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GREEN'S FUNCTION OF A CENTERED PARTIAL DIFFERENCE EQUATION

Richard I. Avery [1] **and Douglas R. Anderson** [2]

[1] College of Arts and Sciences, Dakota State University
Madison, SD 57042 USA
e-mail: rich.avery@dsu.edu
[2] Department of Mathematics and Computer Science, Concordia College
Moorhead, MN 56562 USA
e-mail: andersod@cord.edu

Honoring the Career of John Graef on the Occasion of His Sixty-Seventh Birthday

Abstract

Applying a variation of Jacobi iteration we obtain the Green's function for the centered partial difference equation

$$\Delta_{ww}u(x_{w-1},y_z)+\Delta_{zz}u(x_w,y_{z-1})+f(u(x_w,y_z))=0,$$

which is the result of applying the finite difference method to an associated nonlinear partial differential equation of the form

$$u_{xx}+u_{yy}+h(u)=0.$$

We show that approximations of the partial differential equation can be found by applying fixed point theory instead of the standard techniques associated with solving a system of nonlinear equations.

Key words and phrases: Finite difference method, partial difference equations, Green's function.
AMS (MOS) Subject Classifications: 39A12, 37H10.

1 Introduction

The finite difference method is a classical technique used to approximate the solution of a partial differential equation. An application of the technique yields a nonlinear system of equations which can be considered as a nonlinear partial difference equation. Instead of attempting to solve or approximate the solution of the nonlinear system, we will convert the partial difference equation to a fixed point problem whose inversion involves the Green's function of the partial difference equation. The Green's function can be interpreted combinatorially through an adjacency matrix. Finding the Green's function for a partial difference equation has been done; for example, see [1, 2, 3, 4, 5, 6, 7, 8, 12, 13], however none of the techniques are simple nor are the formulas easy to work with when finding solutions which correspond to approximations to partial differential equations. Also, it is difficult to find bounds on the Green's function to establish the existence of solutions for the classical partial difference equation when these techniques are used. In this paper we apply an elementary iterative technique based on Jacobi iteration to obtain the Green's functions. A technique that is easy to computationally apply to approximate the solution of a partial differential equation using partial difference equations techniques. Since the Green's function involves the sum of nonnegative terms (with a combinatorial interpretation) it is easy to establish elementary bounds which can be used in existence of solutions arguments. A basic understanding of difference equations, Green's functions, and discretization of partial differential equations is assumed; see Kelley and Peterson [10] for background results.

2 Preliminaries

For designation purposes, we will employ interval notation to denote sets of integers, such as $[1, N] = \{1, 2, \ldots, N\}$, etc.

The partial difference equation

$$\Delta_{ww}u(x_{w-1}, y_z) + \Delta_{zz}u(x_w, y_{z-1}) + f(u(x_w, y_z)) = 0,$$

when $w \in [1, N-1]$ and $z \in [1, N-1]$, with boundary conditions

$$u(x, y) = 0 \text{ for all } x = x_0, x = x_N, y = y_0, \text{ or } y = y_N,$$

is the result of applying the finite element method to the nonlinear partial differential equation

$$u_{xx} + u_{yy} + h(u) = 0,$$

with boundary conditions $u(0, y) = 0$, $0 \leq y \leq 1$, and $u(x, 0) = 0$, $0 \leq x \leq 1$, where $N^2 f(u) = h(u)$, $x_w = \frac{w}{N}$, $y_z = \frac{z}{N}$. Thus, this is a problem that is of interest to a large audience extending beyond mathematicians. Note, the arguments throughout can be modified to suit any bounded domain. The Green's function is the function whose domain is $[0, N]^2 \times [1, N-1]^2$ such that for all $(r, s) \in [1, N-1]^2$,

$$g(w, z, r, s) = 0, \quad \text{if} \quad (w, z) \in \partial[0, N]^2,$$

and

$$\Delta_{ww}g(w-1, z, r, s) + \Delta_{zz}g(w, z-1, r, s) = -\delta_{w,r}\delta_{z,s}$$

for all $w \in [1, N-1]$ and $z \in [1, n-1]$. Applying classical techniques it can be shown that any solution of the partial difference equation is a fixed point of the operator T

defined by

$$Tu(x_w, y_z) = \sum_{r=1}^{N-1} \sum_{s=1}^{N-1} g(w, z, r, s) f(u(x_r, y_s)).$$

3 Finding the Green's Function

For each fixed $(r, s) \in [1, N-1]^2$, the Green's function is a solution of

$$\Delta_{ww} g(w-1, z, r, s) + \Delta_{zz} g(w, z-1, r, s) = -\delta_{w,r}\delta_{z,s},$$

when $w \in [1, N-1]$ and $z \in [1, N-1]$, which we can write as

$$\begin{aligned} g(w, z, r, s) &= \left(\frac{1}{4}\right) [g(w-1, z, r, s) + g(w+1, z, r, s) \\ &+ g(w, z-1, r, s) + g(w, z+1, r, s) + \delta_{w,r}\delta_{z,s}]. \end{aligned}$$

Thus, for each fixed $(r, s) \in [1, N-1]^2$, the Green's function is a solution of a linear system of equations which we can write as

$$G_{r,s} = \left(\frac{A}{4}\right) G_{r,s} + \frac{D_{r,s}}{4}.$$

The column matrix $G_{r,s}$ consists of the Green's function terms, that is, for each $(r, s) \in [1, N-1]^2$, the element in the ith row of $G_{r,s}$ is $g(w, z, r, s)$, where $i = (w-1)(N-1)+z$. Similarly, for each $(r, s) \in [1, N-1]^2$, the element in the ith row of $D_{r,s}$ is $\delta_{w,r}\delta_{z,s}$, where $i = (w-1)(N-1) + z$. The matrix A is the $(N-1)^2 \times (N-1)^2$ adjacency matrix for the Laplacian molecule. That is, if

$$i = (w_i - 1)(N-1) + z_i \text{ and } j = (w_j - 1)(N-1) + z_j,$$

then the (i, j)th entry of A is 1, if (w_i, z_i) is adjacent to (w_j, z_j), and 0 otherwise. The vertices (w_i, z_i) and (w_j, z_j) are adjacent if

$$w_i = w_j \text{ and } |z_i - z_j| = 1, \quad \text{or} \quad z_i = z_j \text{ and } |w_i - w_j| = 1.$$

Since A is an adjacency matrix, for all natural numbers k, the (i, j)th entry of A^k is the number of distinct paths of length k from node i to node j that never leave the bounded lattice $[1, N-1]^2$ (see [11] for details), where node i corresponds to the point (w_i, z_i) in the lattice, and the node j corresponds to the point (w_j, z_j) in the lattice, again where

$$i = (w_i - 1)(N-1) + z_i \text{ and } j = (w_j - 1)(N-1) + z_j.$$

For each natural number t and $(i,j) \in [1,(N-1)^2]^2$, let $a_{i,j}^{(t)}$ represent the i,jth entry of A^t. Then the maximum row sum matrix norm is defined by

$$\|A^t\|_\infty = \max_{i\in[1,(N-1)^2]} \sum_{j=1}^{(N-1)^2} |a_{i,j}^{(t)}|.$$

In the theorem below we verify that $I - \frac{A}{4}$ is invertible.

Theorem 3.1 *If A is the adjacency matrix corresponding to the centered Laplacian with zero boundary conditions, then $I - \frac{A}{4}$ is invertible. Moreover,*

$$\left(I - \frac{A}{4}\right)^{-1} = \sum_{k=0}^{\infty} \frac{A^k}{4^k}.$$

Proof. Let $i \in [1,(N-1)^2]$ and (w_i, z_i) be the lattice point corresponding to the i^{th} node. For any natural number t, the number of walks in an infinite lattice of length t from node i is

$$4^t = (1+1+1+1)^t = \sum_{r=0}^{t}\sum_{l=0}^{t-r}\sum_{u=0}^{t-r-l} \binom{t}{r}\binom{t-r}{l}\binom{t-r-l}{u},$$

and if $t \geq \lceil \frac{N}{2} \rceil$, then at least one of these walks has left the bounded lattice $[1, N-1]^2$, since a walk in the direction (all the steps either left, right, up or down) of the closest boundary will leave the bounded lattice $[1, N-1]^2$. Hence, if $t \geq \lceil \frac{N}{2} \rceil$, then for all $i \in [1, N-1]^2$,

$$\sum_{j=1}^{(N-1)^2} |a_{i,j}^{(t)}| \leq 4^t - 1;$$

that is, the number of walks from node i to any other node in the bounded lattice in t steps that does not leave the bounded lattice is at most $4^t - 1$. Therefore,

$$\left\| \frac{A^{\lceil \frac{N}{2} \rceil}}{4^{\lceil \frac{N}{2} \rceil}} \right\|_\infty \leq \frac{4^{\lceil \frac{N}{2} \rceil} - 1}{4^{\lceil \frac{N}{2} \rceil}}$$

which implies that

$$I - \frac{A^{\lceil \frac{N}{2} \rceil}}{4^{\lceil \frac{N}{2} \rceil}}$$

is invertible. Since

$$I - \frac{A^{\lceil \frac{N}{2} \rceil}}{4^{\lceil \frac{N}{2} \rceil}} = \left(I - \frac{A}{4}\right)\left(I + \frac{A}{4} + \frac{A^2}{4^2} + \cdots + \frac{A^{\lceil \frac{N}{2} \rceil - 1}}{4^{\lceil \frac{N}{2} \rceil - 1}}\right)$$

the invertibility of $I - \frac{A^{\lceil \frac{N}{2} \rceil}}{4^{\lceil \frac{N}{2} \rceil}}$ guarantees the invertibility of $I - \frac{A}{4}$ as well as the invertibility of $I + \frac{A}{4} + \frac{A^2}{4^2} + \cdots + \frac{A^{\lceil \frac{N}{2} \rceil - 1}}{4^{\lceil \frac{N}{2} \rceil - 1}}$. Moreover, it is well known that if $\left(I - \frac{A}{4}\right)^{-1}$ exists, then

$$\left(I - \frac{A}{4}\right)^{-1} = \sum_{k=0}^{\infty} \frac{A^k}{4^k}.$$

Now that we have that $I - \frac{A}{4}$ is invertible and that

$$\left(I - \frac{A}{4}\right)^{-1} = \sum_{k=0}^{\infty} \frac{A^k}{4^k},$$

then, for each fixed $(r, s) \in [1, N-1]^2$, the Green's function is given by

$$G_{r,s} = \left(\sum_{k=0}^{\infty} \frac{A^k}{4^k} \right) \frac{D_{r,s}}{4}.$$

Note, that given that $I - \frac{A}{4}$ is invertible, then the Green's function could be found by Jacobi Iteration. That is, if we let $X_0 = \frac{D_{r,s}}{4}$, and for $n \geq 1$, let

$$X_n = \left(\frac{A}{4}\right) X_{n-1} + \frac{D_{r,s}}{4}$$

then by iteration, we have

$$X_n = \left(\sum_{k=0}^{n} \frac{A^k}{4^k} \right) \left(\frac{D_{r,s}}{4} \right)$$

and hence

$$G_{r,s} = \lim_{n \to \infty} X_n.$$

Example 3.1 *Suppose $N = 4$. Then, the adjacency matrix A corresponding to the centered Laplacian with zero boundary conditions (4×4 lattice with the boundary removed which results in a 3×3 lattice) is*

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

and since

$$G_{r,s} = \left(\sum_{k=0}^{\infty} \frac{A^k}{4^k} \right) \left(\frac{D_{r,s}}{4} \right),$$

lower and upper bounds on $\sum_{k=0}^{\infty} \frac{A^k}{4^k}$ *will be crucial in existence of solutions arguments as well as iterative arguments to find an approximation of the original partial differential equation problem.*

4 Alternative Inversion Technique

In the literature concerning the existence of positive solutions of various boundary value problems, the most common procedure is to apply fixed point theorems to operators analogous to the operator T above. Rather than taking that approach in this paper, we will establish the existence of a fixed point for a related operator S.

Let P be a space of functions defined on $[0, N]^2$, and define

$$S : P \to P$$

by

$$Sv(w, z) = f\left(\sum_{r=1}^{N-1} \sum_{s=1}^{N-1} g(w, z, r, s) v(r, s) \right),$$

for $(w, z) \in [0, N]^2$. We now show that if $v \in P$ is a fixed point of S, then

$$u(w, z) := \sum_{r=1}^{N-1} \sum_{s=1}^{N-1} g(w, z, r, s) v(r, s)$$

is a fixed point of T (and hence is a solution of the partial difference equation). To see this, assume $v \in P$ is a fixed point of S. Then,

$$\begin{aligned}
Tu(w, z) &= \sum_{r=1}^{N-1} \sum_{s=1}^{N-1} g(w, z, r, s) f(u(r, s)) \\
&= \sum_{r=1}^{N-1} \sum_{s=1}^{N-1} g(w, z, r, s) f\left(\sum_{\tau=1}^{N-1} \sum_{\psi=1}^{N-1} g(r, s, \tau, \psi) v(\tau, \psi) \right) \\
&= \sum_{r=1}^{N-1} \sum_{s=1}^{N-1} g(w, z, r, s) S(v(r, s)) \\
&= \sum_{r=1}^{N-1} \sum_{s=1}^{N-1} g(w, z, r, s) v(r, s) \\
&= u(w, z).
\end{aligned}$$

Conversely, assume $u \in P$ is a fixed point of T. Then,

$$v(w,z) := f(u(w,z))$$

is a fixed point of S. To see this, assume $u \in P$ is a fixed point of T. Then,

$$\begin{aligned} Sv(w,z) &= f\left(\sum_{r=1}^{N-1}\sum_{s=1}^{N-1} g(w,z,r,s)v(r,s)\right) \\ &= f\left(\sum_{r=1}^{N-1}\sum_{s=1}^{N-1} g(w,z,r,s)f(u(r,s))\right) \\ &= f\left(Tu(w,z)\right) \\ &= f(u(w,z)) = v(w,z). \end{aligned}$$

It follows easily that there is a one-to-one correspondence between fixed points of T and the fixed points of S. By applying the definition, in an appropriate function space context, it can be shown that, if f is continuous, then both S and T are completely continuous.

5 Application

In this section we will state the definitions that are used in the remainder of the paper.

Definition 5.1 *Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a cone if it satisfies the following two conditions:*

(i) $x \in P, \lambda \geq 0$ *implies* $\lambda x \in P$*;*

(ii) $x \in P, -x \in P$ *implies* $x = 0$.

Every cone $P \subset E$ induces an ordering, $\leq$, in E given by

$$x \leq y \text{ if and only if } y - x \in P.$$

Definition 5.2 *An operator is called completely continuous if it is continuous and maps bounded sets into pre-compact sets.*

Definition 5.3 *Let P be a cone in a real Banach space E and $D \subseteq E$. Then the operator $A : D \to E$ is said to be increasing on D provided $x_1, x_2 \in D$ with $x_1 \leq x_2$ implies $Ax_1 \leq Ax_2$.*

Definition 5.4 *A cone P of a real Banach space E is said to be normal if there exists a positive constant δ such that $\|x + y\| \geq \delta$ for all $x, y \in P$ with $\|x\| = \|y\| = 1$.*

The following theorem is an elementary fact about normal cones. A proof can be found in [9].

Theorem 5.1 *Let P be a cone in a real Banach space E. The cone P is normal if and only if the norm of the Banach space E is semi-monotone; that is, there exists a constant $N > 0$ such that $0 \leq x \leq y$ implies that $\|x\| \leq N\|y\|$.*

The next two theorems concerning the convergence of Picard iterates are restatements of theorems that can be found in [9].

Theorem 5.2 *Let P be a normal cone in a real Banach space E and $A : P \to E$ be a completely continuous operator. If $u \in P$ with $Au \leq u$ and there exists a $v \in P$ such that $v \leq A^n u$ for all $n \in N$ and A is increasing on $[v, u]$, then*

$$\{A^n u\}_{n=1}^{\infty}$$

is a decreasing sequence bounded below by $v \in P$, and there exists a fixed point $u^* \in P$ of A such that

$$u^* = \lim_{n\to\infty} A^n u$$

with

$$v \leq u^* \leq A^n u \leq A^{n-1} u \leq \cdots \leq Au.$$

Theorem 5.3 *Let P be a normal cone in a real Banach space E and $A : P \to E$ be a completely continuous operator. If $u \in P$ with $u \leq Au$ and there exists a $v \in P$ such that $v \geq A^n u$ for all $n \in N$ and A is increasing on $[u, v]$, then*

$$\{A^n u\}_{n=1}^{\infty}$$

is an increasing sequence bounded above by $v \in P$, and there exists a fixed point $u^* \in P$ of A such that

$$u^* = \lim_{n\to\infty} A^n u$$

with

$$v \geq u^* \geq A^n u \geq A^{n-1} u \geq \cdots \geq Au.$$

We now present our solutions result as a fixed point application.

Theorem 5.4 *Suppose there exist positive real numbers m and M, with $0 < m < M$, and an increasing continuous function $f : [m, \infty) \to [0, \infty)$ such that*

$$f(x) \leq M \ \textit{ for } \ x \leq \max_{(w,z)\in[1,N-1]^2} \sum_{r=1}^{N-1} \sum_{s=1}^{N-1} g(w, z, r, s)\, M$$

and

$$m \le f(x) \;\; for \quad \min_{(w,z)\in[1,N-1]^2} \sum_{r=1}^{N-1}\sum_{s=1}^{N-1} g(w,z,r,s)m \le x.$$

Then,

$$v_l = \lim_{n\to\infty} v_{l,n} \;\; and \;\; v_u = \lim_{n\to\infty} v_{u,n}$$

are fixed points of S*, where for all natural numbers* n*,* $v_{l,n+1} = S(v_{l,n})$ *and* $v_{u,n+1} = S(v_{u,n})$*, with*

$$v_{l,0}(w,z) = \begin{cases} m, & (w,z)\in[1,N-1]^2, \\ f(0), & \quad otherwise, \end{cases}$$

and

$$v_{u,0}(w,z) = \begin{cases} M, & (w,z)\in[1,N-1]^2, \\ f(0), & \quad otherwise. \end{cases}$$

Moreover, this implies that

$$u^*(w,z) = \sum_{r=1}^{N-1}\sum_{s=1}^{N-1} g(w,z,r,s)v_u(r,s)$$

and

$$u_*(w,z) = \sum_{r=1}^{N-1}\sum_{s=1}^{N-1} g(w,z,r,s)v_l(r,s)$$

are fixed points of the operator T *and are therefore solutions (maybe equal) of the original partial difference equation.*

Proof. Let

$$E := \{u : u : [0,N]^2 \to \mathbb{R}\}$$

and

$$P := \{u : u \in E \text{ and } u \ge 0\}.$$

Then P is a normal cone in the Banach space E with the maximum norm

$$\|u\| = \max_{(w,z)\in[0,N]^2} u(w,z).$$

For any $v \in P$, with $\min_{(w,z)\in[1,N-1]^2} v(w,z) \ge m$,

$$\min_{(w,z)\in[1,N-1]^2} \sum_{r=1}^{N-1}\sum_{s=1}^{N-1} g(w,z,r,s)m \le \min_{(w,z)\in[1,N-1]^2} \sum_{r=1}^{N-1}\sum_{s=1}^{N-1} g(w,z,r,s)v(r,s),$$

and for any $v \in P$, with $\max_{(w,z)\in[1,N-1]^2} v(w,z) \le M$,

$$\max_{(w,z)\in[1,N-1]^2} \sum_{r=1}^{N-1}\sum_{s=1}^{N-1} g(w,z,r,s)v(r,s) \le \max_{(w,z)\in[1,N-1]^2} \sum_{r=1}^{N-1}\sum_{s=1}^{N-1} g(w,z,r,s)M.$$

Therefore, for all $(w,z) \in [1, N-1]^2$, we have

$$v_{l,1}(w,z) = S(v_{l,0}(w,z)) \geq m = v_{l,0}(w,z)$$

and

$$v_{u,1}(w,z) = S(v_{u,0}(w,z)) \leq M = v_{u,0}(w,z).$$

Note that, if $(w,z) \notin [1, N-1]^2$ (that is, (w,z) is on the boundary), then

$$v_{l,1}(w,z) = f(0) = S(v_{l,0}(w,z)) \text{ and } v_{u,1}(w,z) = f(0) = S(v_{u,0}(w,z)).$$

Also, for any $v_1, v_2 \in P$ with $v_{l,0} \leq v_1 \leq v_2 \leq v_{u,0}$, and for all $(w,z) \in [0,N]^2$, we have

$$\sum_{r=1}^{N-1}\sum_{s=1}^{N-1} g(w,z,r,s)v_1(r,s) \leq \sum_{r=1}^{N-1}\sum_{s=1}^{N-1} g(w,z,r,s)v_2(r,s).$$

Thus, since f is increasing,

$$f\left(\sum_{r=1}^{N-1}\sum_{s=1}^{N-1} g(w,z,r,s)v_1(r,s)\right) \leq f\left(\sum_{r=1}^{N-1}\sum_{s=1}^{N-1} g(w,z,r,s)v_2(r,s)\right),$$

and hence $S(v_1) \leq S(v_2)$. Therefore, by Theorems 5.2 and 5.3, we have that

$$v_l = \lim_{n\to\infty} v_{l,n} \text{ and } v_u = \lim_{n\to\infty} v_{u,n}$$

are fixed points of S. Moreover, we have that

$$v_{l,0} \leq v_{l,1} \leq \cdots \leq v_{l,n} \leq \cdots \leq v_l \leq v_u \leq \cdots \leq v_{u,n} \leq \cdots \leq v_{u,1} \leq v_{u,0}.$$

Example 5.1 *Consider the partial difference equation*

$$\Delta_{ww} u\left(\frac{w-1}{4}, \frac{z}{4}\right) + \Delta_{zz} u\left(\frac{w}{4}, \frac{z-1}{4}\right) + 10^2 \arctan\left(u\left(\frac{w}{4}, \frac{z}{4}\right)\right) = 0,$$

when $w, z \in \{1,2,3\}$*, with boundary conditions*

$$u(0,t) = u(1,t) = u(t,0) = u(t,1) = 0,$$

for all $t = \frac{k}{4}$*, where* $k \in \{0,1,2,3,4\}$*, which is the result of applying the finite difference method to the boundary value problem*

$$u_{xx} + u_{yy} + 40^2 \arctan(u) = 0$$

on the unit square with zero boundary conditions taking horizontal and vertical steps of size one fourth. For all $(w,z) \in [1,3]^2$*, there are at least two adjacent vertices; hence,*

$$\sum_{r=1}^{3}\sum_{s=1}^{3} g(w,z,r,s)(40\pi) \geq \left(\frac{1}{4}\right)\left(1+\frac{2}{4}\right)(40\pi) = 15\pi,$$

which can also be obtained by summing the entries in the ith row of

$$\left(\sum_{k=0}^{1} \frac{A^k}{4^k}\right)\left(\frac{1}{4}\right),$$

where $i = w(4-1) + z$, *since as noted in Example 3.1,*

$$G_{r,s} = \left(\sum_{k=0}^{\infty} \frac{A^k}{4^k}\right)\left(\frac{D_{r,s}}{4}\right).$$

One can find a solution by iteration to the partial difference equation that approximates the solution of the partial differential equation applying Theorem 5.4, with $M = 50\pi$ *and* $m = 40\pi$, *since for all* $(w, z) \in [1, 3]^2$,

$$40\pi < 100\arctan(15\pi) < 100\arctan\left(\sum_{r=1}^{3}\sum_{s=1}^{3} g(w,z,r,s)(40\pi)\right)$$

and

$$100\arctan\left(\sum_{r=1}^{3}\sum_{s=1}^{3} g(w,z,r,s)(50\pi)\right) < 100\left(\frac{\pi}{2}\right) = 50\pi.$$

Using Maple 12 we have obtained the following table:

n	$v_{l,n}(2,2)$	$v_{u,n}(2,2)$
1	156.3722891740	156.5137544776
2	156.5105038907	156.5111569644
3	156.5111409414	156.5111441617
4	156.5111440813	156.5111440975
5	156.5111440970	156.5111440971

thus

$$156.5111440970 \le v_l(2,2) \le v_u(2,2) \le 156.5111440971.$$

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