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Exact boundary behavior of the unique positive solution for singular second-order differential equations

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Abstract. In this paper, we give the exact asymptotic behavior of the unique positive solution to the following singular boundary value problem

$$\begin{cases} -\frac{1}{A}(Au')' = p(x)g(u), & x \in (0,1), \\ u > 0, & \text{in } (0,1), \\ \lim_{x\to 0^+}(Au')(x) = 0, & u(1) = 0, \end{cases}$$

where A is a continuous function on $[0,1)$, positive and differentiable on $(0,1)$ such that $\frac{1}{A}$ is integrable in a neighborhood of 1, $g \in C^1((0,\infty),(0,\infty))$ is nonincreasing on $(0,\infty)$ with $\lim_{t\to 0} g'(t)\int_0^t \frac{1}{g(s)}\,ds = -C_g \le 0$ and p is a nonnegative continuous function in $(0,1)$ satisfying

$$0 < p_1 = \liminf_{x\to 1} \frac{p(x)}{h(1-x)} \le \limsup_{x\to 1} \frac{p(x)}{h(1-x)} = p_2 < \infty,$$

where $h(t) = ct^{-\lambda}\exp(\int_t^\eta \frac{z(s)}{s}\,ds)$, $\lambda \le 2$, $c > 0$ and z is continuous on $[0,\eta]$ for some $\eta > 1$ such that $z(0) = 0$.

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1 Introduction

In this paper, we give the exact asymptotic behavior near the boundary of the unique positive solution to the following singular problem

$$\begin{cases} -\frac{1}{A}(Au')' = p(x)g(u), & x \in (0,1), \\ u > 0, & \text{in } (0,1), \end{cases} \tag{1.1}$$

subject to the boundary conditions

$$\lim_{x\to 0^+} (Au')(x) = 0, \qquad u(1) = 0. \tag{1.2}$$

The functions A, p and g satisfy the following assumptions.

($\mathbf{H}_1$) A is a continuous function on $[0,1)$, positive and differentiable on $(0,1)$ such that $\frac{1}{A}$ is integrable in a neighborhood of 1 and $\lim_{x\to 1} \frac{(x-1)A'(x)}{A(x)} = \alpha < 1$.

($\mathbf{H}_2$) p is a nonnegative continuous function in $(0,1)$ satisfying

$$0 < p_1 = \liminf_{x\to 1} \frac{p(x)}{h(1-x)} \le \limsup_{x\to 1} \frac{p(x)}{h(1-x)} = p_2 < \infty,$$

where $h(t) = t^{-\lambda}L(t)$, $\lambda \le 2$ such that $\int_0^\eta s^{1-\lambda}L(s)\,ds < \infty$ for some $\eta > 1$ and L belongs to the class of Karamata functions $\mathcal{K}$ (see Definition 1.1).

($\mathbf{H}_3$) The function $g\colon (0,\infty) \to (0,\infty)$ is nonincreasing, continuously differentiable such that

$$\lim_{t\to 0} g'(t) \int_0^t \frac{1}{g(s)}\,ds = -C_g \quad \text{with } C_g \ge 0.$$

($\mathbf{H}_4$) $\lambda + (2-\lambda)C_g - \alpha - 1 > 0$.

Observe that $C_g \in [0,1]$. Indeed, since the function g is nonincreasing, we obtain for $t > 0$

$$0 < g(t) \int_0^t \frac{1}{g(s)}\,ds \le t.$$

This implies that $\lim_{t\to 0} g(t) \int_0^t \frac{1}{g(s)}\,ds = 0$. Now, since for $t > 0$

$$\int_0^t g'(s) \int_0^s \frac{1}{g(r)}\,dr = g(t) \int_0^t \frac{1}{g(s)}\,ds - t,$$

we get

$$\lim_{t\to 0} \frac{g(t)}{t} \int_0^t \frac{1}{g(s)}\,ds = 1 - C_g.$$

Hence $C_g \in [0,1]$.

The functions $t^{-1}\ln(1+t)$, $\ln\left(\ln(e+\frac{1}{t})\right)$, $t^{-\nu}\ln\left(1+\frac{1}{t}\right)$, $\exp\{\left(\ln(1+\frac{1}{t})\right)^\nu\}$, $\nu \in (0,1)$ satisfy the assumption (H_3), as well as the function

$$\begin{cases} t^2 e^{\frac{1}{t}}, & \text{if } 0 < t < \frac{1}{2}, \\ \frac{1}{4}e^2, & \text{if } t \ge \frac{1}{2}. \end{cases}$$

When $A \equiv 1$, problems of type (1.1) with various boundary conditions arise in the study of boundary layer equations for the class of pseudoplastic fluids and have been studied for both bounded and unbounded intervals of $\mathbb{R}$ (see [4,5,23,27] and the references therein).

When $A(t) = t^{n-1}$ $(n \geq 2)$, the operator $u \to \frac{1}{A}(Au')'$ appears as the radial part of the Laplace operator Δ (see [24]). Our setting includes the scalar curvature equation and the relativistic pendulum equation, which correspond to $A(t) = (1+t^2)^{\frac{-1}{2}}$, resp. $A(t) = (1-t^2)^{\frac{-1}{2}}$. For various existence, uniqueness and asymptotic behavior results of such problem, we refer the reader to [8–11,14,21,25,26] and the references therein. However, we emphasize that in problem (1.1) the function A could be singular at $t = 1$.

On the other hand, the singular nonlinear problem

$$\begin{cases} -\frac{1}{A}(Au')' = f(x,u), \quad x \in (0,1), \\ u > 0, \quad \text{in}(0,1), \end{cases} \tag{1.3}$$

subject to different boundary conditions has been considered by many authors, where A is a continuous function on $[0,1)$, positive and differentiable on $(0,1)$ satisfying some appropriate conditions (see for example [1,2,13,16,17,19]). In [15, Theorem 5], Mâagli and Masmoudi investigated equation (1.3) with boundary value conditions $u'(0) = u(1) = 0$. They supposed that f is a nonnegative continuous function on $(0,1) \times (0,\infty)$ and nonincreasing with respect to the second variable. Under some appropriate conditions on the function A, they proved the existence of a unique positive solution u in $C([0,1]) \cap C^2((0,1))$ to (1.3) and gave estimates on such a solution. In particular they extended some results of [1,2] and [19]. Our aim in this paper is to establish the exact boundary behavior of the unique solution to problem (1.1)–(1.2).

To state our results, we need some notations.

Definition 1.1. The class $\mathcal{K}$ is the set of all Karamata functions L defined on $(0,\eta]$ by

$$L(t) := c \exp\left(\int_t^\eta \frac{z(s)}{s}\,ds\right),$$

for some $\eta > 1$ and where $c > 0$ and $z \in C([0,\eta])$ such that $z(0) = 0$.

Note that functions belonging to the class $\mathcal{K}$ are in particular slowly varying functions. The theory of such functions was initiated by Karamata in a fundamental paper [12].

We also point out that the first use of the Karamata theory in the study of the growth rate of solutions near the boundary is done in the paper of Cîrstea and Rădulescu [7].

Remark 1.2. A function L is in $\mathcal{K}$ if and only if L is a positive function in $C^1((0,\eta])$, for some $\eta > 1$, such that $\lim_{t\to 0^+} \frac{tL'(t)}{L(t)} = 0$.

Typical examples of functions belonging to the class $\mathcal{K}$ (see [3,18,22]) are:

$$L(t) = \prod_{k=1}^{m}\left(\log_k\left(\frac{\omega}{t}\right)\right)^{\xi_k},$$

$$L(t) = 2 + \sin\left(\log_2\left(e + \frac{1}{t}\right)\right),$$

$$L(t) = \exp\left\{\prod_{k=1}^{m}\left(\log_k\left(\frac{\omega}{t}\right)\right)^{\nu_k}\right\},$$

where $\log_k x = \log \circ \log \circ \cdots \circ \log x$ (k times), $\xi_k \in \mathbb{R}$, $\nu_k \in (0,1)$ and ω is a sufficiently large positive real number such that L is defined and positive on $(0, \eta]$.

Throughout this paper, we denote by ψ_g the unique solution determined by

$$\int_0^{\psi_g(t)} \frac{1}{g(s)}\, ds = t, \qquad t \in [0, \infty), \tag{1.4}$$

and we mention that

$$\lim_{t \to 0} t g'(\psi_g(t)) = -C_g. \tag{1.5}$$

Our first result is the following.

Theorem 1.3. *Assume that hypotheses (H_1)–(H_4) are fulfilled. Then problem (1.1)–(1.2) has a unique positive solution $u \in C([0,1]) \cap C^2((0,1))$ satisfying*

(i) *if $\lambda < 2$, then*

$$\left(\frac{\xi_1}{2-\lambda}\right)^{1-C_g} \le \liminf_{x \to 1} \frac{u(x)}{\psi_g((1-x)^{2-\lambda} L(1-x))} \le \limsup_{x \to 1} \frac{u(x)}{\psi_g((1-x)^{2-\lambda} L(1-x))} \le \left(\frac{\xi_2}{2-\lambda}\right)^{1-C_g};$$

(ii) *if $\lambda = 2$, then*

$$\xi_1^{1-C_g} \le \liminf_{x \to 1} \frac{u(x)}{\psi_g\left(\int_0^{1-x} \frac{L(t)}{t}\, dt\right)} \le \limsup_{x \to 1} \frac{u(x)}{\psi_g\left(\int_0^{1-x} \frac{L(t)}{t}\, dt\right)} \le \xi_2^{1-C_g},$$

where $\xi_1 = \frac{p_1}{\lambda + (2-\lambda)C_g - \alpha - 1}$ and $\xi_2 = \frac{p_2}{\lambda + (2-\lambda)C_g - \alpha - 1}$.

An immediate consequence of Theorem 1.3 is the following.

Corollary 1.4. *Let u be the unique solution of problem (1.1)–(1.2). Then, we have the following exact boundary behavior.*

(a) *When $C_g = 1$, then we have*

- (i) $\lim_{x \to 1} \frac{u(x)}{\psi_g((1-x)^{2-\lambda} L(1-x))} = 1$ *if $\lambda < 2$;*
- (ii) $\lim_{x \to 1} \frac{u(x)}{\psi_g(\int_0^{1-x} \frac{L(t)}{t}\, dt)} = 1$ *if $\lambda = 2$.*

(b) *When $C_g < 1$ and $p_1 = p_2 = p_0$, then we have*

- (i) $\lim_{x \to 1} \frac{u(x)}{\psi_g((1-x)^{2-\lambda} L(1-x))} = \left(\frac{p_0}{(2-\lambda)\left(\lambda + (2-\lambda)C_g - \alpha - 1\right)}\right)^{1-C_g}$ *if $\lambda < 2$;*
- (ii) $\lim_{x \to 1} \frac{u(x)}{\psi_g(\int_0^{1-x} \frac{L(t)}{t}\, dt)} = \left(\frac{p_0}{1-\alpha}\right)^{1-C_g}$ *if $\lambda = 2$.*

Example 1.5. Let g be the function defined by

$$g(t) = \begin{cases} t^2 e^{\frac{1}{t}}, & \text{if } 0 < t < \frac{1}{2}, \\ \frac{1}{4} e^2, & \text{if } t \ge \frac{1}{2}, \end{cases}$$

and p be a nonnegative continuous function in $(0,1)$ satisfying

$$\lim_{x\to 1}\frac{p(x)}{h(1-x)} = p_0 \in (0,\infty),$$

where $h(t) = t^{-\lambda}L(t)$, $\lambda \le 2$ and $L \in \mathcal{K}$ such that $\int_0^\eta s^{1-\lambda}L(s)\,ds < \infty$. Then, we have $C_g = 1$ and $\psi_g(\xi) = \frac{-1}{\log(\xi)}$ for $\xi \in (0, e^{-2})$. Let u be the unique solution of (1.1)–(1.2). Then, we have the following exact behavior.

(*i*) $\lim_{x\to 1} u(x)\log\left(\frac{1}{(1-x)^{2-\lambda}L(1-x)}\right) = 1$ if $\lambda < 2$;

(*ii*) $\lim_{x\to 1} u(x)\log\left(\frac{1}{\int_0^{1-x}\frac{L(t)}{t}\,dt}\right) = 1$ if $\lambda = 2$.

In order to establish our second result, we consider the special case where $g(t) = t^{-\gamma}$ with $\gamma \ge 0$ and $\lambda = (\alpha+1)+(\alpha-1)\gamma$. Note that in this case $C_g = \frac{\gamma}{\gamma+1}$ and $\lambda + (2-\lambda)C_g - \alpha - 1 = 0$. We assume the following hypotheses.

($\mathbf{H}_5$) A is a continuous function on $[0,1)$, positive and differentiable on $(0,1)$ such that $A(x) = (1-x)^\alpha B(x)$ with $\alpha < 1$ and $\frac{(1-x)^\zeta B'(x)}{B(x)}$ is bounded in a neighborhood of 1 for some $\zeta \in (0,1)$.

($\mathbf{H}_6$) p is a nonnegative continuous function in $(0,1)$ satisfying

$$\begin{aligned} 0 < p_1 = \liminf_{x\to 1}\frac{p(x)}{(1-x)^{\gamma-1-\alpha(1+\gamma)}L(1-x)} \\ \le \limsup_{x\to 1}\frac{p(x)}{(1-x)^{\gamma-1-\alpha(1+\gamma)}L(1-x)} = p_2 < \infty, \end{aligned}$$

where $\gamma \ge 0$ and $L \in \mathcal{K}$ with $\int_0^\eta \frac{L(s)}{s}\,ds = \infty$.

Our second result is the following.

Theorem 1.6. *Assume that hypotheses* (H_5) *and* (H_6) *are fulfilled. Then the problem*

$$\begin{cases} -\frac{1}{A}(Au')' = p(x)u^{-\gamma}, \quad x \in (0,1), \\ u > 0, \quad in\ (0,1), \\ \lim_{x\to 0^+}(Au')(x) = 0, \ u(1) = 0, \end{cases} \tag{1.6}$$

has a unique positive solution $u \in C([0,1]) \cap C^2((0,1))$ *satisfying*

$$(b_1)^{\frac{1}{\gamma+1}} \le \liminf_{x\to 1}\frac{u(x)}{(1-x)^{1-\alpha}\left(\int_{1-x}^\eta \frac{L(t)}{t}\,dt\right)^{\frac{1}{\gamma+1}}} \le \limsup_{x\to 1}\frac{u(x)}{(1-x)^{1-\alpha}\left(\int_{1-x}^\eta \frac{L(t)}{t}\,dt\right)^{\frac{1}{\gamma+1}}} \le (b_2)^{\frac{1}{\gamma+1}},$$

where $b_1 = \frac{(\gamma+1)p_1}{1-\alpha}$ *and* $b_2 = \frac{(\gamma+1)p_2}{1-\alpha}$.

In particular if $p_1 = p_2 = p_0$, *then*

$$\lim_{x\to 1}\frac{u(x)}{(1-x)^{1-\alpha}\left(\int_{1-x}^\eta \frac{L(t)}{t}\,dt\right)^{\frac{1}{\gamma+1}}} = \left(\frac{(\gamma+1)p_0}{1-\alpha}\right)^{\frac{1}{\gamma+1}}.$$

The content of this paper is organized as follows. In Section 2, we present some fundamental properties of Karamata regular variation theory. In Section 3, exploiting the results of the previous section, we prove Theorems 1.3 and 1.6 by constructing a convenient pair of subsolution and supersolution.

2 On the Karamata class $\mathcal{K}$

In this section, we collect some properties of Karamata functions.

Proposition 2.1 ([18,22]).

(*i*) *Let $L_1, L_2 \in \mathcal{K}$ and $q \in \mathbb{R}$. Then the functions*

$$L_1 + L_2, \qquad L_1 L_2 \quad \text{and} \quad L_1^q \quad \text{belong to the class } \mathcal{K}.$$

(*ii*) *Let L be a function in $\mathcal{K}$ and $\varepsilon > 0$.*

Then we have

$$\lim_{t\to 0^+} t^\varepsilon L(t) = 0.$$

Lemma 2.2 ([18,22]). *Let $\mu \in \mathbb{R}$ and L be a function in $\mathcal{K}$ defined on $(0,\eta]$. Then the following hold.*

(*i*) *If $\mu < -1$, then $\int_0^\eta s^\mu L(s)\,ds$ diverges and $\int_t^\eta s^\mu L(s)\,ds \underset{t\to 0^+}{\sim} -\frac{t^{\mu+1}L(t)}{\mu+1}$.*

(*ii*) *If $\mu > -1$, then $\int_0^\eta s^\mu L(s)\,ds$ converges and $\int_0^t s^\mu L(s)\,ds \underset{t\to 0^+}{\sim} \frac{t^{\mu+1}L(t)}{\mu+1}$.*

The proof of the next lemma can be found in [6].

Lemma 2.3. *Let L be a function in $\mathcal{K}$ defined on $(0,\eta]$. Then we have*

$$\lim_{t\to 0^+} \frac{L(t)}{\int_t^\eta \frac{L(s)}{s}\,ds} = 0.$$

If further $\int_0^\eta \frac{L(s)}{s}\,ds$ converges, then we have

$$\lim_{t\to 0^+} \frac{L(t)}{\int_0^t \frac{L(s)}{s} ds} = 0.$$

Remark 2.4. Let L be a function in $\mathcal{K}$ defined on $(0,\eta]$, then using Remark 1.2 and Lemma 2.3, we deduce that

$$t \to \int_t^\eta \frac{L(s)}{s}\,ds \in \mathcal{K}.$$

Definition 2.5. A positive measurable function f is called normalized regularly varying at zero with index $\rho \in \mathbb{R}$ and we write $f \in \mathrm{NRVZ}_\rho$ if $f(s) = s^\rho L(s)$ for $s \in (0,\eta)$ with $L \in \mathcal{K}$.

Using the definition of Karamata class and the previous lemmas, we obtain the following.

Lemma 2.6 ([25]).

(*i*) *If $f \in \mathrm{NRVZ}_\rho$, then $\lim_{t\to 0} \frac{f(\xi t)}{f(t)} = \xi^\rho$, uniformly for $\xi \in [c_1, c_2] \subset (0,\infty)$.*

(*ii*) *A positive measurable function f belongs to NRVZ_ρ if and only if $\lim_{t\to 0} \frac{tf'(t)}{f(t)} = \rho$.*

(*iii*) *Let $L \in \mathcal{K}$ and $\lambda \le 2$ such that $\int_0^\eta s^{1-\lambda} L(s)\,ds < \infty$. Then the function $\theta(t) := \int_0^t s^{1-\lambda} L(s)\,ds$ belongs to $\mathrm{NRVZ}_{2-\lambda}$.*

(*iv*) *The function $\psi_g \in \mathrm{NRVZ}_{(1-C_g)}$.*

(*v*) *The function $\psi_g \circ \theta \in \mathrm{NRVZ}_{(2-\lambda)(1-C_g)}$.*

(*vi*) *Let $f \in \mathrm{NRVZ}_\rho$ and m_1, m_2 be two positive functions on $(0,\infty)$ such that*

$$\lim_{t\to 0^+} m_1(t) = \lim_{t\to 0^+} m_2(t) = 0 \quad \text{and} \quad \lim_{t\to 0^+} \frac{m_1(t)}{m_2(t)} = 1, \quad \text{then} \quad \lim_{t\to 0^+} \frac{f(m_1(t))}{f(m_2(t))} = 1.$$

3 Proofs of Theorems 1.3 and 1.6

In the sequel, we denote by

$$v_0(x) = \int_x^1 \frac{t}{A(t)}\,dt, \quad \text{for } x \in (0,1),$$

and we let $L_A u := \frac{1}{A}(Au')' = u'' + \frac{A'}{A}u'$. Note that since the function A satisfies (H_1), the function $v_0(x)$ is well defined and we have $L_A v_0 = -\frac{1}{A}$.

The following result will play a crucial role in the proof of our main result.

Lemma 3.1. *Assume (H_1), then there exists $L_0 \in \mathcal{K}$ such that*

$$v_0(x) \underset{x\to 1}{\sim} \frac{(1-x)^{1-\alpha}}{(1-\alpha)L_0(1-x)}. \tag{3.1}$$

Proof. It is clear that

$$v_0(x) \underset{x\to 1}{\sim} \int_x^1 \frac{1}{A(t)}\,dt. \tag{3.2}$$

On the other hand, by (H_1), we have $\lim_{x\to 1} \frac{(1-x)A'(x)}{A(x)} = \lim_{t\to 0} \frac{tA'(1-t)}{A(1-t)} = -\alpha > -1$.

So by Lemma 2.6, we deduce that the function $f(t) := A(1-t)$ belongs to NRVZ_α. Therefore, there exists $L_0 \in \mathcal{K}$ such that

$$f(t) := A(1-t) = t^\alpha L_0(t), \quad \text{for } t \in (0,\delta).$$

Hence by using this fact, Proposition 2.1 (i) and Lemma 2.2 (ii), we deduce that

$$\int_x^1 \frac{1}{A(t)}\,dt = \int_0^{1-x} \frac{1}{A(1-t)}\,dt = \int_0^{1-x} t^{-\alpha}\frac{1}{L_0(t)}\,dt \underset{x\to 1}{\sim} \frac{(1-x)^{1-\alpha}}{(1-\alpha)L_0(1-x)}. \tag{3.3}$$

Combining (3.2) and (3.3), we obtain the required result. This completes the proof. □

Proof of Theorem 1.3. Let $\varepsilon \in (0, \frac{p_1}{2})$ and put $\xi_i = \frac{p_i}{\lambda+(2-\lambda)C_g-\alpha-1}$ for $i = 1,2$, $\tau_1 = \xi_1 - \varepsilon\frac{\xi_1}{p_1}$ and $\tau_2 = \xi_2 + \varepsilon\frac{\xi_2}{p_2}$. Clearly, we have $\frac{\xi_1}{2} < \tau_1 < \tau_2 < \frac{3}{2}\xi_2$.

Let $\theta(t) = \int_0^t s^{1-\lambda}L(s)\,ds$ and define

$$v_i(x) = \psi_g\left(\tau_i \int_0^{1-x} s^{1-\lambda}L(s)\,ds\right) = \psi_g(\tau_i\theta(1-x)), \quad \text{for } x \in (0,1) \text{ and } i \in \{1,2\}.$$

By a simple calculus, we obtain for $i \in \{1,2\}$,

$$\begin{aligned} L_A v_i(x) + p(x)g(v_i(x)) &= v_i''(x) + \frac{A'(x)}{A(x)}v_i'(x) + p(x)g(v_i(x)) \\ &= g(v_i(x))(1-x)^{-\lambda}L(1-x) \\ &\quad \times \Bigg[\tau_i\left(\tau_i(1-x)^{2-\lambda}L(1-x)g'(v_i(x)) + (2-\lambda)C_g\right) \\ &\quad + \tau_i\left(\frac{(x-1)A'(x)}{A(x)} - \alpha + \frac{(1-x)L'(1-x)}{L(1-x)}\right) \\ &\quad - \tau_i\left(\lambda + (2-\lambda)C_g - \alpha - 1\right) + p_i \\ &\quad + \left(\frac{p(x)}{(1-x)^{-\lambda}L(1-x)} - p_i\right)\Bigg]. \end{aligned}$$

So, for the fixed $\varepsilon > 0$, there exists $\delta_\varepsilon \in (0,1)$ such that for $x \in (\delta_\varepsilon, 1)$ and $i \in \{1,2\}$, we have

$$\left|\tau_i\left(\frac{(x-1)A'(x)}{A(x)} - \alpha + \frac{(1-x)L'(1-x)}{L(1-x)}\right)\right| \\ \leq \frac{3}{2}\xi_2\left(\left|\frac{(x-1)A'(x)}{A(x)} - \alpha\right| + \left|\frac{(1-x)L'(1-x)}{L(1-x)}\right|\right) \leq \frac{\varepsilon}{4},$$

$$p_1 - \frac{\varepsilon}{2} \leq \frac{p(x)}{(1-x)^{-\lambda}L(1-x)} \leq p_2 + \frac{\varepsilon}{2}$$

and

$$\left|\tau_i\left(\tau_i(1-x)^{2-\lambda}L(1-x)g'(v_i(x)) + (2-\lambda)C_g\right)\right| \\ \leq \frac{3}{2}\xi_2\left|\tau_i(1-x)^{2-\lambda}L(1-x)g'(v_i(x)) + (2-\lambda)C_g\right| \leq \frac{\varepsilon}{4}.$$

Indeed, the last inequality follows from (1.5) and the fact that from Lemmas 2.2 and 2.3, we have $\lim_{x\to 1}\frac{(1-x)^{2-\lambda}L(1-x)}{\theta(1-x)} = 2-\lambda$. This implies that for each $x \in (\delta_\varepsilon, 1)$, we have

$$L_A v_1(x) + p(x)g(v_1(x)) \\ \geq g(v_1(x))(1-x)^{-\lambda}L(1-x)\left[-\varepsilon + p_1 - \tau_1\left(\lambda + (2-\lambda)C_g - \alpha - 1\right)\right] = 0$$

and

$$L_A v_2(x) + p(x)g(v_2(x)) \\ \leq g(v_2(x))(1-x)^{-\lambda}L(1-x)\left[\varepsilon + p_2 - \tau_2\left(\lambda + (2-\lambda)C_g - \alpha - 1\right)\right] = 0.$$

Let $u \in C([0,1]) \cap C^2((0,1))$ be the unique solution of (1.1)–(1.2) (see [15, Theorem 5]). Then, there exists $M > 0$ such that

$$v_1(\delta_\varepsilon) - Mv_0(\delta_\varepsilon) \leq u(\delta_\varepsilon) \leq v_2(\delta_\varepsilon) + Mv_0(\delta_\varepsilon). \tag{3.4}$$

We claim that

$$v_1(x) - Mv_0(x) \leq u(x) \leq v_2(x) + Mv_0(x) \quad \text{for each } x \in (\delta_\varepsilon, 1). \tag{3.5}$$

Assume for instance that the left inequality of (3.5) is not true. Then, there exists $x_0 \in (\delta_\varepsilon, 1)$ such that

$$v_1(x_0) - Mv_0(x_0) - u(x_0) > 0.$$

By (3.4), the continuity of the functions v_1, v_0 and u on $[\delta_\varepsilon, 1)$ and that $\lim_{x\to 1} v_1(x) = \lim_{x\to 1} v_0(x) = \lim_{x\to 1} u(x) = 0$, we deduce that there exists $x_1 \in (\delta_\varepsilon, 1)$ such that

$$0 < v_1(x_1) - Mv_0(x_1) - u(x_1) = \max_{x\in[\delta_\varepsilon,1]}\left(v_1(x) - Mv_0(x) - u(x)\right).$$

This implies that $v_1'(x_1) - Mv_0'(x_1) - u'(x_1) = 0$ and

$$L_A(v_1 - Mv_0 - u)(x_1) = (v_1 - Mv_0 - u)''(x_1) \leq 0.$$

On the other hand, using the fact that p is nonnegative and the monotonicity of g, we obtain

$$\begin{aligned} L_A(v_1 - Mv_0 - u)(x_1) &= L_A(v_1)(x_1) + \frac{M}{A(x_1)} - L_A u(x_1) \\ &\geq p(x_1)\left[g(u(x_1)) - g\left(v_1\left(x_1\right)\right)\right] + \frac{M}{A(x_1)} \\ &\geq p(x_1)\left[g(u(x_1) + Mv_0(x_1)) - g(v_1(x_1))\right] + \frac{M}{A(x_1)} \\ &> 0. \end{aligned}$$

This yields to a contradiction. In the same way, we prove the right inequality of (3.5). Now, since $\psi_g \circ \theta \in \mathrm{NRVZ}_{(2-\lambda)(1-C_g)}$, there exists $\widehat{L} \in \mathcal{K}$ such that $\psi_g(\theta(t)) = t^{(2-\lambda)(1-C_g)}\widehat{L}(t)$ for $t \in (0, \eta)$. Moreover, since $\lambda + (2-\lambda)C_g - \alpha - 1 > 0$, it follows by Proposition 2.1 that $\lim_{t\to 0} \frac{t^{1-\alpha}}{t^{(2-\lambda)(1-C_g)}\widehat{L}(t)} = 0$. This implies that

$$\lim_{t\to 0} \frac{t^{1-\alpha}}{\psi_g(\tau_i \int_0^t s^{1-\lambda} L(s)\,ds)} = \lim_{t\to 0} \frac{t^{1-\alpha}}{\psi_g(\tau_i \theta(t))} = \lim_{t\to 0} \frac{\psi_g(\theta(t))}{\psi_g(\tau_i\ \theta(t)} \frac{t^{1-\alpha}}{\psi_g(\theta(t))} = 0$$

uniformly in $\tau_i \in [\frac{\xi_1}{2}, \frac{3\xi_2}{2}] \subset (0, \infty)$.

This together with (3.1) and Proposition 2.1 implies that

$$\lim_{x\to 1} \frac{v_0(x)}{\psi_g(\tau_1\theta(1-x))} = \lim_{x\to 1} \frac{v_0(x)}{\psi_g(\tau_2\theta(1-x))} = 0.$$

So, we get by (3.5)

$$\limsup_{x\to 1} \frac{u(x)}{v_2(x)} \leq 1 \leq \liminf_{x\to 1} \frac{u(x)}{v_1(x)}.$$

Using this fact and assertions (iv), (i) and (vi) of Lemma 2.6, we deduce that

$$\liminf_{x\to 1} \frac{u(x)}{\psi_g(\theta(1-x))} \geq \liminf_{x\to 1} \frac{u(x)}{v_1(x)} \frac{v_1(x)}{\psi_g(\theta(1-x))} \geq \liminf_{x\to 1} \frac{\psi_g(\tau_1\theta(1-x))}{\psi_g(\theta(1-x))} = \tau_1^{1-C_g}.$$

By letting ε to zero, we obtain that $\xi_1^{1-C_g} \leq \liminf_{x\to 1} \frac{u(x)}{\psi_g(\theta(1-x))}$.

Similarly, we obtain that $\limsup_{x\to 1} \frac{u(x)}{\psi_g(\theta(1-x))} \leq \xi_2^{1-C_g}$.

This proves in particular assertion (ii) of Theorem 1.3. Now, for $\lambda < 2$ we have by Lemma 2.2

$$\theta(1-x) \underset{x\to 1}{\sim} \frac{(1-x)^{2-\lambda} L(1-x)}{2-\lambda}.$$

Hence it follows by assertions (iv), (i) and (vi) of Lemma 2.6 that for $\lambda < 2$, we have

$$\begin{aligned} \lim_{x\to 1} \frac{\psi_g(\theta(1-x))}{\psi_g((1-x)^{2-\lambda} L(1-x))} &= \lim_{x\to 1} \frac{\psi_g(\theta(1-x))}{\psi_g((2-\lambda)\theta(1-x))} \frac{\psi_g((2-\lambda)\theta(1-x))}{\psi_g((1-x)^{2-\lambda} L(1-x))} \\ &= \frac{1}{(2-\lambda)^{1-C_g}}. \end{aligned}$$

□

Proof of Theorem 1.6. We recall that $g(t) = t^{-\gamma}$ with $\gamma \geq 0$ and $\lambda = \alpha + 1 + (\alpha - 1)\gamma$. In this case $C_g = \frac{\gamma}{\gamma+1}$ and $\lambda + (2-\lambda)C_g - \alpha - 1 = 0$. Let $L \in \mathcal{K}$ be the function given in hypothesis (H_6). Put

$$k(t) = \int_t^{\eta} \frac{L(s)}{s}\, ds \quad \text{and} \quad v_i(x) = \left((1+\gamma)\tau_i \int_0^{1-x} s^{\gamma-\alpha(1+\gamma)} k(s)\, ds \right)^{1/(\gamma+1)} \quad \text{for } i \in \{1,2\}.$$

Then we have

$$\begin{aligned} L_A v_i(x) + p(x) g(v_i(x)) &= v_i''(x) + \frac{A'(x)}{A(x)} v_i'(x) + p(x) g(v_i(x)) \\ &= g(v_i(x))(1-x)^{\gamma-\alpha(1+\gamma)-1} L(1-x) \\ &\quad \times \Bigg[\tau_i \frac{k(1-x)}{L(1-x)} \left(\tau_i (1-x)^{\gamma-\alpha(1+\gamma)+1} k(1-x) g'(v_i(x)) + \gamma(1-\alpha) \right) \\ &\quad + \tau_i \frac{k(1-x)}{L(1-x)} \left(\frac{(x-1)A'(x)}{A(x)} - \alpha \right) \\ &\quad - \tau_i + p_i + \left(\frac{p(x)}{(1-x)^{\gamma-\alpha(1+\gamma)-1} L(1-x)} - p_i \right) \Bigg] \\ &= g(v_i(x))(1-x)^{\gamma-\alpha(1+\gamma)-1} L(1-x) \\ &\quad \times \Bigg[\tau_i \left(\frac{k(1-x)}{L(1-x)} \left(\tau_i (1-x)^{\gamma-\alpha(1+\gamma)+1} k(1-x) g'(v_i(x)) + \gamma(1-\alpha) \right) - \frac{\gamma}{\gamma+1} \right) \\ &\quad + \tau_i \frac{k(1-x)}{L(1-x)} \left(\frac{(x-1)A'(x)}{A(x)} - \alpha \right) \\ &\quad + \frac{\gamma}{\gamma+1} \tau_i - \tau_i + p_i + \left(\frac{p(x)}{(1-x)^{\gamma-\alpha(1+\gamma)-1} L(1-x)} - p_i \right) \Bigg] \\ &= g(v_i(x))(1-x)^{\gamma-\alpha(1+\gamma)-1} L(1-x) \\ &\quad \times \Bigg[\tau_i \left(\frac{k(1-x)}{L(1-x)} \left(\tau_i (1-x)^{\gamma-\alpha(1+\gamma)+1} k(1-x) g'(v_i(x)) + \gamma(1-\alpha) \right) - \frac{\gamma}{\gamma+1} \right) \\ &\quad + \tau_i \frac{k(1-x)}{L(1-x)} \left(\frac{(x-1)B'(x)}{B(x)} \right) \\ &\quad - \frac{\tau_i}{\gamma+1} + p_i + \left(\frac{p(x)}{(1-x)^{\gamma-\alpha(1+\gamma)-1} L(1-x)} - p_i \right) \Bigg]. \end{aligned}$$

Since $g(t) = t^{-\gamma}$, then, by integration by parts, we obtain

$$\begin{aligned} &\tau_i (1-x)^{\gamma-\alpha(1+\gamma)+1} k(1-x) g'(v_i(x)) + \gamma(1-\alpha) \\ &\quad = -\gamma \tau_i (1-x)^{\gamma-\alpha(1+\gamma)+1} k(1-x) (v_i(x))^{-(1+\gamma)} + \gamma(1-\alpha) \\ &\quad = \gamma \left((1-\alpha) - \frac{(1-x)^{\gamma-\alpha(1+\gamma)+1} k(1-x)}{(1+\gamma) \int_0^{1-x} s^{\gamma-\alpha(1+\gamma)} k(s)\, ds} \right) \\ &\quad = \gamma \left(\frac{(1-\alpha)(1+\gamma) \int_0^{1-x} s^{\gamma-\alpha(1+\gamma)} k(s)\, ds - (1-x)^{\gamma-\alpha(1+\gamma)+1} k(1-x)}{(1+\gamma) \int_0^{1-x} s^{\gamma-\alpha(1+\gamma)} k(s)\, ds} \right) \\ &\quad = \gamma \left(\frac{-\int_0^{1-x} s^{\gamma-\alpha(1+\gamma)+1} k'(s)\, ds}{(1+\gamma) \int_0^{1-x} s^{\gamma-\alpha(1+\gamma)} k(s)\, ds} \right) \\ &\quad = \frac{\gamma}{\gamma+1} \frac{\int_0^{1-x} s^{\gamma-\alpha(1+\gamma)} L(s)\, ds}{\int_0^{1-x} s^{\gamma-\alpha(1+\gamma)} k(s)\, ds}. \end{aligned}$$

On the other hand, by Remark 2.4, we have k in $\mathcal{K}$. This together with Lemma 2.2, implies that

$$\lim_{x\to 1}\frac{k(1-x)}{L(1-x)}\left(\tau_i(1-x)^{\gamma-\alpha(1+\gamma)+1}k(1-x)g'(v_i(x))+\gamma(1-\alpha)\right)-\frac{\gamma}{\gamma+1}=0.$$

Now since $\frac{(1-x)^{\zeta}B'(x)}{B(x)}$ is bounded in a neighborhood of 1 and by Proposition 2.1, we have $\frac{k}{L}\in\mathcal{K}$ and $\lim_{x\to 1}\frac{(1-x)^{1-\zeta}k(1-x)}{L(1-x)}=0$, we deduce that

$$\lim_{x\to 1}\frac{k(1-x)}{L(1-x)}\left(\frac{(1-x)B'(x)}{B(x)}\right)=\lim_{x\to 1}\frac{(1-x)^{1-\zeta}k(1-x)}{L(1-x)}\left(\frac{(1-x)^{\zeta}B'(x)}{B(x)}\right)=0.$$

Let $\varepsilon\in(0,\frac{p_1}{2})$ and put $\tau_1=(\gamma+1)(p_1-\varepsilon)$ and $\tau_2=(\gamma+1)(p_2+\varepsilon)$.

So, for the fixed $\varepsilon>0$, there exists $\delta_\varepsilon\in(0,1)$ such that for $x\in(\delta_\varepsilon,1)$, we have

$$\begin{aligned}&L_Av_1(x)+p(x)g(v_1(x))\\&\quad\geq g(v_1(x))(1-x)^{\gamma-\alpha(1+\gamma)-1}L(1-x)\left[-\frac{\varepsilon}{3}-\frac{\varepsilon}{3}-\frac{\tau_1}{\gamma+1}+p_1-\frac{\varepsilon}{3}\right]=0\end{aligned}$$

and

$$\begin{aligned}&L_Av_2(x)+p(x)g(v_2(x))\\&\quad\leq g(v_2(x))(1-x)^{\gamma-\alpha(1+\gamma)-1}L(1-x)\left[\frac{\varepsilon}{3}+\frac{\varepsilon}{3}-\frac{\tau_2}{\gamma+1}+p_2+\frac{\varepsilon}{3}\right]=0.\end{aligned}$$

This implies that v_1 and v_2 are respectively a subsolution and a supersolution of the equation $-L_Au+p(x)u^{-\gamma}=0$ in $(\delta_\varepsilon,1)$.

Let $u\in C([0,1])\cap C^2((0,1)$ be the unique solution of (1.1)–(1.2) (see [15, Theorem 5]). As in the proof of Theorem 1.3, we choose $M>0$ such that

$$v_1-Mv_0\leq u\leq v_2+Mv_0\quad\text{in }(\delta_\varepsilon,1).$$

Moreover, thanks to assumption (H_6), we have $\lim_{t\to 0}k(t)=\infty$. So, using Lemma 2.2, we obtain for $i\in\{1,2\}$

$$\lim_{x\to 1}\frac{(1-x)^{1-\alpha}}{\left((1+\gamma)\tau_i\int_0^{1-x}s^{\gamma-\alpha(1+\gamma)}k(s)\,ds\right)^{\frac{1}{\gamma+1}}}=\lim_{x\to 1}\frac{(1-x)^{1-\alpha}}{(1-x)^{1-\alpha}\left(\frac{\tau_i}{1-\alpha}k(1-x)\right)^{\frac{1}{\gamma+1}}}=0.$$

This together with the fact that $v_0(x)\underset{x\to 1}{\sim}\frac{(1-x)^{1-\alpha}}{(1-\alpha)B(1)}$ gives that

$$\lim_{x\to 1}\frac{v_0(x)}{v_1(x)}=0=\lim_{x\to 1}\frac{v_0(x)}{v_2(x)}.$$

So we have $\limsup_{x\to 1}\frac{u(x)}{v_2(x)}\leq 1\leq\liminf_{x\to 1}\frac{u(x)}{v_1(x)}$. This implies that

$$\liminf_{x\to 1}\frac{u(x)}{\left((1+\gamma)\int_0^{1-x}s^{\gamma-\alpha(1+\gamma)}k(s)\,ds\right)^{\frac{1}{\gamma+1}}}\geq\tau_1^{\frac{1}{\gamma+1}}.$$

Since $\tau_1=(\gamma+1)(p_1-\varepsilon)$, then by letting ε tends to zero, we get

$$\liminf_{x\to 1}\frac{u(x)}{\left((1+\gamma)\int_0^{1-x}s^{\gamma-\alpha(1+\gamma)}k(s)\,ds\right)^{\frac{1}{\gamma+1}}}\geq((\gamma+1)p_1)^{\frac{1}{\gamma+1}}.$$

Similarly, we obtain that

$$\liminf_{x\to 1} \frac{u(x)}{\left((1+\gamma)\int_0^{1-x} s^{\gamma-\alpha(1+\gamma)}k(s)\,ds\right)^{\frac{1}{\gamma+1}}} \le ((\gamma+1)p_2)^{\frac{1}{\gamma+1}}.$$

On the other hand, since

$$(1+\gamma)\int_0^{1-x} s^{\gamma-\alpha(1+\gamma)}k(s)\,ds \underset{x\to 1}{\sim} \frac{(1-x)^{(1+\gamma)(1-\alpha)}}{(1-\alpha)}k(1-x) = \frac{(1-x)^{(1+\gamma)(1-\alpha)}}{(1-\alpha)}\int_{1-x}^{\eta}\frac{L(t)}{t}\,dt,$$

we deduce that

$$(b_1)^{\frac{1}{\gamma+1}} \le \liminf_{x\to 1}\frac{u(x)}{(1-x)^{1-\alpha}\left(\int_{1-x}^{\eta}\frac{L(t)}{t}\,dt\right)^{\frac{1}{\gamma+1}}} \le \limsup_{x\to 1}\frac{u(x)}{(1-x)^{1-\alpha}\left(\int_{1-x}^{\eta}\frac{L(t)}{t}\,dt\right)^{\frac{1}{\gamma+1}}} \le (b_2)^{\frac{1}{\gamma+1}},$$

where $b_1 = \frac{(\gamma+1)p_1}{1-\alpha}$ and $b_2 = \frac{(\gamma+1)p_2}{1-\alpha}$.

In particular, if $p_1 = p_2 = p_0$, then

$$\lim_{x\to 1}\frac{u(x)}{(1-x)^{1-\alpha}\left(\int_{1-x}^{\eta}\frac{L(t)}{t}\,dt\right)^{\frac{1}{\gamma+1}}} = \left(\frac{(\gamma+1)p_0}{1-\alpha}\right)^{\frac{1}{\gamma+1}}.$$

This completes the proof. □

4 Application

We consider the following singular problem

$$\begin{cases} -\frac{1}{A}(Au')' + \frac{\beta}{u}(u')^2 = p(x)g(u), & x\in(0,1), \\ u>0, \quad \text{in } (0,1), \\ \lim_{x\to 0^+}(Au')(x) = 0, \quad u(1)=0, \end{cases} \tag{4.1}$$

where $\beta < 1$ and $\lim_{x\to 1}\frac{p(x)}{(1-x)^{-\lambda}L(1-x)} = p_0$ with $\lambda \le 2$ and $L\in\mathcal{K}$ such that $\int_0^{\eta} s^{1-\lambda}L(s)\,ds < \infty$. We assume (H_1) and that g satisfies the following hypotheses.

$(\mathbf{A}_1)$ The function $t \to t^{-\beta}g(t)$ is nonincreasing, continuously differentiable form $(0,\infty)$ into $(0,\infty)$.

$(\mathbf{A}_2)$ $\lim_{t\to 0} g'(t)\int_0^t \frac{1}{g(s)}\,ds = -C_g$ with $\max(0, \frac{\beta}{\beta-1}) \le C_g \le 1$.

$(\mathbf{A}_3)$ $1-\alpha-(2-\lambda)(1-\beta)(1-C_g) > 0$.

Note that for $\gamma \ge 0$ and $-\gamma < \beta < 1$, the function $g(t) = t^{-\gamma}L_0(t)$, where $L_0 \in \mathcal{K}$, satisfies assumptions (A_1) and (A_2).

Put $u = v^{\frac{1}{1-\beta}}$. Then v satisfies

$$\begin{cases} -\frac{1}{A}(Av')' = (1-\beta)p(x)g(v^{\frac{1}{1-\beta}})v^{\frac{-\beta}{1-\beta}}, & x\in(0,1), \\ v>0, \quad \text{in } (0,1), \\ \lim_{x\to 0^+}(Av')(x) = 0, \quad v(1)=0, \end{cases} \tag{4.2}$$

The function $f(t) = (1-\beta)g(t^{\frac{1}{1-\beta}})t^{\frac{-\beta}{1-\beta}}$ is nonincreasing on $(0,\infty)$ and a simple computation shows that $\psi_g = (\psi_f)^{\frac{1}{1-\beta}}$ and

$$\lim_{t\to 0} f'(t)\int_0^t \frac{1}{f(s)}\,ds = (1-\beta)(1-C_g) - 1 =: -C_f \quad \text{with } 0 \le C_f \le 1.$$

Applying Corollary 1.4 to problem (4.2), we deduce that there exists a unique solution v to (4.2) such that

(a) if $C_f = 1$, then

(i) $\lim_{x\to 1} \frac{v(x)}{\psi_f((1-x)^{2-\lambda}L(1-x))} = 1$ if $\lambda < 2$;

(ii) $\lim_{x\to 1} \frac{v(x)}{\psi_f\left(\int_0^{1-x}\frac{L(t)}{t}\,dt\right)} = 1$ if $\lambda = 2$;

(b) if $C_f < 1$, then

(i) $\lim_{x\to 1} \frac{v(x)}{\psi_f((1-x)^{2-\lambda}L(1-x))} = \left(\frac{p_0}{(2-\lambda)\left(\lambda+(2-\lambda)C_f-\alpha-1\right)}\right)^{1-C_f}$ if $\lambda < 2$;

(ii) $\lim_{x\to 1} \frac{v(x)}{\psi_f\left(\int_0^{1-x}\frac{L(t)}{t}\,dt\right)} = \left(\frac{p_0}{1-\alpha}\right)^{1-C_f}$ if $\lambda = 2$.

This implies that

(a) If $C_g = 1$, then

(i) $\lim_{x\to 1} \frac{u(x)}{\psi_g((1-x)^{2-\lambda}L(1-x))} = 1$ if $\lambda < 2$;

(ii) $\lim_{x\to 1} \frac{u(x)}{\psi_g\left(\int_0^{1-x}\frac{L(t)}{t}\,dt\right)} = 1$ if $\lambda = 2$.

(b) If $C_g < 1$, then

(i) $\lim_{x\to 1} \frac{u(x)}{\psi_g((1-x)^{2-\lambda}L(1-x))} = \left(\frac{p_0}{(2-\lambda)\left(1-\alpha-(2-\lambda)(1-\beta)(1-C_g)\right)}\right)^{1-C_g}$ if $\lambda < 2$;

(ii) $\lim_{x\to 1} \frac{u(x)}{\psi_g\left(\int_0^{1-x}\frac{L(t)}{t}\,dt\right)} = \left(\frac{p_0}{1-\alpha}\right)^{1-C_g}$ if $\lambda = 2$.

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