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Stability radius of second order linear structured differential inclusions

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Abstract. For arbitrary second order square matrices A, B, C; A Hurwitz stable, the minimum positive value R for which the differential inclusion

$$\dot{x} \in F_R(x) := \{(A + B\Delta C)x,\ \Delta \in \mathbb{R}^{2\times 2},\ \|\Delta\| \leq R\}$$

fails to be asymptotically stable is calculated, where $\|\cdot\|$ denotes the operator norm of a matrix.

Keywords: robust stability, stability radius, differential inclusions, ordinary differential equations, qualitative theory, perturbation theory.

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1 Introduction

Let A be a second order stable matrix (it means that all eigenvalues of A have negative real part), and let B, C be second order arbitrary matrices and R a positive real number. For each vector x of the plane we consider the set of vectors

$$F_R(x) := \{(A + B\Delta C)x,\ \Delta \in \mathbb{R}^{2\times 2},\ \|\Delta\| \leq R\}, \tag{1.1}$$

where $\|\cdot\|$ denotes the operator norm of a matrix. The object of investigation in this work is the global asymptotic stability (g.a.s.) of the parameter-dependent differential inclusion

$$\dot{x} \in F_R(x), \tag{1.2}$$

and the main problem considered is the computation of the number

$$R_i(A, B, C) = \inf\{R > 0 : \dot{x} \in F_R(x) \text{ is not g.a.s.}\}. \tag{1.3}$$

The number $R_i(A, B, C)$ is closely related to the robustness of stability of the linear system $\dot{x} = Ax$ under real perturbations of different classes. As in [5] we consider perturbed systems

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of the type

$$
\begin{aligned}
\Sigma_\Delta &: \dot{x}(t) = Ax(t) + B\Delta Cx(t) \\
\Sigma_N &: \dot{x}(t) = Ax(t) + BN(Cx(t)) \\
\Sigma_{\Delta(t)} &: \dot{x}(t) = Ax(t) + B\Delta(t)Cx(t) \\
\Sigma_{N(t)} &: \dot{x}(t) = Ax(t) + BN(Cx(t), t)
\end{aligned} \tag{1.4}
$$

where

- Δ belongs to the class of matrices $\mathbb{R}^{2\times 2}$ provided with the operator norm;
- N belongs to the class $P_n(\mathbb{R})$ of functions $N : \mathbb{R}^2 \to \mathbb{R}^2, N(0) = 0, N$ is differentiable at 0, is locally Lipschitz and there exists $\gamma \geq 0$ such that $\|N(x)\| \leq \gamma\|x\|$ for all $x \in \mathbb{R}^2$ provided with the norm

$$\|N\|_n = \inf\{\gamma > 0;\ \forall x \in \mathbb{R}^2 : \|N(x)\| \leq \gamma\|x\|\};$$

- $\Delta(\cdot)$ belongs to the class $P_t(\mathbb{R})$ of functions of the space $L^\infty(R_+, R^{2\times 2})$ provided with the norm

$$\|\Delta\|_t = \operatorname*{ess\,sup}_{t\in\mathbb{R}_+} \|\Delta(t)\|;$$

- $N(\cdot,\cdot)$ belongs to the class $P_{nt}(\mathbb{R})$ of functions $N(\cdot,\cdot) : \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}^2, N(0,t) = 0$ for all $t \in \mathbb{R}_+, N(x,t)$ is locally Lipschitz in x continuous in t and there exists $\gamma \geq 0$ such that $\|N(x,t)\| \leq \gamma\|x\|$ for all $x \in \mathbb{R}^2, t \in \mathbb{R}_+$ provided with the norm

$$\|N\|_{nt} = \inf\{\gamma > 0; \forall t \in \mathbb{R}_+\ \forall x \in \mathbb{R}^2 : \|N(x,t)\| \leq \gamma\|x\|\}.$$

Following [5] (see [3, 4] also), we define the stability radii of A with respect to the considered perturbations classes

$$
\begin{aligned}
R(A,B,C) &= \inf\{\|\Delta\|;\ \Delta \in \mathbb{R}^{2\times 2},\ \Sigma_\Delta \text{ is not } g.a.s.\} \\
R_n(A,B,C) &= \inf\{\|N\|_n;\ N \in P_n(\mathbb{R}),\ \Sigma_N \text{ is not } g.a.s.\} \\
R_t(A,B,C) &= \inf\{\|\Delta\|_t;\ \Delta \in P_t(\mathbb{R}),\ \Sigma_\Delta \text{ is not } g.a.s.\} \\
R_{nt}(A,B,C) &= \inf\{\|N\|_{nt};\ N \in P_{nt}(\mathbb{R}),\ \Sigma_N \text{ is not } g.a.s.\}
\end{aligned}
$$

For the defined stability radii in [5] it has been shown that for arbitrary triple (A,B,C) of matrices

$$R(\cdot) \geq R_n(\cdot) \geq R_t(\cdot) \geq R_{nt}(\cdot). \tag{1.5}$$

Effective methods for the calculation of the complex stability radius are exposed in [5], and for the real time invariant linear structured stability radius a method is given in [7]. For the real time-varying and nonlinear structured stability radii we do not have general methods, but for the class of positive systems the problem has been solved in [6]. The problem of the calculation of the real linear structured time-varying stability radius of second order systems taken Frobenius norm as the perturbation norm is considered in works [8] and [9].

The main result of this work is a characterization of the number $R_i(A,B,C)$ in terms of the radius $R(A,B,C)$ and a pair of extremal elliptic integrals associated with the differential inclusion (1.1)–(1.2) in case that the differential inclusion has orbits that are spirals in the plane of faces turning around the origin in positive or negative sense. We also prove that:

$$R(\cdot) \geq R_n(\cdot) = R_t(\cdot) = R_{nt}(\cdot) = R_i(\cdot) \tag{1.6}$$

for all triple (A, B, C) of second order matrices, where A is Hurwitz stable.

The organization of the paper is as follows: in Section 2 we give a formula for the computation of $R(A, B, C)$. In Section 3 we enunciate a Filippov's Theorem [1] about the asymptotic stability of differential inclusions, which will help us in the fundamentation of the results. In Section 4 we apply this theorem and obtain conditions for the stability of our differential inclusion (1.1)–(1.2) in terms of the number $R(A, B, C)$ and two elliptic integrals. In Section 5 we prove the relations (1.6) and in Section 7 we give some examples for the applications of the main results of this work. The results of this work are a continuation of the paper [2], where the problem of the calculation of the number $R_i(A)$ is solved when the perturbations of the linear inclusion are unstructured, i.e., the matrices B, C are the second order identity matrix.

2 Computation of $R(A, B, C)$

Lemma 2.1. *Let A, B, C be arbitrary second order matrices and A Hurwitz stable. Then*

$$R(A, B, C) = \inf \left\{ \frac{-\operatorname{tr} A}{s_1 + s_2}, \left(\sigma_1(CA^{-1}B)\right)^{-1} \right\}, \tag{2.1}$$

where $\operatorname{tr}(M)$ *denotes the trace of the matrix M, s_1, s_2 are the singular values of the matrix BC, and $\sigma_1(CA^{-1}B)$ denotes the greatest singular value of the matrix $CA^{-1}B$. (The singular values of a square matrix M are the square roots of the eigenvalues of the symmetric matrix M^*M.)*

Proof. $\lambda^2 - \operatorname{tr}(A + B\Delta C)\lambda + \det(A + B\Delta C)$ is the characteristic polynomial of the matrix $A + B\Delta C$. The roots of this polynomial have negative real parts if and only if $\operatorname{tr}(A + B\Delta C) > 0$ and $\det(A + B\Delta C) > 0$. From this it follows that

$$\begin{aligned} R(A, B, C) &= \inf \{ \|\Delta\| : A + B\Delta C \text{ is not Hurwitz matrix} \} \\ &= \inf \{ \|\Delta\| : \operatorname{tr}(A + B\Delta C) = 0 \text{ or } \det(A + B\Delta C) = 0 \}. \end{aligned} \tag{2.2}$$

Let $BC = U^* \operatorname{diag}(s_1, s_2) V$ the singular value decomposition of the matrix BC, where U and V are orthogonal matrices and denote $\overline{\Delta} = V\Delta U^*$, then we have

$$\begin{aligned} tr(A + B\Delta C) &= \operatorname{tr}(A + BC\Delta) = \operatorname{tr} A + \operatorname{tr}\left(U^* \operatorname{diag}(s_1, s_2) V\Delta\right) \\ &= \operatorname{tr} A + \operatorname{tr}\left(\operatorname{diag}(s_1, s_2) V\Delta U^*\right) \\ &= \operatorname{tr} A + \operatorname{tr}\left(\operatorname{diag}(s_1, s_2)\overline{\Delta}\right). \end{aligned}$$

This equality implies that

$$\inf \{ \|\Delta\| : \operatorname{tr}(A + B\Delta C) = 0 \} = \frac{-\operatorname{tr} A}{s_1 + s_2}. \tag{2.3}$$

A is a Hurwitz matrix, so $\det(A + B\Delta C) = 0 \Leftrightarrow \det(I + A^{-1}B\Delta C) = 0$. This equality is equivalent to the existence of $\mathbf{0} \neq \mathbf{v} \in \mathbb{R}^2$ such that $(I + A^{-1}B\Delta C)\mathbf{v} = \mathbf{0}$, and this last assertion is equivalent to the existence of $\mathbf{0} \neq \mathbf{w} \in \mathbb{R}^2$ such that $(I + CA^{-1}B\Delta)\mathbf{w} = \mathbf{0}$. Then from this fact and making use of the equality $\inf\{\|\Delta\| : \det(I + CA^{-1}B\Delta) = 0\} = (\sigma_1(CA^{-1}B))^{-1}$ which is a direct consequence of Lemma 1 of [7] we obtain

$$\inf\{\|\Delta\| : \det(A + B\Delta C) = 0\} = \left(\sigma_1(CA^{-1}B)\right)^{-1}. \tag{2.4}$$

The assertion of the lemma follows from (2.2), (2.3) and (2.4). □

3 Filippov's theorem

In this section we enunciate Filippov's Theorem [1], which will be the fundamental tool in the analysis of the stability of the differential inclusion (1.1)–(1.2). Let:

$$\dot{x} \in F(x), \qquad x \in \mathbb{R}^2 \tag{3.1}$$

be a differential inclusion which satisfies the following properties

(i) for all x the set $F(x)$ is nonempty, bounded, closed and convex;

(ii) $F(x)$ is upper semi-continuous with respect to the set's inclusion as function of x;

(iii) $F(cx) = cF(x)$ for all x and $c \geq 0$.

Let ρ, φ be the polar coordinates of the point $x = (x_1, x_2)$. Then we can write $F(x) = \rho\widetilde{F}(\varphi)$ and the differential inclusion (1.1)–(1.2) takes the form

$$\frac{\dot{\rho}(t)}{\rho} = y_1(t)$$
$$\dot{\varphi}(t) = y_2(t),$$

where $(y_1(t), y_2(t)) \in \widetilde{F}(\varphi(t))$.

We will use the notations

$$\widetilde{F}^+(\varphi) := \{(y_1, y_2) \in \widetilde{F}(\varphi) : y_2 > 0\};$$
$$\widetilde{F}^-(\varphi) := \{(y_1, y_2) \in \widetilde{F}(\varphi) : y_2 < 0\}.$$

For φ such that $\widetilde{F}^+(\varphi) \neq \phi$ (resp. $\widetilde{F}^-(\varphi) \neq \phi$) we put

$$K^+(\varphi) := \sup_{(y_1,y_2)\in\widetilde{F}^+(\varphi)} \frac{y_1}{\|y_2\|} \qquad \left(\text{resp. } K^-(\varphi) := \sup_{(y_1,y_2)\in\widetilde{F}^-(\varphi)} \frac{y_1}{\|y_2\|}\right). \tag{3.2}$$

Theorem 3.1 (Filippov's Theorem). *The differential inclusion (3.1) satisfying the conditions (i)–(iii) is asymptotically stable if and only if for all $x \neq 0$ the set $F(x)$ does not have common points with the ray cx, $0 \leq c < +\infty$ and when the set $\widetilde{F}^+(\varphi)$ (resp. $\widetilde{F}^-(\varphi)$) for almost all φ is nonempty, the inequality*

$$\int_0^{2\pi} K^+(\varphi)d\varphi < 0 \qquad \left(\textit{resp. } \int_0^{2\pi} K^+(\varphi)d\varphi < 0\right)$$

holds.

4 Application of Filippov's theorem

From the definition (1.1) we have that for all $R > 0$ the set $F_R(x)$ for all $x \in \mathbb{R}^2$ is non empty, bounded, closed and convex set of the plane. So the differential inclusion (1.1)–(1.2) satisfies properties (i)–(iii) and Filippov's Theorem is applicable.

The following lemma allows us to write the set $F_R(x)$ in the form we will use in the application of Filippov's theorem.

Lemma 4.1. *For all $R > 0$ and $x \in \mathbb{R}^2$ it holds that*

$$\left\{\Delta Cx,\ \Delta \in \mathbb{R}^{2\times 2},\ \|\Delta\| \le R\right\} = \left\{r\|Cx\| \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} : (r,\theta) \in [0,R]\times[0,2\pi)\right\}.$$

Proof. Let $z = \Delta Cx$, $\Delta \in \mathbb{R}^{2\times 2}, \|\Delta\| \le R$, then $\|z\| = \|\Delta Cx\| \le R\|Cx\|$. Thus there exist r: $0 \le r \le R$, and $\theta \in [0, 2\pi)$ such that $z = r\|Cx\| \left(\begin{smallmatrix}\cos\theta\\ \sin\theta\end{smallmatrix}\right)$ so that we obtained:

$$z \in \left\{r\|Cx\| \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} : 0 \le r \le R;\ 0 \le \theta < 2\pi\right\}.$$

Let now $z = r\|Cx\| \left(\begin{smallmatrix}\cos\theta\\ \sin\theta\end{smallmatrix}\right)$, $0 \le r \le R$; $0 \le \theta < 2\pi$ then there exists $\widetilde{\Delta} \in \mathbb{R}^{2\times 2}$ such that $\widetilde{\Delta}Cx = r\|Cx\| \left(\begin{smallmatrix}\cos\theta\\ \sin\theta\end{smallmatrix}\right)$, so $\|\widetilde{\Delta}Cx\| \le R\|Cx\|$ and from the well known theorem of Hahn–Banach, $\widetilde{\Delta} \in \mathbb{R}^{2\times 2}$ may be chosen such that $\|\widetilde{\Delta}\| \le R$. So we have:

$$z = r\|Cx\| \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} \in \left\{\Delta Cx,\ \Delta \in \mathbb{R}^{2\times 2},\ \|\Delta\| \le R\right\},$$

and the lemma is proved. $\square$

As a direct consequence of this lemma the inclusion (1.1)–(1.2) can be written in the form

$$\dot{x} \in \left\{Ax + r\|Cx\|B \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} : 0 \le r \le R;\ 0 \le \theta < 2\pi\right\} = F_R(x). \tag{4.1}$$

Lemma 4.2. *Let be $A \in \mathbb{R}^{2\times 2}$ a Hurwitz stable matrix, and B, C arbitrary matrices of $\mathbb{R}^{2\times 2}$. Then*

$$R_{nt}(\cdot) \ge R_i(\cdot). \tag{4.2}$$

Proof. Let $N(x,t) \in P_{nt}(\mathbb{R}), \|N(x,t)\|_{nt} = R_0$. Then for all $t \in \mathbb{R}$, $x \in \mathbb{R}^2$, $N(Cx,t) = r(t)\|Cx\| \left(\begin{smallmatrix}\cos\theta(t)\\ \sin\theta(t)\end{smallmatrix}\right)$ for suitable $0 \le r(t) \le R_0$, $0 \le \theta(t) < 2\pi$, and so all solutions of the perturbed system $\dot{x} = Ax + BN(Cx,t)$ is a solution of the differential inclusion (4.1) with $R = R_0$, from what follows the inequality (4.2) $\square$

So from this lemma and (1.5) we can restrict the analysis of the asymptotic stability of the differential inclusion (1.1)–(1.2) for $R < R(A, B, C)$.

Changing in (4.1) to polar coordinates:

$$\begin{gathered}\frac{\dot{\rho}(t)}{\rho} = y_1(t)\\ \dot{\varphi}(t) = y_2(t),\\ (y_1(t), y_2(t)) \in \widetilde{F}_R(\varphi),\end{gathered} \tag{4.3}$$

$$
\begin{aligned}
\widetilde{F}_R(\varphi) &:= \{(y_1(\varphi,\theta,r),\ y_2(\varphi,\theta,r)),\ 0 \le r \le R;\ 0 \le \theta < 2\pi\} \\
y_1(\varphi,\theta,r) &:= f_1(\varphi) + r\sqrt{g(\varphi)}\,(p_1(\varphi)\cos(\theta) + p_2(\varphi)\sin(\theta)) \\
y_2(\varphi,\theta,r) &:= f_2(\varphi) + r\sqrt{g(\varphi)}\,(p_3(\varphi)\cos(\theta) + p_4(\varphi)\sin(\theta)),
\end{aligned}
$$

where:

$$
\begin{aligned}
f_1(\varphi) &:= a_{11}\cos^2(\varphi) + (a_{12}+a_{21})\sin(\varphi)\cos(\varphi) + a_{22}\sin^2(\varphi), \\
f_2(\varphi) &:= a_{21}\cos^2(\varphi) + (a_{22}-a_{11})\sin(\varphi)\cos(\varphi) - a_{12}\sin^2(\varphi), \\
g(\varphi) &:= (c_{11}\cos(\varphi) + c_{12}\sin(\varphi))^2 + (c_{21}\cos(\varphi) + c_{22}\sin(\varphi))^2,
\end{aligned}
$$

$$
\begin{aligned}
p_1(\varphi) &:= b_{11}\cos(\varphi) + b_{21}\sin(\varphi), \\
p_2(\varphi) &:= b_{12}\cos(\varphi) + b_{22}\sin(\varphi), \\
p_3(\varphi) &:= -b_{11}\sin(\varphi) + b_{21}\cos(\varphi), \\
p_4(\varphi) &:= -b_{12}\sin(\varphi) + b_{22}\cos(\varphi).
\end{aligned}
$$

In the following we make use of the notations:

$$
h_1(\varphi) := p_3(\varphi)f_1(\varphi) - p_1(\varphi)f_2(\varphi) \tag{4.4}
$$

$$
h_2(\varphi) := p_2(\varphi)f_2(\varphi) - p_4(\varphi)f_1(\varphi) \tag{4.5}
$$

and easily we can verify that:

$$
\begin{aligned}
h_1(\varphi) &= d_1\cos(\varphi) + d_2\sin(\varphi), \\
h_2(\varphi) &= d_3\cos(\varphi) + d_4\sin(\varphi),
\end{aligned}
$$

where:

$$
\begin{aligned}
d_1 &:= b_{21}a_{11} - b_{11}a_{21}, \\
d_2 &:= b_{21}a_{12} - b_{11}a_{22}, \\
d_3 &:= b_{12}a_{21} - b_{22}a_{11}, \\
d_4 &:= b_{12}a_{22} - b_{22}a_{12},
\end{aligned}
$$

For the corresponding sets $\widetilde{F}^+(\varphi)$ and $\widetilde{F}^-(\varphi)$ that appear in the Filippov's theorem we have:

$$
\begin{aligned}
\widetilde{F}_R^+(\varphi) &= \{(y_1,y_2) \in \widetilde{F}_R(\varphi) : y_2 > 0\}, \\
\widetilde{F}_R^-(\varphi) &= \{(y_1,y_2) \in \widetilde{F}_R(\varphi) : y_2 < 0\}.
\end{aligned}
$$

Denote:

$$
\begin{aligned}
R^+(A,B,C) &:= \begin{cases} 0, & \text{if } f_2(\varphi) > 0 \quad \forall \varphi \in [0,2\pi), \\ \sup\limits_{f_2(\varphi)<0} \dfrac{-f_2(\varphi)}{\sqrt{p_3^2+p_4^2}\sqrt{g(\varphi)}} & \text{otherwise.} \end{cases} \\
R^-(A,B,C) &:= \begin{cases} 0, & \text{if } f_2(\varphi) < 0 \quad \forall \varphi \in [0,2\pi), \\ \sup\limits_{f_2(\varphi)>0} \dfrac{f_2(\varphi)}{\sqrt{p_3^2+p_4^2}\sqrt{g(\varphi)}} & \text{otherwise.} \end{cases}
\end{aligned} \tag{4.6}
$$

Lemma 4.3. *Let $R < R(A,B,C)$. Then*

a) The set $F_R(x)$ does not have common points with the ray $cx, 0 \le c < +\infty$ for all $x \neq 0$.

b) The set $\widetilde{F}_R^+(\varphi) \neq \phi$ for all $\varphi \in [0,2\pi)$ if and only if $R \in (R^+(\cdot), R(\cdot))$.

c) The set $\widetilde{F}_R^-(\varphi) \neq \phi$ for all $\varphi \in [0,2\pi)$ if and only if $R \in (R^-(\cdot), R(\cdot))$.

Proof. a) The set $F_R(x) := \{(A+B\Delta C)x, \Delta \in \mathbb{R}^{2\times 2}, \|\Delta\| \le R\}$, with $R < R(A,B,C)$ does not have common points with the ray $cx, 0 \le c < +\infty$ for all $x \neq 0$ because the matrix $A+B\Delta C$ is stable for $\|\Delta\| < R(A,B,C)$.

b) $\widetilde{F}_R^+(\varphi) \neq \phi$ for all $\varphi \in [0,2\pi)$ if and only if for all $\varphi \in [0,2\pi)$ there is $\theta \in [0,2\pi)$ such that $f_2(\varphi) + r\sqrt{g(\varphi)}\,(p_3(\varphi)\cos(\theta) + p_4(\varphi)\sin(\theta)) > 0$. This is true if and only if for all $\varphi \in [0,2\pi)$, $f_2(\varphi) + r\sqrt{g(\varphi)}\sqrt{p_3^2+p_4^2} > 0$, which, according to definitions (4.6), is equivalent to the assertion b) of this lemma.

c) $\widetilde{F}_R^-(\varphi) \neq \phi$ for all $\varphi \in [0,2\pi)$ if and only if for all $\varphi \in [0,2\pi)$ there is $\theta \in [0,2\pi)$ such that $f_2(\varphi) + r\sqrt{g(\varphi)}\,(p_3(\varphi)\cos(\theta) + p_4(\varphi)\sin(\theta)) < 0$. This is true if and only if for all $\varphi \in [0,2\pi)$, $f_2(\varphi) - r\sqrt{g(\varphi)}\sqrt{p_3^2+p_4^2} < 0$, which, according to definitions (4.6), is equivalent to the assertion c) of this lemma. □

In what follows with the aim of shorten the expressions for the functions $f_i,\ i=1,2$, g and $p_i,\ i=1,2,3,4$ we omit the φ argument.

We denote:

$$K(\theta,\varphi,r) := \frac{f_1 + r\sqrt{g}\,(p_1\cos(\theta) + p_2\sin(\theta))}{f_2 + r\sqrt{g}\,(p_3\cos(\theta) + p_4\sin(\theta))}, \tag{4.7}$$

then for $R \in (R^+(A,B,C), R(A,B,C))$ the function $K^+(\varphi)$ that appears in the Filippov's theorem can be written as

$$K_R^+(\varphi) = \sup_{(r,\theta)\in[0,R]\times[0,2\pi)} \{K(\theta,\varphi,r) : f_2 + r\sqrt{g}\,(p_3\cos(\theta) + p_4\sin(\theta)) > 0\}. \tag{4.8}$$

Similarly for $R \in (R^-(A,B,C), R(A,B,C))$ the function $K^-(\varphi)$ can be written as

$$K_R^-(\varphi) = \sup_{(r,\theta)\in[0,R]\times[0,2\pi)} \{-K(\theta,\varphi,r) : f_2 + r\sqrt{g}\,(p_3\cos(\theta) + p_4\sin(\theta)) < 0\}. \tag{4.9}$$

Lemma 4.4. *a) For $R \in (R^+(A,B,C), R(A,B,C))$ we have*

$$K_R^+(\varphi) = \frac{f_1\sqrt{h_1^2+h_2^2-R^2\mu^2 g} - R\sqrt{g}\,(p_1h_1 - p_2h_2)}{f_2\sqrt{h_1^2+h_2^2-R^2\mu^2 g} - R\sqrt{g}\,(p_3h_1 - p_4h_2)}. \tag{4.10}$$

b) For $R \in (R^-(A,B,C), R(A,B,C))$ we have

$$K_R^-(\varphi) = \frac{f_1\sqrt{h_1^2+h_2^2-R^2\mu^2 g} + R\sqrt{g}\,(p_1h_1 - p_2h_2)}{-f_2\sqrt{h_1^2+h_2^2-R^2\mu^2 g} - R\sqrt{g}\,(p_3h_1 - p_4h_2)}. \tag{4.11}$$

Proof. First for arbitrary $R \in (R^+(A,B,C), R(A,B,C))$ we prove (4.10). Let $\varphi \in [0,2\pi)$, $r \in [0,R]$, and $\theta_0 \in [0,2\pi)$ be such that $y_2(\theta_0,\varphi,r) = 0$. Then $y_1(\theta_0,\varphi,r) < 0$ and so the limit of $K(\theta,\varphi,r)$ for $\theta \to \theta_0$ and $y_2(\theta,\varphi,r) > 0$ is $-\infty$ and therefore for the calculation of the supremum in (4.8) we can consider only points in the interior of the set for $y_2(\theta,\varphi,r) > 0$. So the supremum is taken for a value θ for which the partial derivative of $K(\theta,\varphi,r)$ with respect to θ is zero. From this condition using notations (4.4)–(4.5) and after simplifications we obtain

$$h_1\sin(\theta) + h_2\cos(\theta) + \mu\sqrt{g}\,r = 0,$$

and solving this equation for $\sin(\theta)$ and $\cos(\theta)$ we obtain

$$\cos(\theta) = \frac{-r\mu\sqrt{g}h_2 \pm h_1\sqrt{h_1^2+h_2^2-r^2\mu^2 g}}{h_1^2+h_2^2}, \tag{4.12}$$

$$\sin(\theta) = \frac{-r\mu\sqrt{g}h_1 \mp h_2\sqrt{h_1^2+h_2^2-r^2\mu^2 g}}{h_1^2+h_2^2}. \tag{4.13}$$

Substituting in the expression (4.7) of $K(\theta,\varphi,r)$ we obtain

$$K(\varphi,r) = \frac{f_1\sqrt{h_1^2+h_2^2-r^2\mu^2 g} \pm r\sqrt{g}(p_1h_1-p_2h_2)}{f_2\sqrt{h_1^2+h_2^2-r^2\mu^2 g} \pm r\sqrt{g}(p_3h_1-p_4h_2)}. \tag{4.14}$$

and taken into account that

$$\frac{\delta}{\delta\alpha}\left(\frac{f_1\sqrt{h_1^2+h_2^2-r^2\mu^2 g} + \alpha r\sqrt{g}(p_1h_1-p_2h_2)}{f_2\sqrt{h_1^2+h_2^2-r^2\mu^2 g} + \alpha r\sqrt{g}(p_3h_1-p_4h_2)}\right)$$
$$= \frac{(-h_1^2-h_2^2)r\sqrt{g}\sqrt{h_1^2+h_2^2-r^2\mu^2 g}}{(f_2\sqrt{h_1^2+h_2^2-r^2\mu^2 g} + \alpha r\sqrt{g}(p_3h_1-p_4h_2))^2} < 0$$

we conclude that the maximum value of $K(\varphi,r)$ according to (4.14) is given by the expression (4.10). So we have proved the assertion a) of the lemma. Assertion b) follows from (4.9) and the results obtained in the proof of part a). □

Theorem 4.5. *The differential inclusion (4.1) depending on the parameter R is asymptotically stable if and only if the following conditions hold:*

i) $R \in [0, R(A,B,C))$;

ii) when $R \in (R^+(A,B,C), R(A,B,C))$
(resp. $R \in (R^-(A,B,C), R(A,B,C))$)

$$I^+(R) := \int_0^{2\pi} K_R^+(\varphi)d\varphi < 0, \tag{4.15}$$

$$\left(\text{resp. } I^-(R) := \int_0^{2\pi} K_R^-(\varphi)d\varphi < 0\right) \tag{4.16}$$

where the functions $K_R^+(\varphi)$ and $K_R^-(\varphi)$ are defined by the expressions (4.10) and (4.11).

Proof. The assertion of the theorem follows directly from Filippov's Theorem and Lemmas 4.3 and 4.4. □

5 Remarks to the Theorem 4.5

Remark 5.1. Theorem 4.5 gives a method for the calculation of the number $R_i(A,B,C)$ for arbitrary triples of matrices $(A,B,C) \in \mathbb{R}^{2\times 2}$, where A is Hurwitz stable. We have formulas (2.1), (4.14), (4.10) and (4.11) for the computation of the numbers $R(A,B,C)$, $R^+(A,B,C)$, $R^-(A,B,C)$ and the functions $K_R^+(\varphi)$, $K_R^-(\varphi)$. The condition about the sign of integrals $I^+(R)$ and $I^-(R)$ must be checked only when $R < R^+(A,B,C)$, respectively $R < R^-(A,B,C)$. Note that from expressions (4.8), (4.9) we have that the integrals $I^+(R)$, $I^-(R)$ are monotone increasing functions of R, and so the value of R for which the integrals are equal to zero can be obtained using bisection method. Note also that $R_i(A,B,C) = R(A,B,C)$ if and only if $I^+(R(A,B,C)) \leq 0$ in the case $R^+(A,B,C) < R(A,B,C)$ and $I^-(R(A,B,C)) \leq 0$ in the case $R^-(A,B,C) < R(A,B,C)$.

Remark 5.2. The differential inclusion (4.3) can be written as

$$\frac{1}{\rho}\frac{d\rho}{d\varphi} \in \{K(\theta,\varphi,r) : (r,\theta) \in [0,R] \times [0,2\pi)\},$$

where $K(\theta,\varphi,r)$ is given by (4.7). Then for $R \in (R^+(A,B,C), R(A,B,C))$ the equation

$$\frac{1}{\rho}\frac{d\rho}{d\varphi} = K_R{}^+(\varphi),$$

where $K_R{}^+(\varphi)$ is given by (4.10), is the equation in polar coordinates corresponding to the differential system obtained from inclusion (4.1), after the substitution of $\cos(\theta)$ and $\sin(\theta)$ by (4.12) and (4.13), respectively. This last system has the form

$$\dot{x} = Ax + R\|Cx\|B \begin{pmatrix} \frac{-R\mu\sqrt{G}H_2 - H_1\sqrt{H_1^2+H_2^2-R^2\mu^2 G}}{H_1^2+H_2^2} \\ \frac{-R\mu\sqrt{G}H_1 + H_2\sqrt{H_1^2+H_2^2-R^2\mu^2 G}}{H_1^2+H_2^2} \end{pmatrix}, \tag{5.1}$$

where

$$H_1(x) := d_1x_1 + d_2x_2, \qquad H_2(x) := d_3x_1 + d_4x_2,$$
$$G(x) := (c_{11}x_1 + c_{12}x_2)^2 + (c_{21}x_1 + c_{22}x_2)^2.$$

Put

$$N_R^+(x) = R\|x\| \begin{pmatrix} \frac{-R\mu\|x\|H_2 - H_1\sqrt{H_1^2+H_2^2-R^2\mu^2\|x\|^2}}{H_1^2+H_2^2} \\ \frac{-R\mu\|x\|H_1 + H_2\sqrt{H_1^2+H_2^2-R^2\mu^2\|x\|^2}}{H_1^2+H_2^2} \end{pmatrix}. \tag{5.2}$$

Then the function $N_R^+(x)$ belongs to the class $P_n(\mathbb{R})$ defined in the introduction of this work and has norm which is equal to R. Furthermore, the differential system (5.1) can be written as

$$\dot{x} = Ax + R\|Cx\|BN_R^+(x).$$

We can conclude that for $R \in (R^+(A,B,C), R(A,B,C))$ the system (5.1), which we will name the positive extremal system of the differential inclusion (1.1)–(1.2) is the perturbation of the nominal linear system $\dot{x} = Ax$ with the nonlinear perturbation $N_R^+(x)$ of the class $P_n(\mathbb{R})$ which has norm equal to R. Furthermore, the trajectories of this system are spirals which turn around the origin in positive sense and the value of the integral $I^+(R)$ is the Lyapunov exponent of the solutions of this system (note that the homogenity of the system and the rotations of the solutions around the origin implies that all solution of the system have the same Lyapunov exponent). So the condition $I^+(R) < 0$ is true if and only if the system (5.1) is asymptotically stable.

Similarly for $R \in (R^-(A,B,C), R(A,B,C))$ the differential inclusion (1.1)–(1.2) has a negative extremal system

$$\dot{x} = Ax + R\|Cx\|B \begin{pmatrix} \frac{-R\mu\sqrt{G}H_2 + H_1\sqrt{H_1^2+H_2^2-R^2\mu^2G}}{H_1^2+H_2^2} \\ \frac{-R\mu\sqrt{G}H_1 - H_2\sqrt{H_1^2+H_2^2-R^2\mu^2G}}{H_1^2+H_2^2} \end{pmatrix}. \tag{5.3}$$

This system can be written as

$$\dot{x} = Ax + R\|Cx\|BN_R^-(x),$$

where

$$N_R^-(x) = R\|x\| \begin{pmatrix} \frac{-R\mu\|x\|H_2 + H_1\sqrt{H_1^2+H_2^2-R^2\mu^2\|x\|^2}}{H_1^2+H_2^2} \\ \frac{-R\mu\|x\|H_1 - H_2\sqrt{H_1^2+H_2^2-R^2\mu^2\|x\|^2}}{H_1^2+H_2^2} \end{pmatrix} \tag{5.4}$$

is a perturbation of the class $P_n(\mathbb{R})$ which has norm equal to R. Furthermore the trajectories of this system are spirals which turn around the origin in negative sense and the value of the integral $I^-(R)$ is the Lyapunov exponent of the solutions of this system (note that the homogenity of the system and the rotations of the solutions around the origin implies that all solution of the system have the same Lyapunov exponent). So the condition $I^-(R) < 0$ is true if and only if the system (5.3) is asymptotically stable.

Lemma 5.3. *Let be $A \in \mathbb{R}^{2\times 2}$ a stable matrix, and B, C arbitrary matrices of $\mathbb{R}^{2\times 2}$. Then*

$$R(\cdot) \geq R_n(\cdot) = R_t(\cdot) = R_{nt}(\cdot) = R_i(\cdot). \tag{5.5}$$

Proof. From Lemma 4.2 we have

$$R_{nt}(A,B,C) \geq R_i(A,B,C). \tag{5.6}$$

In the case $R_i(A,B,C) = R(A,B,C)$ from the inequalities (1.5) and (5.6) it follows that all the considered stability radii are equals and then the assertion of the lemma is true. In the case $R_i(A,B,C) < R(A,B,C)$ from Remark 5.2 of Theorem 4.5 there exists $N_{R_i(A,B,C)}(x)$ nonlinear perturbation of the class $P_n(\mathbb{R})$ and norm $R_i(A,B,C)$ such that the perturbed system $\dot{x} = Ax + BN_{R_i(A,B,C)}(Cx)$ is not g.a.s., so $R_n(A,B,C) \leq R_i(A,B,C)$, and from that and (1.5), (5.6) the assertion of the lemma follows. □

6 Algorithm

Given the second order matrices A, B, C; where A is Hurwitz stable:

1. calculate the numbers: $R(\cdot), R^+(), R^-(\cdot)$;
2. if $R^+(\cdot) \geq R(\cdot)$, or $I^+(R(\cdot)) \leq 0$, put $R_i^+(\cdot) = R(\cdot)$; in other case put $R_i^+(\cdot) = R$, where R is such that $I^+(R) = 0$ (R can be computed using bisection method);

3. if $R^{-}(\cdot) \geq R(.)$, or $I^{-}(R(\cdot)) \leq 0$, put $R_i^{-}(\cdot) = R(\cdot)$; in other case put $R_i^{-}(\cdot) = R$, where R is such that $I^{-}(R) = 0$ (R can be computed using bisection method);
4. put $R_i(\cdot) = \min\left\{R_i^{+}(\cdot), R_i^{-}(\cdot)\right\}$.

7 Examples

In this section we give applications of the main results of this work to the calculation of the stability radius $R_i(A)$.

Example 7.1. In this example $\max\{R^{+}(A,B,C),\ R^{-}(A,B,C)\} < R(A,B,C)$ and $R_i(A,B,C) < R(A,B,C)$. Let

$$A = \begin{bmatrix} -110 & -49.5 \\ 90.15 & -109 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.65 & 0.059 \\ 0.007 & 0.6 \end{bmatrix}, \qquad C = \begin{bmatrix} 0.67 & 0.071 \\ 0.00054 & 0.64 \end{bmatrix}.$$

Then $R(A,B,C) = 221.321$, $R^{+}(A,B,C) = 0$, $R^{-}(A,B,C) = 176.397$, and $I^{-}(R(A,B,C)) < 0$, $I^{+}(R(A,B,C)) > 0$, so from the algorithm exposed above we have that $R_i(A,B,C) < R(A,B,C)$ and $R_i(A,B,C)$ is the root of the equation: $I^{+}(R) = 0$. Using *Mathematica*, we calculate the integral $I^{+}(218) = -0.0148651 < 0$ and $I^{+}(218.5) = 0.0173011 > 0$ from what follows that $R_i(A,B,C) \in (218, 218.5)$. Finally using bisection method we obtained:

$$I^{+}(218.21875) = -0.000921 < 0, \qquad I^{+}(218.234375) = 0.0000815 > 0$$

and approximately we can take $R_i(A,B,C) = 218.21875$.

Example 7.2. In this example $\max\{R^{+}(A,B,C), R^{-}(A,B,C)\} < R(A,B,C)$ and $R_i(A,B,C) = R(A,B,C)$. Let

$$A = \begin{bmatrix} -218 & -9 \\ 91 & -220 \end{bmatrix}, \qquad B = \begin{bmatrix} 1.1 & 0.6 \\ 0 & 1.02 \end{bmatrix}, \qquad C = \begin{bmatrix} 0.8 & 0.1 \\ 0.002 & 0.9 \end{bmatrix}.$$

Then $R(A,B,C) = 144.352$, $R^{+}(A,B,C) = 0$, $R^{-}(A,B,C) = 116.889$, and $I^{-}(R(A,B,C)) < 0$, $I^{+}(R(A,B,C)) < 0$, so from the algorithm exposed above we have that $R_i(A,B,C) = R(A,B,C) = 144.352$.

Example 7.3. In this example $R^{+}(A,B,C) > R(A,B,C)$, $R^{-}(A,B,C) < R(A,B,C)$ and $R_i(A,B,C) = R(A,B,C)$. Let

$$A = \begin{bmatrix} -6 & 6 \\ -4 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1.1 & 0.012 \\ 0.021 & 1.13 \end{bmatrix}, \qquad C = \begin{bmatrix} 1.02 & 0.21 \\ 0.12 & 0.95 \end{bmatrix}.$$

Then $R(A,B,C) = 0.989071$, $R^{+}(A,B,C) = 9.89183$, $R^{-}(A,B,C) = 0$, and $I^{-}(R(A,B,C)) < 0$, so from the algorithm exposed above we have that $R_i(A,B,C) = R(A,B,C) = 0.989071$.

Example 7.4. In this example $R^{+}(A,B,C) > R(A,B,C)$, $R^{-}(A,B,C) < R(A,B,C)$ and $R_i(A,B,C) < R(A,B,C)$. Let

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -0.5 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Then $R(A,B,C) = 1$, $R^{+}(A,B,C) = \infty$, $R^{-}(A,B,C) = 0$, and $I^{-}(1) > 0$, so from the algorithm exposed above we have that $R_i(A,B,C) < 1$ and $R_i(A,B,C)$ is the root of the equation:

$I^-(R) = 0$. Using *Mathematica*, we calculate the integral $I^-(0.75) = 0.0294657 > 0$ and $I^-(0.7) = -0.399255 < 0$ from what follows that $R_i(A) \in (0.7, 0.75)$. Finally using bisection method we obtained:

$$I^-(0.7292) = -0.0001334 < 0, \qquad I^-(0.729688) = 0.000548289 > 0,$$

and approximately we can take $R_i(A) = 0.7292$.

Example 7.5. In this example $\min\{R^+(A,B,C), R^-(A,B,C)\} > R(A,B,C)$ and $R_i(A,B,C) = R(A,B,C)$. Let

$$A = \begin{bmatrix} -9 & 6 \\ -4 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1.1 & 0.012 \\ 0.021 & 1.13 \end{bmatrix}, \qquad C = \begin{bmatrix} 1.02 & 0.21 \\ 0.12 & 0.95 \end{bmatrix}.$$

Then the calculations give $R(A,B,C) = 0.407454$, $R^+(A,B,C) = 11.4806$, $R^-(A,B,C) = 0.463946$, so from the algorithm exposed above we do not need to calculate the integrals and we have that $R_i(A,B,C) = R(A,B,C) = 0.407454$.

8 Conclusion

In this paper we have solved the problem of the computation of the number $R_i(A,B,C)$. We have characterize the triples of second order matrices (A,B,C) for which the equality $R_i(A,B,C) = R(A,B,C)$ holds. In the case when this numbers are not equal, the results allow with arbitrary accuracy calculate $R_i(A,B,C)$ using the bisection method to find the zero of the integral $I^+(R)$ or $I^-(R)$. We have proved also that $R_n(\cdot) = R_t(\cdot) = R_{nt}(\cdot) = R_i(\cdot)$ for all Hurwitz stable matrix A and second order matrices B, C. These results, to our knowledge, are not reported in the mathematical literature.

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