that v is oscillatory in Ω. □

The following corollary is an immediate result of Theorem 4.2.

Corollary 4.3. *If the elliptic equation*

$$\nabla u + \left(C(x) + \sum_{i=1}^{N} H_1(\beta_i,\ 1\ \gamma_i, D_i(x), E_i(x)) - \nabla\cdot B(x) - |B(x)|^2 \right) u = 0$$

is nodally oscillatory in Ω, then every solution $v \in C^2(\Omega, R)$ of

$$\nabla v + 2B(x)\cdot\nabla v + C(x)v + \sum_{i=1}^{\ell} D_i(x)|v|^{\beta_i-1}v + \sum_{j=1}^{\ell} E_j(x)|v|^{\gamma_j-1}v = 0$$

is oscillatory in Ω.

Note that Corollary 5.3 in [20] can also be given as an application of Theorem 4.2.

Definition 4.4. A function $v \in C(\Omega, R)$ is said to be oscillatory in Ω if v has a zero in Ω_r for any $r > 0$.

Assume that the matrix functions $A_k(x) \in C(\Omega; R^{n\times n})$, $k = 1, 2, \ldots, n$ are symmetric and positive definite in Ω, $C(x) \in C(\Omega, R)$; $D_i(x), E_j(x) \in C(\Omega, R)$, $(i = 1, 2\ldots, \ell;\ j = 1, 2, \ldots, m)$ and $f(x) \in C(\Omega, R)$. The domain $D_P(\Omega)$ of P is defined to be the set of all functions v of class $C^1(\Omega, R)$ with the property $A_k(x)|\sqrt{A_k(x)}\nabla v|^{\alpha-1}\nabla v \in C^1(\Omega, R^n) \cap C(\Omega, R^n)$.

Theorem 4.5. *Assume that for any $r > 0$ there exists a bounded and piecewise smooth domain G with $\bar{G} \subset \Omega_r$ which can be divided into subdomains G_1 and G_2 by an $(n-1)$ dimensional hypersurface in such a way that $f(x) \geq 0$ and G_1 and $f(x) \leq 0$ in G_2. Assume furthermore that $D_i(x) \geq 0$, $(i = 1, 2, \ldots, \ell)$ and $E_j(x) \geq 0$, $(j = 1, 2, \ldots, m)$ in G and that there are nontrivial functions $u_k \in C^1(\bar{G}_k, R)$ such that $u_k = 0$ on ∂G_k and $\tilde{M}_{G_k}[u_k] \leq 0$, $(k = 1, 2)$ where $\tilde{M}_G$ is defined by (3.19). Then every solution $v \in D_P(\Omega)$ of $P(v) = f(x)$ is oscillatory in Ω.*

Proof. We need to apply Theorem 3.16 to make sure that v has a zero in any domain G as mentioned in the hypotheses of Theorem 4.5. Theorem 3.16 implies that every solution of $P(v) = f(x)$ has a zero on $\bar{G} \subset \Omega_r$, that is, v is oscillatory in Ω. □

Example 4.6. Consider the forced quasilinear elliptic equation

$$\frac{\partial}{\partial x_1}\left[\left\{2\left(\frac{\partial v}{\partial x_1}\right)^2 + 5\left(\frac{\partial v}{\partial x_2}\right)^2\right\}\frac{\partial v}{\partial x_1}\right] + \frac{\partial}{\partial x_2}\left[2\left\{\left(\frac{\partial v}{\partial x_1}\right)^2 + 5\left(\frac{\partial v}{\partial x_2}\right)^2\right\}\frac{\partial v}{\partial x_2}\right]$$
$$+ 4\left(2\left(\frac{\partial v}{\partial x_1}\right)^2 + 5\left(\frac{\partial v}{\partial x_2}\right)^2\right)\frac{\partial v}{\partial x_1} + K(\sin x_1 \sin x_2)|v|^{\beta-1}v = \cos x_1 \sin x_2, \quad (x_1, x_2) \in \Omega,$$

where β and K are the constants with $\beta > 3$, $K > 0$, and Ω is an unbounded domain in R^2 containing a horizontal strip such that $[2\pi, \infty) \times [0, \pi] \subset \Omega$. Here $m = n = 2$, $\ell = 1$, $\alpha = 3$, $\beta_1 = \beta$, $A_1(x) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $A_2(x) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 4 \end{smallmatrix}\right)$, it can be easily shown that $\sqrt{A_1(x)} + \sqrt{A_2(x)} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 3 \end{smallmatrix}\right)$ positive definite matrix. $B(x) = (1,0)^T$ $C(x) \equiv 0$, $D_1(x) = K \sin x_1 \sin x_2$, $D_i(x) \equiv 0$, $(i = 2, \ldots, \ell)$, $E_j(x) \equiv 0$, $j = 1, 2, \ldots, m$, $f(x) = \cos x_1 \sin x_2$. For any fixed $j \in N$, we consider the rectangular region,

$$G^j = (2j\pi, (2j+1)\pi) \times (0, \pi),$$

which is divided into two subdomains

$$G_1^j = (2j\pi, (2j + (1/2))\pi) \times (0, \pi),$$
$$G_2^j = ((2j + (1/2))\pi, (2j+1)\pi) \times (0, \pi),$$

by the vertical line $x_1 = (2j + (1/2))\pi$. We observe that $f(x) \geq 0$ in G_1^j and $f(x) \leq 0$ in G_2^j. Letting $u_k = \sin 2x_1 \sin x_2$, $k = 1, 2$, we get $u_k = 0$ on ∂G_k^j $(k = 1, 2)$ and after some computations we obtain

$$\begin{aligned} M_{G_k^j[u_k]} = \int_{G_k^j} \Bigg[& \left[\left(\frac{\partial u_k}{\partial x_1} - u_k \right)^2 + \left(\frac{\partial u_k}{\partial x_2} \right)^2 \right]^2 + \left[\left(\frac{\partial u_k}{\partial x_1} - u_k \right)^2 + \left(2\frac{\partial u_k}{\partial x_2} \right)^2 \right]^2 \\ & + \frac{\beta}{3} \left(\frac{\beta - 3}{3} \right)^{\frac{3-\beta}{\beta}} (K(\sin x_1 \sin x_2))^{\frac{3}{\beta}} |\cos x_1 \sin x_2|^{\frac{\beta-3}{\beta}} u_k^4 \Bigg] dx \\ = {} & \frac{673}{128}\pi^2 - \frac{128}{15} K^{\frac{3}{\beta}} \frac{\beta}{3} \left(\frac{\beta - 3}{3} \right)^{\frac{3-\beta}{\beta}} B\left(\frac{5}{2} + \frac{3}{2\beta}, 3 - \frac{3}{2\beta} \right), \end{aligned}$$

where $B(s, t)$ is the beta function. If we take

$$K \geq \left[\frac{10095}{16384}\pi^2 \left(\frac{\beta}{3} \left(\frac{\beta - 3}{3} \right)^{\frac{3-\beta}{\beta}} B\left(\frac{5}{2} + \frac{3}{2\beta}, 3 - \frac{3}{2\beta} \right) \right)^{-1} \right]^{\frac{\beta}{3}}$$

then $M_{G_k^j}[u_k] \leq 0$ for $k = 1, 2$ and for any $j \in N$. Therefore, Theorem 4.5 implies that every solution v considered forced elliptic equation is oscillatory in Ω for all sufficiently large $K > 0$. For example if we take $\beta = 4$ and $K = 330$, then the above inequality is valid.

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