Electronic Journal of Qualitative Theory of Differential Equations
2015, No. **27**, 1–18; http://www.math.u-szeged.hu/ejqtde/

A frictional contact problem with wear involving elastic-viscoplastic materials with damage and thermal effects

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Received 6 January 2015, appeared 25 May 2015
Communicated by Michal Fečkan

Abstract. We consider a mathematical problem for quasistatic contact between a thermo-elastic-viscoplastic body with damage and an obstacle. The contact is frictional and bilateral with a moving rigid foundation which results in the wear of the contacting surface. We employ the thermo-elastic-viscoplastic with damage constitutive law for the material. The damage of the material caused by elastic deformations. The evolution of the damage is described by an inclusion of parabolic type. The problem is formulated as a coupled system of an elliptic variational inequality for the displacement, a parabolic variational inequality for the damage and the heat equation for the temperature. We establish a variational formulation for the model and we prove the existence of a unique weak solution to the problem. The proof is based on a classical existence and uniqueness result on parabolic inequalities, differential equations and fixed point arguments.

Keywords: damage field, temperature, thermo-elastic-viscoplastic, variational inequality, wear.

2010 Mathematics Subject Classification: 74H20, 74H25, 74M15, 74F05, 74R20.

1 Introduction

Scientific research and recent papers in mechanics are articulated around two main components, one devoted to the laws of behavior and other devoted to boundary conditions imposed on the body. The boundary conditions reflect the binding of the body with the outside world.

Recent researches use coupled laws of behavior between mechanical and electric effects or between mechanical and thermal effects. For the case of coupled laws of behavior between mechanical and electric effects, general models for electro-elastic materials can be found in

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[18, 19], the study of an electro-viscoelastic body is considered in [13], a frictional contact problem for an electro elastic-viscopalastic body with damage is studied in [1]. For the case of coupled laws of behavior between mechanical and thermal effects, the transmission problem in thermo-viscoplasticity is studied in [16], contact problem with adhesion for thermo-elastic-viscoplastic is considered in [3], thermo-elastic-viscoplastic materials with damage for displacement-traction, and Neumann and Fourier boundary conditions was studied in [15].

Situations of contact between deformable bodies are very common in the industry and everyday life. Contact of braking pads with wheels, tires with roads, pistons with skirts or the complex metal forming processes are just a few examples. The constitutive laws with internal variables have been used in various publications in order to model the effect of internal variables in the behavior of real bodies like metals, rocks, polymers and so on, for which the rate of deformation depends on the internal variables. Some of the internal state variables considered by many authors are the spatial display of dislocation, the work-hardening of materials, the temperature and the damage field, see for example [1, 2, 11, 15–17] and references therein for the case of hardening, temperature and other internal state variables.

Wear is one of the processes which reduce the lifetime of modern machine elements. It represents the unwanted removal of materials from surfaces of contacting bodies occurring in relative motion. Wear arises when a hard rough surface slides against a softer surface, digs into it, and its asperities plough a series of grooves. When two surfaces come into contact, rearrangement of the surface asperities takes place. When they are in relative motion, some of the peaks will break, and therefore, the harder surface removes the softer material. This phenomenon involves the wear of the contacting surfaces. Material loss of wearing solids, the generation and circulation of free wear debris are the main effects of the wear process. The loose particles form a thin layer on the body surface. Tribological experiments show that this layer has a great influence on contact phenomena and the wear particles between sliding surface affect the frictional behavior. Realistically, wear cannot be totally eliminated.

Generally, a mathematical theory of friction and wear should be a generalization of experimental facts and it must be in agreement with the laws of thermodynamics of irreversible processes. The first attempts of a thermodynamical description of the friction and wear processes were provided in [12]. General models of quasi-static frictional contact with wear between deformable bodies were derived in [22] from thermodynamic considerations.

The aim of this paper is to study the coupling of a thermo-elastic-viscoplastic problem with damage and wear. For this, we consider a rate-type constitutive equation with two internal variables of the form

$$\boldsymbol{\sigma}(t) = \mathcal{A}(\varepsilon(\dot{\mathbf{u}}(t))) + \mathcal{B}(\varepsilon(\mathbf{u}(t)), \beta(t)) + \int_0^t \mathcal{G}\Big(\boldsymbol{\sigma}(s) - \mathcal{A}(\varepsilon(\dot{\mathbf{u}}(s))), \varepsilon(\mathbf{u}(s)), \theta(s)\Big) ds, \tag{1.1}$$

in which $\mathbf{u}$, $\boldsymbol{\sigma}$ represent, respectively, the displacement field and the stress field where the dot above denotes the derivative with respect to the time variable, θ represents the temperature, β is the damage field, $\mathcal{A}$ and $\mathcal{B}$ are nonlinear operators describing the purely viscous and the elastic properties of the material, respectively, and $\mathcal{G}$ is a nonlinear constitutive function which describes the visco-plastic behavior of the material. The differential inclusion used for the evolution of the damage field is

$$\dot{\beta} - k_1 \Delta\beta + \partial\varphi_K(\beta) \ni S(\varepsilon(\mathbf{u}), \beta),$$

where $\varphi_K(\beta)$ denotes the subdifferential of the indicator function of the set K of admissible damage functions defined by

$$K = \{\xi \in V : 0 \leqslant \xi(x) \leqslant 1 \text{ a.e. } x \in \Omega\},$$

and S is a given constitutive function which describes the sources of the damage in the system. When $\beta = 1$, the material is undamaged, when $\beta = 0$, the material is completely damaged, and for $0 < \beta < 1$ there is partial damage. General models of mechanical damage, which were derived from thermodynamical considerations and the principle of virtual work, can be found in [22] and references therein. The models describe the evolution of the material damage which results from the excess tension or compression in the body as a result of applied forces and tractions. Mathematical analysis of one-dimensional damage models can be found in [15].

Damage may be initiated and evolves in both the elastic and plastic deformation processes. Particularly, damage in the elastic deformation state is termed elastic damage which is the case for most brittle materials while damage in the plastic deformation state is termed plastic damage which is mainly for ductile materials. In this paper we use the damage caused by elastic deformations for mechanical and mathematical reasons. Mechanically we use elastic damage because the brittle materials are more susceptible to the damage and wear, mathematically it is easier to treat the case of an internal variable outside the integral compared to the case when the internal variable inside the integral term. The differential inclusion used for the evolution of the temperature field is

$$\dot{\theta} - k_0 \Delta\theta = \psi(\boldsymbol{\sigma}, \varepsilon(\dot{\mathbf{u}}), \theta) + q.$$

Dynamic and quasistatic contact problems are the topic of numerous papers, e.g. [1, 2, 13]. A model of damage coupled to wear was studied in [10]. However, the mathematical problem modelled the quasi-static evolution of damage in thermo-viscoplastic materials has been studied in [18], the dynamic evolution of damage in elastic-thermo-viscoplastic materials was studied in [15].

Most papers related with wear process use laws of behavior of mechanical kind or mechanical nature with electric effects. In this paper we deal the case of laws of behavior coupled between mechanical and thermal effects. In practice the thermal effect facilitates wear which makes this paper closer to the reality.

The paper is organized as follows. In Section 2 we introduce the notations and give some preliminaries. In Section 3 we present the mechanical problem, list the assumptions on the data, give the variational formulation of the problem. In Section 4 we state our main existence and uniqueness result Theorem 4.1.

2 Notations and preliminaries

In this short section, we present the notation we shall use and some preliminary material. For more details, we refer the reader to [8]. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$, while "$\cdot$ " and $\|\cdot\|$ denotes the absolute value if it is applied to a scalar or the Euclidean norm if it applied to a vector on $\mathbb{S}^d$ and $\mathbb{R}^d$, respectively.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with outer Lipschitz boundary Γ and let ν denote the unit outer normal on $\partial\Omega = \Gamma$. We shall use the notation

$$H = L^2(\Omega)^d = \left\{\mathbf{u} = (u_i) : u_i \in L^2(\Omega)\right\}, \qquad \mathcal{H} = \left\{\boldsymbol{\sigma} = \left(\sigma_{ij}\right) : \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\right\},$$
$$H^1(\Omega)^d = \{\mathbf{u} = (u_i) \in H : u_i \in H^1(\Omega)\}, \qquad \mathcal{H}_1 = \{\boldsymbol{\sigma} \in \mathcal{H} : \operatorname{Div} \boldsymbol{\sigma} \in H\}.$$

Here $\varepsilon\colon H^1(\Omega)^d \to \mathcal{H}$ and $\operatorname{Div}\colon \mathcal{H}_1 \to H$ are the deformation and divergence operators, respectively, defined by

$$\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \qquad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \qquad \mathrm{Div}\,\boldsymbol{\sigma} = (\sigma_{ij,j}).$$

Here and below, the indices i and j run from 1 to d, the summation convention over repeated indices is used and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable. The spaces H, $\mathcal{H}$, $H^1(\Omega)^d$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by:

$$(\mathbf{u},\mathbf{v})_H = \int_\Omega u_i v_i\,dx, \quad \mathbf{u},\mathbf{v} \in H, \qquad (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij}\,dx, \quad \forall \boldsymbol{\sigma},\boldsymbol{\tau} \in \mathcal{H},$$
$$(\mathbf{u},\mathbf{v})_{H^1(\Omega)^d} = \int_\Omega \mathbf{u}.\mathbf{v}\,dx + \int_\Omega \nabla\mathbf{u}.\nabla\mathbf{v}\,dx, \quad \forall \mathbf{u},\mathbf{v} \in H^1(\Omega)^d,$$

where

$$\nabla\mathbf{v} = (v_{i,j}), \qquad \forall \mathbf{v} \in H^1(\Omega)^d,$$
$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} + (\mathrm{Div}\,\boldsymbol{\sigma}, \mathrm{Div}\,\boldsymbol{\tau})_H, \qquad \forall \boldsymbol{\sigma},\boldsymbol{\tau} \in \mathcal{H}_1.$$

The associated norms are denoted by $\|\cdot\|_H$, $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{H^1}$ and $\|\cdot\|_{\mathcal{H}_1}$, respectively. Let $H_\Gamma = (H^{1/2}(\Gamma))^d$ and $\gamma\colon H^1(\Gamma))^d \to H_\Gamma$ be the trace map. For every element $\mathbf{v} \in H^1(\Omega)^d$, we also use the notation $\mathbf{v}$ to denote the trace map $\gamma\mathbf{v}$ of $\mathbf{v}$ on Γ, and we denote by v_ν and $\mathbf{v}_\tau$ the normal and tangential components of $\mathbf{v}$ on Γ given by

$$v_\nu = \mathbf{v}\cdot\nu, \qquad \mathbf{v}_\tau = \mathbf{v} - v_\nu \nu. \tag{2.1}$$

Similarly, for a regular (say $\mathcal{C}^1$) tensor field $\boldsymbol{\sigma} : \Omega \to \mathbb{S}^d$ we define its normal and tangential components by

$$\sigma_\nu = (\boldsymbol{\sigma}\nu)\cdot\boldsymbol{\nu}, \qquad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\nu - \sigma_\nu\boldsymbol{\nu}, \tag{2.2}$$

and for all $\boldsymbol{\sigma} \in \mathcal{H}_1$ the following Green's formula holds

$$(\boldsymbol{\sigma}, \varepsilon(\mathbf{v}))_{\mathcal{H}} + (\mathrm{Div}\,\boldsymbol{\sigma}, \mathbf{v})_H = \int_\Gamma \boldsymbol{\sigma}\nu.\mathbf{v}\,da, \qquad \forall \mathbf{v} \in H^1(\Omega)^d.$$

We recall the following standard result for parabolic variational inequalities used in Section 4 (see [4, p. 124]).

Let V and H be real Hilbert spaces such that V is dense in H and the injection map is continuous. The space H is identified with its own dual and with a subspace of the dual V' of V. We write

$$V \subset H \subset V',$$

and we say that the inclusions above define a Gelfand triple. We denote by $\|\cdot\|_V$, $\|\cdot\|_H$ and $\|\cdot\|_{V'}$, the norms on the spaces V, H and V' respectively, and we use $(\cdot,\cdot)_{V'\times V}$ for the duality pairing between V' and V. Note that if $f \in H$ then

$$(f, \mathbf{v})_{V'\times V} = (f, \mathbf{v})_H, \qquad \forall \mathbf{v} \in H.$$

Theorem 2.1. *Let $V \subset H \subset V'$ be a Gelfand triple. Let K be a nonempty, closed, and convex set of V. Assume that $\mathfrak{a}(\cdot,\cdot)\colon V \times V \to \mathbb{R}$ is a continuous and symmetric bilinear form such that for some constants $\zeta > 0$ and c_0,*

$$\mathfrak{a}(\mathbf{v},\mathbf{v}) = c_0\|\mathbf{v}\|_H^2 \geqslant \zeta\|\mathbf{v}\|_V^2, \qquad \forall \mathbf{v} \in H.$$

Then, for every $\mathbf{u}_0 \in K$ *and* $f \in L^2(0,T;H)$*, there exists a unique function* $\mathbf{u} \in H^1(0,T;H) \cap L^2(0,T;V)$ *such that* $\mathbf{u}(0) = \mathbf{u}_0$*,* $\mathbf{u}(t) \in K$ *for all* $t \in [0,T]$*, and for almost all* $t \in (0,T)$*,*

$$(\dot{\mathbf{u}}(t), \mathbf{v} - \mathbf{u}(t))_{V' \times V} + \mathfrak{a}(\mathbf{u}(t), \mathbf{v} - \mathbf{u}(t)) \geqslant (f(t), \mathbf{v} - \mathbf{u}(t))_H, \qquad \forall \mathbf{v} \in K,$$

Finally, for any real Hilbert space X, we use the classical notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, where $1 \leqslant p \leqslant \infty$ and $k > 1$. For $T > 0$ we denote by $\mathcal{C}(0,T;X)$ and $\mathcal{C}^1(0,T;X)$ the space of continuous and continuously differentiable functions from $[0,T]$ to X, respectively, with the norms

$$\|\mathbf{f}\|_{\mathcal{C}(0,T;X)} = \max_{t \in [0,T]} \|\mathbf{f}(t)\|_X,$$
$$\|\mathbf{f}\|_{\mathcal{C}^1(0,T;X)} = \max_{t \in [0,T]} \|\mathbf{f}(t)\|_X + \max_{t \in [0,T]} \|\dot{\mathbf{f}}(t)\|_X,$$

respectively. Moreover, we use the dot above to indicate the derivative with respect to the time variable and if X_1 and X_2 are real Hilbert spaces, then $X_1 \times X_2$ denotes the product Hilbert space endowed with the canonical inner product $(\cdot,\cdot)_{X_1 \times X_2}$.

The mechanical problem may be formulated as follows.

3 Mechanical and variational formulations

The physical setting is the following. A body occupies the domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with outer Lipschitz surface Γ. The body undergoes the action of body forces of density f_0 and external heat source q. It also undergoes the mechanical and thermal constraint on the boundary. We consider a partition of Γ into three disjoint parts Γ_1, Γ_2 and Γ_3. We assume that $\text{meas}(\Gamma_1) > 0$. Let $T > 0$ and let $[0,T]$ be the time interval of interest.

We admit a possible external heat source applied in $\Omega \times (0,T)$, given by the function q. The body is clamped on $\Gamma_1 \times (0,T)$ and so the displacement field vanishes there. Surface tractions of density f_2 act on $\Gamma_2 \times (0,T)$ and a volume force of density f_0 is applied in $\Omega \times (0,T)$. Finally, on the part Γ_3 the body may come into frictional and bilateral contact with a moving rigid foundation which results in the wear of the contacting surface. We suppose that the body forces and tractions vary slowly in time, and therefore, the accelerations in the system may be neglected. Neglecting the inertial terms in the equation of motion leads to a quasistatic approach to the process. For the body we use a thermo-elastic-viscoplastic constitutive law with damage given by (1.1) to model the material's behavior.

We now briefly describe the boundary conditions on the contact surface Γ_3, based on the model derived in [22]. We introduce the wear function $w\colon \Gamma_3 \times [0,T] \to \mathbb{R}^+$ which measures the wear of the surface. The wear is identified as the normal depth of the material that is lost. Since the body is in bilateral contact with the foundation, it follows that

$$u_\nu = -w \quad \text{on } \Gamma_3. \tag{3.1}$$

Thus the location of the contact evolves with the wear. We point out that the effect of the wear is the recession on Γ_3 and therefore, it is natural to expect that $u_\nu \leqslant 0$ on Γ_3, which implies $w \geqslant 0$ on Γ_3.

The evolution of the wear of the contacting surface is governed by a simplified version of Archard's law (see [22]) which we now describe. The rate form of Archard's law is

$$\dot{w} = -k\sigma_\nu \|\dot{\mathbf{u}}_\tau - \mathbf{v}^*\|,$$

where $k > 0$ is a wear coefficient, $\mathbf{v}^*$ is the tangential velocity of the foundation and $\|\dot{\mathbf{u}}_\tau - \mathbf{v}^*\|$ represents the slip speed between the contact surface and the foundation.

We see that the rate of wear is assumed to be proportional to the contact stress and the slip speed. For the sake of simplicity we assume in the rest of the section that the motion of the foundation is uniform, i.e., $\mathbf{v}^*$ does not vary in time. Denote $v^* = \|\mathbf{v}^*\| > 0$.

We assume that v^* is large so that we can neglect in the sequel $\dot{\mathbf{u}}_\tau$ compared with $\mathbf{v}^*$ to obtain the following version of Archard's law

$$\dot{w} = -kv^*\sigma_\nu. \tag{3.2}$$

The use of the simplified law (3.2) for the evolution of the wear avoids some mathematical difficulties in the study of the quasistatic thermo-elastic-viscoplastic contact problem.

We can now eliminate the unknown function w from the problem. In this manner, the problem decouples, and once the solution of the frictional contact problem has been obtained, the wear of the surface can be obtained by integration of (3.2). Let $\zeta = kv^*$ and $\alpha = \frac{1}{\zeta}$. Using (3.1) and (3.2) we have

$$\sigma_\nu = \alpha\dot{u}_\nu. \tag{3.3}$$

We model the frictional contact between the thermo-elastic-viscoplastic body and the foundation with Coulomb's law of dry friction. Since there is only sliding contact, it

$$\|\sigma_\tau\| = \mu\|\sigma_\nu\|, \qquad \sigma_\tau = -\lambda(\dot{\mathbf{u}}_\tau - \mathbf{v}^*), \qquad \lambda \geqslant 0, \tag{3.4}$$

where $\mu > 0$ is the coefficient of friction. These relations set constraints on the evolution of the tangential stress; in particular, the tangential stress is in the direction opposite to the relative sliding velocity $\|\dot{\mathbf{u}}_\tau - \mathbf{v}^*\|$.

Naturally, the wear increases in time, i.e. $\dot{w} \geqslant 0$. Hence, it follows from (3.1) and (3.2) that $\dot{u}_\nu \leqslant 0$ and $\sigma_\nu \leqslant 0$ on Γ_3. Thus, the conditions (3.3) and (3.4) imply

$$-\sigma_\nu = \alpha\|\dot{u}_\nu\|, \qquad \|\sigma_\tau\| = -\mu\sigma_\nu, \qquad \sigma_\tau = -\lambda(\dot{\mathbf{u}}_\tau - \mathbf{v}^*), \qquad \lambda \geqslant 0. \tag{3.5}$$

To simplify the notation, we do not indicate explicitly the dependence of various functions on the variables $\mathbf{x} \in \Omega \cup \Gamma$ and $t \in [0, T]$. Then, the classical formulation of the mechanical problem of a frictional bilateral contact with wear may be stated as follows.

Problem $\mathcal{P}$

Find the displacement field $\mathbf{u}\colon \Omega \times [0, T] \to \mathbb{R}^d$, the stress field $\sigma\colon \Omega \times [0, T] \to \mathbb{S}^d$, the damage field $\beta : \Omega \times [0, T] \to \mathbb{R}$ and the temperature $\theta : \Omega \times [0, T] \to \mathbb{R}$ such that

$$\begin{aligned}\sigma(t) &= \mathcal{A}(\varepsilon(\dot{\mathbf{u}}(t))) + \mathcal{B}(\varepsilon(\mathbf{u}(t)), \beta(t)) \\ &\quad + \int_0^t \mathcal{G}\Big(\sigma(s) - \mathcal{A}(\varepsilon(\dot{\mathbf{u}}(s))), \varepsilon(\mathbf{u}(s)), \theta(s)\Big)\, ds \quad \text{in } \Omega \text{ a.e. } t \in (0, T),\end{aligned} \tag{3.6}$$

$$\operatorname{Div}\sigma + \mathbf{f}_0 = 0 \quad \text{in } \Omega \times (0, T), \tag{3.7}$$

$$\dot{\theta} - k_0\Delta\theta = \psi(\sigma, \varepsilon(\dot{\mathbf{u}}), \theta) + q \quad \text{in } \Omega \times (0, T), \tag{3.8}$$

$$\dot{\beta} - k_1\Delta\beta + \partial\varphi_K(\beta) \ni S(\varepsilon(\mathbf{u}), \beta) \quad \text{in } \Omega \times (0, T), \tag{3.9}$$

$$\mathbf{u} = 0 \quad \text{on } \Gamma_1 \times (0, T), \tag{3.10}$$

$$\sigma\nu = \mathbf{f}_2 \quad \text{on } \Gamma_2 \times (0, T), \tag{3.11}$$

$$\begin{cases} \sigma_\nu = -\alpha\|\dot{u}_\nu\|, \quad \|\boldsymbol{\sigma}_\tau\| = -\mu\sigma_\nu \\ \boldsymbol{\sigma}_\tau = -\lambda\left(\dot{\mathbf{u}}_\tau - \mathbf{v}^*\right), \quad \lambda \geqslant 0, \end{cases} \quad \text{on } \Gamma_3 \times (0,T), \tag{3.12}$$

$$k_0 \frac{\partial \theta}{\partial \nu} + B\theta = 0 \quad \text{on } \Gamma \times (0,T), \tag{3.13}$$

$$\frac{\partial \beta}{\partial \nu} = 0 \quad \text{on } \Gamma \times (0,T), \tag{3.14}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \qquad \theta(0) = \theta_0, \qquad \beta(0) = \beta_0 \quad \text{in } \Omega. \tag{3.15}$$

Here, equation (3.6) is the thermo-elastic-viscoplastic constitutive law with damage introduced in the first section. Equation (3.7) represents is the steady equation for the stress field. Equation (3.8) represents the energy conservation where ψ is a nonlinear constitutive function which represents the heat generated by the work of internal forces and q is a given volume heat source. Inclusion (3.9) describes the evolution of damage field, governed by the source damage function ϕ, where $\partial_K \varphi(\varsigma)$ is the subdifferential of indicator function of the set K of admissible damage functions.

Equalities (3.10) and (3.11) are the displacement-traction boundary conditions, respectively. (3.12) describes the frictional bilateral contact with wear described above on the potential contact surface Γ_3. (3.13), (3.14) represent, respectively on Γ, a Fourier boundary condition for the temperature and a homogeneous Neumann boundary condition for the damage field on Γ. The functions $\mathbf{u}_0$, θ_0 and β_0 in (3.15) are the initial data.

In the study of the mechanical problem $\mathcal{P}$, we consider the following assumptions.

The *viscosity operator* $\mathcal{A}\colon \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\begin{cases} \text{(a)} & \text{There exists } L_{\mathcal{A}} > 0 \text{ such that} \\ & \|\mathcal{A}(\mathbf{x},\boldsymbol{\varepsilon}_1) - \mathcal{A}(\mathbf{x},\boldsymbol{\varepsilon}_2)\| \leqslant L_{\mathcal{A}}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \text{ for all } \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(b)} & \text{There exists } m_{\mathcal{A}} > 0 \text{ such that} \\ & (\mathcal{A}(\mathbf{x},\boldsymbol{\varepsilon}_1) - \mathcal{A}(\mathbf{x},\boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geqslant m_{\mathcal{A}}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \\ & \text{for all } \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(c)} & \text{The mapping } \mathbf{x} \mapsto \mathcal{A}(\mathbf{x},\boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega, \text{ for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ \text{(d)} & \text{The mapping } \mathbf{x} \mapsto \mathcal{A}(\mathbf{x},\mathbf{0}) \text{ belongs to } \mathcal{H}. \end{cases} \tag{3.16}$$

The *elasticity operator* $\mathcal{B}\colon \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$ satisfies

$$\begin{cases} \text{(a)} & \text{There exists } L_{\mathcal{B}} > 0 \text{ such that} \\ & \|\mathcal{B}(\mathbf{x},\boldsymbol{\varepsilon}_1,\alpha_1) - \mathcal{B}(\mathbf{x},\boldsymbol{\varepsilon}_2,\alpha_2)\| \leqslant L_{\mathcal{B}}(\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| + \|\alpha_1 - \alpha_2\|) \\ & \text{for all } \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ for all } \alpha_1,\alpha_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(b)} & \text{The mapping } \mathbf{x} \mapsto \mathcal{B}(\mathbf{x},\boldsymbol{\varepsilon},\alpha) \text{ is Lebesque measurable on } \Omega, \\ & \text{for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d \text{ and } \alpha \in \mathbb{R}. \\ \text{(c)} & \text{The mapping } \mathbf{x} \mapsto \mathcal{B}(\mathbf{x},0,0) \text{ belongs to } \mathcal{H}. \end{cases} \tag{3.17}$$

The *plasticity operator* $\mathcal{G}\colon \Omega\times\mathbb{S}^d\times\mathbb{S}^d\times\mathbb{R}\to\mathbb{S}^d$ satisfies

$$\begin{cases} \text{(a)} & \text{There exists a constant } L_{\mathcal{G}}>0 \text{ such that} \\ & \|\mathcal{G}(x,\boldsymbol{\sigma}_1,\boldsymbol{\varepsilon}_1,\theta_1)-\mathcal{G}(x,\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_2,\theta_2)\|\leqslant L_{\mathcal{G}}(\|\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2\|+\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|+\|\theta_1-\theta_2\|) \\ & \forall\ \boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2\in\mathbb{S}^d,\ \forall\ \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d,\ \forall\theta_1,\theta_2\in\mathbb{R} \text{ a.e. } \mathbf{x}\in\Omega. \\ \text{(b)} & \text{The mapping } \mathbf{x}\to\mathcal{G}(\mathbf{x},\boldsymbol{\sigma},\boldsymbol{\varepsilon},\theta) \text{ is Lebesgue measurable on } \Omega, \\ & \text{for all } \boldsymbol{\sigma},\boldsymbol{\varepsilon}\in\mathbb{S}^d,\ \text{for all } \theta\in\mathbb{R}. \\ \text{(c)} & \text{The mapping } \mathbf{x}\to\mathcal{G}(x,0,0,0)\in\mathcal{H}. \end{cases} \tag{3.18}$$

The *nonlinear constitutive function* $\psi\colon \Omega\times\mathbb{S}^d\times\mathbb{S}^d\times\mathbb{R}\to\mathbb{R}$ satisfies

$$\begin{cases} \text{(a)} & \text{There exists a constant } L_{\psi}>0 \text{ such that} \\ & \|\psi(x,\boldsymbol{\sigma}_1,\varepsilon_1,\theta_1)-\psi(x,\boldsymbol{\sigma}_2,\varepsilon_2,\theta_2)\|\leqslant L_{\psi}(\|\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2\|+\|\varepsilon_1-\varepsilon_2\|+\|\theta_1-\theta_2\|) \\ & \forall\ \boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2\in\mathbb{S}^d,\ \forall\ \varepsilon_1,\varepsilon_2\in\mathbb{S}^d,\ \forall\theta_1,\theta_2\in\mathbb{R} \text{ a.e. } \mathbf{x}\in\Omega. \\ \text{(b)} & \text{The mapping } \mathbf{x}\to\psi(x,\boldsymbol{\sigma},\varepsilon,\theta) \text{ is Lebesgue measurable on } \Omega, \\ & \text{for all } \boldsymbol{\sigma},\boldsymbol{\varepsilon}\in\mathbb{S}^d,\ \text{for all } \theta\in\mathbb{R}. \\ \text{(c)} & \text{The mapping } \mathbf{x}\to\psi(x,0,0,0)\in L^2(\Omega). \end{cases} \tag{3.19}$$

The *damage source function* $S\colon \Omega\times\mathbb{S}^d\times\mathbb{R}\to\mathbb{R}$ satisfies

$$\begin{cases} \text{(a)} & \text{There exists a constant } M_S>0 \text{ such that} \\ & \|S(\mathbf{x},\varepsilon_1,\alpha_1)-S(\mathbf{x},\varepsilon_2,\alpha_2)\|\leqslant M_S(\|\varepsilon_1-\varepsilon_2\|+\|\alpha_1-\alpha_2\|) \\ & \text{for all } \varepsilon_1,\varepsilon_2\in\mathbb{S}^d,\ \text{for all } \alpha_1,\alpha_2\in\mathbb{R},\ \text{a.e. } \mathbf{x}\in\Omega. \\ \text{(b)} & \text{for all } \boldsymbol{\varepsilon}\in\mathbb{S}^d,\ \alpha\in\mathbb{R},\mathbf{x}\mapsto S(\mathbf{x},\boldsymbol{\varepsilon},\alpha) \text{ is Lebesgue measurable on } \Omega. \\ \text{(c)} & \text{The mapping } \mathbf{x}\mapsto S(\mathbf{x},0,0) \text{ belongs to } L^2(\Omega). \end{cases} \tag{3.20}$$

In order to write a variational formulation of mechanical problem, we introduce the closed subspace of $H^1(\Omega)^d$ defined by

$$V=\left\{\mathbf{v}\in H^1(\Omega)^d : \mathbf{v}=0 \text{ on } \Gamma_1\right\}.$$

Since meas$(\Gamma_1)>0$, Korn's inequality holds and there exists a constant $C_k>0$, depending only on Ω and Γ_1, such that

$$\|\varepsilon(\mathbf{v})\|_{\mathcal{H}}\geqslant C_k\|\mathbf{v}\|_{H^1(\Omega)^d},\qquad \forall\mathbf{v}\in V. \tag{3.21}$$

A proof of Korn's inequality may be found in [20, p. 79]. On the space V we consider the inner product and the associated norm given by

$$(\mathbf{u},\mathbf{v})_V=(\varepsilon(\mathbf{u}),\varepsilon(\mathbf{v}))_{\mathcal{H}},\qquad \|\mathbf{v}\|_V=\|\varepsilon(\mathbf{v})\|_{\mathcal{H}},\qquad \forall\mathbf{u},\mathbf{v}\in V. \tag{3.22}$$

It follows that the norms $\|\cdot\|_{H^1(\Omega)^d}$ and $\|\cdot\|_V$ are equivalent on V and, therefore, the space $(V,(\cdot,\cdot)_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem and (3.22), there exists a constant $C_0>0$, depending only on Ω, Γ_1 and Γ_3, such that

$$\|\mathbf{v}\|_{L^2(\Gamma_3)^d}\leqslant C_0\|\mathbf{v}\|_V,\qquad \forall\mathbf{v}\in V. \tag{3.23}$$

The *body forces*, *surface tractions*, *volume heat source* and the *functions* α and μ, satisfy

$$\mathbf{f}_0 \in L^2(0,T;H), \qquad \mathbf{f}_2 \in L^2(0,T;L^2(\Gamma_2)^d), \qquad q \in L^2(0,T;L^2(\Omega)), \tag{3.24}$$

$$\mathbf{u}_0 \in V, \qquad \theta_0 \in V, \qquad \beta_0 \in K, \tag{3.25}$$

$$\alpha \in L^\infty(\Gamma_3) \qquad \alpha(x) \geqslant \alpha^* > 0, \quad \text{a.e. on } \Gamma_3, \tag{3.26}$$

$$\mu \in L^\infty(\Gamma_3), \qquad \mu(x) > 0, \quad \text{a.e. on } \Gamma_3, \tag{3.27}$$

$$B > 0, \qquad k_i > 0, \qquad i = 0,1. \tag{3.28}$$

We denote by $\mathbf{f}(t) \in V'$ the following element

$$(\mathbf{f}(t), \mathbf{v})_V = \int_\Omega \mathbf{f}_0(t).\mathbf{v}\, dx + \int_{\Gamma_2} \mathbf{f}_2(t).\mathbf{v}\, da, \qquad \forall \mathbf{v} \in V,\ t \in (0,T). \tag{3.29}$$

The use of (3.24) permits to verify that

$$\mathbf{f} \in \mathcal{C}(0,T;V). \tag{3.30}$$

We introduce the following bilinear forms $\mathfrak{a}_i \colon V \times V \to \mathbb{R}$ $(i = 0,1)$,

$$a_0(\zeta,\xi) = k_0 \int_\Omega \nabla\zeta \cdot \nabla\xi\, dx + B \int_\Gamma \zeta\xi\, d\gamma, \tag{3.31}$$

$$a_1(\zeta,\xi) = k_1 \int_\Omega \nabla\zeta \cdot \nabla\xi\, dx. \tag{3.32}$$

Finally, we consider the *wear functional* $j \colon V \times V \to \mathbb{R}$,

$$j(\mathbf{u},\mathbf{v}) = \int_{\Gamma_3} \alpha \|u_\nu\| \left(\mu \left\|\mathbf{v}_\tau - \mathbf{v}^*\right\|\right) + v_\nu)\, da. \tag{3.33}$$

Using the above notation and Green's formula, we derive the following variational formulation of mechanical problem $\mathcal{P}$.

Problem $\mathcal{PV}$

Find the displacement field $\mathbf{u} \colon [0,T] \to V$, the stress field $\boldsymbol{\sigma} \colon [0,T] \to \mathcal{H}_1$, the temperature $\theta \colon [0,T] \to V$, the damage field $\beta \colon [0,T] \to K$ such that

$$\begin{aligned} \boldsymbol{\sigma}(t) &= \mathcal{A}(\varepsilon(\dot{\mathbf{u}}(t))) + \mathcal{B}(\varepsilon(\mathbf{u}(t)), \beta(t)) \\ &\quad + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{A}(\varepsilon(\dot{\mathbf{u}}(s))), \varepsilon(\mathbf{u}(s)), \theta(s))\, ds, \quad \text{a.e. } t \in (0,T), \end{aligned} \tag{3.34}$$

$$\begin{aligned} &(\boldsymbol{\sigma}(t), \varepsilon(\mathbf{v} - \dot{\mathbf{u}}(t)))_{\mathcal{H}} + j(\dot{\mathbf{u}}(t), \mathbf{v}) - j(\dot{\mathbf{u}}(t), \dot{\mathbf{u}}(t)) \geqslant (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V, \\ &\qquad \forall \mathbf{v} \in V,\ \text{a.e. } t \in (0,T), \end{aligned} \tag{3.35}$$

$$\begin{aligned} &(\dot{\theta}(t), \mathbf{v})_{V' \times V} + a_0(\theta(t), \mathbf{v}) \\ &\qquad = (\psi(\boldsymbol{\sigma}(t), \varepsilon(\dot{\mathbf{u}}(t)), \theta(t)), \mathbf{v})_{V' \times V} + (q(t), \mathbf{v})_{L^2(\Omega)}, \qquad \forall \mathbf{v} \in V,\ \text{a.e. } t \in (0,T) \end{aligned} \tag{3.36}$$

$$\begin{aligned} &(\dot{\beta}(t), \zeta - \beta(t))_{L^2(\Omega)} + a_1(\beta(t), \zeta - \beta(t)) \\ &\qquad \geq (S(\varepsilon(\mathbf{u}(t)), \beta(t)), \zeta - \beta(t))_{L^2(\Omega)}, \qquad \forall \zeta \in K,\ \text{a.e. } t \in (0,T), \end{aligned} \tag{3.37}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \qquad \theta(0) = \theta_0, \qquad \beta(0) = \beta_0 \quad \text{in } \Omega, \tag{3.38}$$

Then $\{\mathbf{u}, \boldsymbol{\sigma}, \theta, \beta\}$ which satisfies (3.34)–(3.38) is called a weak solution of the mechanical of frictional bilateral contact with wear.

4 Main results

The main results are stated by the following theorems.

Theorem 4.1. *Assume that* (3.16)–(3.28) *hold and, in addition, the smallness assumption*

$$\|\alpha\|_{L^\infty(\Gamma_3)}\left(\|\mu\|_{L^\infty(\Gamma_3)}+1\right)<\alpha_0, \tag{4.1}$$

where $\alpha_0=\frac{m_{\mathcal{A}}}{C_0^2}$, *such that* $m_{\mathcal{A}}$ *is defined in* (3.16) *and* C_0 *defined by* (3.23)*. Then there exists a unique solution* $\{\mathbf{u},\boldsymbol{\sigma},\theta,\beta\}$ *to problem* $\mathcal{PV}$*. Moreover, the solution has the regularity*

$$\mathbf{u}\in\mathcal{C}^1(0,T;V), \tag{4.2}$$

$$\boldsymbol{\sigma}\in\mathcal{C}(0,T;\mathcal{H}_1), \tag{4.3}$$

$$\theta\in W^{1,2}(0,T;L^2(\Omega))\cap L^2(0,T;V), \tag{4.4}$$

$$\beta\in W^{1,2}(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega)), \tag{4.5}$$

Remark 4.2. We remark that if v^* is large enough then $\alpha=1/(kv^*)$ is sufficiently small and therefore, the condition (4.1) for the unique solvability of Problem $\mathcal{PV}$ is satisfied. We conclude that the mechanical problem (3.6)–(3.15) has a unique weak solution if the tangential velocity of the foundation is large enough. Moreover, having solved the problem (3.6)–(3.15), we find that there exists a unique solution $w\in\mathcal{C}^1(0,T;L^2(\Gamma_3))$, for the auxiliary problem (3.2) and using the initial condition $w(0)=0$ which means that at the initial moment the body is not subject to any prior wear. Moreover, by using the Cauchy–Lipschitz theorem, we find that there exists a unique solution $w\in\mathcal{C}^1(0,T;L^2(\Gamma_3))$, for an auxiliary problem satisfying (3.2) and $w(0)=0$.

Remark 4.3. The element $\{\mathbf{u},\boldsymbol{\sigma},\theta,\beta\}$ which satisfies (3.34)–(3.38) is called a weak solution of the contact problem $\mathcal{PV}$. We conclude that, under the assumptions (3.16)–(3.28) and if (4.1) holds, then the mechanical problem (3.6)–(3.15) has a unique weak solution having the regularity (4.2)–(4.5).

The proof of Theorem 4.1 is carried out in several steps that we prove in what follows. Everywhere in this section we suppose that assumptions of Theorem 4.1 hold, and we consider that C is a generic positive constant which depends on Ω, Γ_1 and Γ_3 and may change from place to place. The proof is based on arguments of elliptic variational inequalities, classical and uniqueness results on parabolic inequalities and fixed point arguments.

First step

Let $\boldsymbol{\eta}\in\mathcal{C}(0,T;\mathcal{H})$ and $\mathbf{g}\in\mathcal{C}(0,T;V)$ we consider the following variational problem.

Problem $\mathcal{PV}_{\eta,\mathbf{g}}$

Find $\mathbf{v}_{\eta,\mathbf{g}}\colon[0,T]\to\mathcal{H}_1$ such that

$$\boldsymbol{\sigma}_{\eta,\mathbf{g}}(t)=\mathcal{A}(\varepsilon(\mathbf{v}_{\eta,\mathbf{g}}(t)))+\boldsymbol{\eta}(t),\qquad\forall t\in[0,T], \tag{4.6}$$

$$(\boldsymbol{\sigma}_{\eta,\mathbf{g}}(t),\varepsilon(\mathbf{v}-\mathbf{v}_{\eta,\mathbf{g}}(t)))_{\mathcal{H}}+j(\mathbf{g}(t),\mathbf{v})-j(\mathbf{g}(t),\mathbf{v}_{\eta,\mathbf{g}}(t))\geqslant(\mathbf{f}(t),\mathbf{v}-\mathbf{v}_{\eta,\mathbf{g}}(t))_V,\quad\forall\mathbf{v}\in V. \tag{4.7}$$

Lemma 4.4. *There exists a unique solution to problem $\mathcal{PV}_{\eta,\mathbf{g}}$ such that*

$$\mathbf{v}_{\eta,\mathbf{g}} \in \mathcal{C}(0,T;V), \qquad \boldsymbol{\sigma}_{\eta,\mathbf{g}} \in \mathcal{C}(0,T;\mathcal{H}_1).$$

Proof. It follows from classical results for elliptic variational inequalities, (see for example [6]) that there exists a unique pair $\{\mathbf{v}_{\eta,\mathbf{g}}, \boldsymbol{\sigma}_{\eta,\mathbf{g}}\}$, $\mathbf{v}_{\eta,\mathbf{g}} \in V$, $\boldsymbol{\sigma}_{\eta,\mathbf{g}} \in \mathcal{H}$, which is a solution of (4.6) and (4.7). Choosing $\mathbf{v} = \mathbf{v}_{\eta,\mathbf{g}}(t) \pm \Phi$ in (4.7), where $\Phi \in \mathcal{D}(\Omega)^d$ is arbitrary, we find

$$(\boldsymbol{\sigma}_{\eta,\mathbf{g}}(t), \boldsymbol{\varepsilon}(\Phi))_{\mathcal{H}} = (\mathbf{f}(t), \Phi)_V$$

Using the definition (3.29) for $\mathbf{f}$, we deduce

$$\operatorname{Div} \boldsymbol{\sigma}_{\eta,\mathbf{g}}(t) + \mathbf{f}_0(t) = 0, \qquad t \in (0,T). \tag{4.8}$$

With the regularity assumption (3.24) on $\mathbf{f}_0$ we see that $\operatorname{Div} \boldsymbol{\sigma}_\eta(t) \in H$. Therefore, $\boldsymbol{\sigma}_\eta(t) \in \mathcal{H}_1$.

Let $t_1, t_2 \in [0,T]$ and denote $\mathbf{v}_{\eta,\mathbf{g}}(t_i) = \mathbf{v}_i$, $\boldsymbol{\sigma}_{\eta,\mathbf{g}}(t_i) = \boldsymbol{\sigma}_i$, $\mathbf{f}(t_i) = \mathbf{f}_i$, $\mathbf{g}(t_i) = \mathbf{g}_i$ and $\eta(t_i) = \eta_i$, for $i = 1,2$. Using the relations (4.6) and (4.7), we find that

$$\begin{aligned}
&(\mathcal{A}(\boldsymbol{\varepsilon}(\mathbf{v}_1)) - \mathcal{A}(\boldsymbol{\varepsilon}(\mathbf{v}_2)), \boldsymbol{\varepsilon}(\mathbf{v}_1) - \boldsymbol{\varepsilon}(\mathbf{v}_2))_{\mathcal{H}} \\
&\quad \leqslant (\mathbf{f}_1 - \mathbf{f}_2, \mathbf{v}_1 - \mathbf{v}_2)_V - (\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2, \boldsymbol{\varepsilon}(\mathbf{v}_1) - \boldsymbol{\varepsilon}(\mathbf{v}_2))_{\mathcal{H}} \\
&\qquad + j(\mathbf{g}_1, \mathbf{v}_2) + j(\mathbf{g}_2, \mathbf{v}_1) - j(\mathbf{g}_1, \mathbf{v}_1) - j(\mathbf{g}_2, \mathbf{v}_2).
\end{aligned} \tag{4.9}$$

Moreover, it follows form (3.21) and (3.16) that

$$(\mathcal{A}(\boldsymbol{\varepsilon}(\mathbf{v}_1)) - \mathcal{A}(\boldsymbol{\varepsilon}(\mathbf{v}_2)), \boldsymbol{\varepsilon}(\mathbf{v}_1) - \boldsymbol{\varepsilon}(\mathbf{v}_2))_{\mathcal{H}} \geqslant C \|\mathbf{v}_1 - \mathbf{v}_2\|_V^2. \tag{4.10}$$

From the definition of the functional j given by (3.33), we have

$$\begin{aligned}
&j(\mathbf{g}_1, \mathbf{v}_2) + j(\mathbf{g}_2, \mathbf{v}_1) - j(\mathbf{g}_1, \mathbf{v}_1) - j(\mathbf{g}_2, \mathbf{v}_2) \\
&\quad = \int_{\Gamma_3} (\alpha\|g_{1\nu}\| - \alpha\|g_{2\nu}\|)\,(\mu\,\|\mathbf{v}_{2\tau}\| - \mu\,\|\mathbf{v}_{1\tau}\|) + v_{2\nu} - v_{1\nu}\, da.
\end{aligned}$$

The relation (3.23) and the assumptions (3.26) and (3.27) imply

$$\begin{aligned}
&\|j(\mathbf{g}_1, \mathbf{v}_2) + j(\mathbf{g}_2, \mathbf{v}_1) - j(\mathbf{g}_1, \mathbf{v}_1) - j(\mathbf{g}_2, \mathbf{v}_2)\| \\
&\quad \leqslant C_0^2\, \|\alpha\|_{L^\infty(\Gamma_3)} \Big(\|\mu\|_{L^\infty(\Gamma_3)} + 1\Big)\, \|\mathbf{g}_1 - \mathbf{g}_2\|_V\, \|\mathbf{v}_1 - \mathbf{v}_2\|_V.
\end{aligned} \tag{4.11}$$

The relation (3.22), the assumption (3.16), and the inequality (4.10) combined with (4.11) give us

$$m_{\mathcal{A}}\, \|\mathbf{v}_1 - \mathbf{v}_2\|_V \leqslant C_0^2\, \|\alpha\|_{L^\infty(\Gamma_3)} \Big(\|\mu\|_{L^\infty(\Gamma_3)} + 1\Big)\, \|\mathbf{g}_1 - \mathbf{g}_2\|_V + \|\mathbf{f}_1 - \mathbf{f}_2\|_V + \|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_{\mathcal{H}}. \tag{4.12}$$

Moreover, from (3.16) and (4.6), we obtain

$$\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|_{\mathcal{H}} \leqslant C\Big(\|\mathbf{v}_1 - \mathbf{v}_2\|_V + \|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_{\mathcal{H}}\Big). \tag{4.13}$$

Now, from (3.30), (4.12) and (4.13), we obtain that $\mathbf{v}_{\eta,\mathbf{g}} \in \mathcal{C}(0,T;V)$ and $\boldsymbol{\sigma}_{\eta,\mathbf{g}} \in \mathcal{C}(0,T;\mathcal{H})$, then it follows from (3.24) and (3.17) that $\boldsymbol{\sigma}_{\eta,\mathbf{g}} \in \mathcal{C}(0,T;\mathcal{H}_1)$. □

Let us consider now the operator $\Lambda_\eta\colon \mathcal{C}(0,T;V) \to \mathcal{C}(0,T;V)$, defined by

$$\Lambda_\eta \mathbf{g} = \mathbf{v}_{\eta,\mathbf{g}}, \qquad \forall \mathbf{g} \in \mathcal{C}(0,T;V). \tag{4.14}$$

We have the following lemma.

Lemma 4.5. *The operator Λ_η has a unique fixed point $\mathbf{g}_\eta \in \mathcal{C}(0,T;V)$.*

Proof. Let $\mathbf{g}_1, \mathbf{g}_2 \in \mathcal{C}(0,T;V)$ and let $\eta \in \mathcal{C}(0,T;\mathcal{H})$. We use the notation $\mathbf{v}_{\eta,\mathbf{g}_i} = \mathbf{v}_i$ and $\sigma_{\eta,\mathbf{g}_i} = \sigma_i$ for $i = 1,2$. Using similar arguments as those in (4.12), we find

$$m_{\mathcal{A}} \|\mathbf{v}_1(t) - \mathbf{v}_2(t)\|_V \leqslant C_0^2 \|\alpha\|_{L^\infty(\Gamma_3)} \left(\|\mu\|_{L^\infty(\Gamma_3)} + 1 \right) \|\mathbf{g}_1(t) - \mathbf{g}_2(t)\|_V, \qquad \forall t \in [0,T], \tag{4.15}$$

From (4.14) and (4.15) we find that

$$\begin{aligned} &\|\Lambda_\eta \mathbf{g}_1(t) - \Lambda_\eta \mathbf{g}_2(t)\|_V \\ &\quad \leqslant \frac{C_0^2}{m_{\mathcal{A}}} \|\alpha\|_{L^\infty(\Gamma_3)} \left(\|\mu\|_{L^\infty(\Gamma_3)} + 1 \right) \|\mathbf{g}_1(t) - \mathbf{g}_2(t)\|_V, \qquad \forall t \in [0,T]. \end{aligned} \tag{4.16}$$

Let

$$\alpha_0 = \frac{C_0^2}{m_{\mathcal{A}}},$$

where α_0 is a positive constant which depends on Ω, Γ_1, Γ_3, and on the operator $\mathcal{A}$. If (4.1) is satisfied we deduce from (4.16) that the operator Λ_η is a contraction. From Banach's fixed point theorem we conclude that the operator Λ_η has a unique fixed point $\mathbf{g}^*_\eta \in \mathcal{C}(0,T;V)$. □

For $\eta \in \mathcal{C}(0,T;\mathcal{H})$, let $\mathbf{g}^*_\eta$ be the fixed point given by the above lemma, i.e. $\mathbf{g}^*_\eta = \mathbf{v}_{\eta,\mathbf{g}^*_\eta}$.

In the sequel we denote by $(\mathbf{v}_\eta, \sigma_\eta) \in \mathcal{C}(0,T;V) \times \mathcal{C}(0,T;\mathcal{H}_1)$ the unique solution of Problem $\mathcal{PV}_{\eta,\mathbf{g}^*_\eta}$, i.e. $\mathbf{v}_\eta = \mathbf{v}_{\eta,\mathbf{g}^*_\eta}$, $\sigma_\eta = \sigma_{\eta,\mathbf{g}^*_\eta}$. Also, we denote by $\mathbf{u}_\eta\colon [0,T] \to V$ the function defined by

$$\mathbf{u}_\eta(t) = \int_0^t \mathbf{v}(s)\, ds + \mathbf{u}_0, \qquad \forall t \in [0,T]. \tag{4.17}$$

From Lemma 4.4 we deduce that $\mathbf{u}_\eta \in C^1(0,T;V)$.

Second step

For $\chi \in \mathcal{C}(0,T;V')$, we consider the following variational problem.

Problem $\mathcal{PV}_\chi$

Find the temperature $\theta_\chi\colon [0,T] \to V$ which is solution of the variational problem

$$(\dot\theta_\chi(t), \mathbf{v})_{V'\times V} + a_0(\theta_\chi(t), \mathbf{v}) = \langle \chi(t) + q(t), \mathbf{v} \rangle_{V'\times V} \qquad \forall \mathbf{v} \in V, \text{ a.e. } t \in (0,T), \tag{4.18}$$

$$\theta_\chi(0) = \theta_0, \quad \text{in } \Omega. \tag{4.19}$$

Lemma 4.6. *For all $\chi \in \mathcal{C}(0,T;V')$, there exists a unique solution θ_χ to the auxiliary problem $\mathcal{PV}_\chi$ satisfying* (4.4).

Proof. By an application of the Friedrichs–Poincaré inequality, we can find a constant $B' > 0$ such that

$$\int_\Omega \|\nabla\zeta\|^2\,dx + \frac{B}{k_0}\int_\Gamma \|\zeta\|^2\,d\gamma \geqslant B'\int_\Omega \|\zeta\|^2\,dx, \qquad \forall \zeta \in V.$$

Thus, we obtain

$$\mathfrak{a}_0(\zeta,\zeta) \geqslant c_1\|\zeta\|_V^2, \qquad \forall\zeta \in V, \tag{4.20}$$

where $c_1 = k_0 \min(1, B')/2$, which implies that $\mathfrak{a}_0$ is V-elliptic. Consequently, based on classical arguments of functional analysis concerning parabolic equations, the variational equation (4.18) has a unique solution θ_χ satisfying (4.4). □

Third step

For $\phi \in \mathcal{C}(0,T;L^2(\Omega))$, we consider the following variational problem.

Problem $\mathcal{PV}_\phi$

Find the damage field $\beta_\phi\colon [0,T] \to K$ such that

$$\begin{aligned}&(\dot\beta_\phi(t),\zeta-\beta_\phi(t))_{L^2(\Omega)} + a_1(\beta_\phi(t),\zeta-\beta_\phi(t)) \geq (\phi,\zeta-\beta_\phi(t))_{L^2(\Omega)},\\ &\forall\zeta \in K \text{ a.e } t \in [0,T],\end{aligned} \tag{4.21}$$

$$\beta_\phi(0) = \beta_0 \quad \text{in } \Omega. \tag{4.22}$$

We apply Theorem 2.1 to problem $\mathcal{PV}_\phi$.

Lemma 4.7. *There exists a unique solution β_ϕ to the auxiliary problem $\mathcal{PV}_\phi$ such that*

$$\beta_\phi \in W^{1,2}\big(0,T;L^2(\Omega)\big) \cap L^2\big(0,T;H^1(\Omega)\big). \tag{4.23}$$

Proof. The inclusion mapping of $(H^1(\Omega), \|\,.\|_{H^1(\Omega)})$ into $(L^2(\Omega), \|\,.\|_{L^2(\Omega)})$ is continuous and its range is dense. We denote by $\left(H^1(\Omega)\right)'$ the dual space of $H^1(\Omega)$ and, identifying the dual of $L^2(\Omega)$ with itself, we can write the Gelfand triple

$$H^1(\Omega) \subset L^2(\Omega) \subset \Big(H^1(\Omega)\Big)'.$$

We use the notation $(\cdot,\cdot)_{(H^1(\Omega))'\times H^1(\Omega)}$ to represent the duality pairing between $\left(H^1(\Omega)\right)'$ and $H^1(\Omega)$. We have

$$(\beta,\xi)_{(H^1(\Omega))'\times H^1(\Omega)} = (\beta,\xi)_{L^2(\Omega)}, \qquad \forall\beta \in L^2(\Omega),\ \xi \in H^1(\Omega)$$

and we note that K is a closed convex set in $H^1(\Omega)$. Then, using the definition (3.32) of the bilinear form a_1 , and the fact that $\beta_\phi \in K$ in (3.25), it is easy to see that Lemma 4.6 is a consequence of Theorem 2.1. □

By taking into account the above results and the properties of the operators $\mathcal{B}$ and $\mathcal{G}$ and of the functions ψ and S, we may consider the operator

$$\begin{aligned}&\Lambda\colon \mathcal{C}(0,T;\mathcal{H}\times V'\times L^2(\Omega)) \to \mathcal{C}(0,T;\mathcal{H}\times V'\times L^2(\Omega)),\\ &\Lambda(\boldsymbol{\eta},\chi,\phi)(t) = (\Lambda_1(\boldsymbol{\eta},\chi,\phi)(t),\Lambda_2(\boldsymbol{\eta},\chi,\phi)(t),\Lambda_3(\boldsymbol{\eta},\chi,\phi)(t)),\end{aligned} \tag{4.24}$$

defined by

$$\begin{aligned}\Lambda_1(\boldsymbol{\eta},\chi,\phi)(t) &= \mathcal{B}\left(\boldsymbol{\varepsilon}\left(\mathbf{u}_\eta(t)\right),\beta_\phi(t)\right)\\ &\quad + \left(\int_0^t \mathcal{G}\Big(\boldsymbol{\sigma}_\eta(s)-\mathcal{A}(\varepsilon(\dot{\mathbf{u}}_\eta(s))),\boldsymbol{\varepsilon}(\mathbf{u}_\eta(s),\theta_\chi(t))\Big)ds\right), \qquad \forall t\in[0,T],\end{aligned} \tag{4.25}$$

$$\Lambda_2(\boldsymbol{\eta},\chi,\phi)(t) = \psi\big(\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)),\beta_\phi(t)\big), \qquad \forall t\in[0,T], \tag{4.26}$$

$$\Lambda_3(\boldsymbol{\eta},\chi,\phi)(t) = S\big(\boldsymbol{\sigma}_\eta,\varepsilon(\dot{\mathbf{u}}_\eta),\theta_\chi\big), \qquad \forall t\in[0,T]. \tag{4.27}$$

We have the following result.

Lemma 4.8. *Let (4.4) be satisfied. Then for $(\boldsymbol{\eta},\chi,\phi)\in\mathcal{C}(0,T;\mathcal{H}\times V'\times L^2(\Omega))$, the mapping $\Lambda(\boldsymbol{\eta},\chi,\phi)\colon [0,T]\to\mathcal{H}\times V'\times L^2(\Omega)$ has a unique element $(\boldsymbol{\eta}^*,\chi^*,\phi^*)\in\mathcal{C}(0,T;\mathcal{H}\times V'\times L^2(\Omega))$ such that $\Lambda(\boldsymbol{\eta}^*,\chi^*,\phi^*)=(\boldsymbol{\eta}^*,\chi^*,\phi^*)$.*

Proof. Let $(\boldsymbol{\eta}_1,\chi_1,\phi_1),(\boldsymbol{\eta}_2,\chi_2,\phi_2)\in\mathcal{C}(0,T;\mathcal{H}\times V'\times L^2(\Omega))$, and $t\in[0,T]$. We use the notation $\mathbf{u}_{\eta_i}=\mathbf{u}_i$, $\dot{\mathbf{u}}_{\eta_i}=\mathbf{v}_{\eta_i}=\mathbf{v}_i$, $\beta_{\phi_i}=\beta_i$, $\theta_{\chi_i}=\theta_i$ and $\boldsymbol{\sigma}_{\eta_i}=\boldsymbol{\sigma}_i$, for $i=1,2$. Using (3.22) and the relations (3.17)–(3.20), we obtain

$$\begin{aligned}&\|\Lambda(\boldsymbol{\eta}_1,\chi_1,\phi_1)(t)-\Lambda(\boldsymbol{\eta}_1,\chi_1,\phi_1)(t)\|_{\mathcal{H}\times V'\times L^2(\Omega)}\\ &\quad\le L_{\mathcal{B}}\Big(\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_V+\|\beta_1(t)-\beta_2(t)\|_{L^2(\Omega)}\Big)\\ &\quad +L_{\mathcal{G}}\int_0^t\Big(\|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}+L_{\mathcal{A}}\|\mathbf{v}_1(s)-\mathbf{v}_2(s)\|_V\\ &\qquad +\|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|_V+\|\theta_1(s)-\theta_2(s)\|_{L^2(\Omega)}\Big)ds\\ &\quad +M_S\Big(\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_V+\|\beta_1(t)-\beta_2(t)\|_{L^2(\Omega)}\Big)\\ &\quad +L_\psi\Big(\|\boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t)\|_{\mathcal{H}}+\|\mathbf{v}_1(t)-\mathbf{v}_2(t)\|_V+\|\theta_1(t)-\theta_2(t)\|_{L^2(\Omega)}\Big).\end{aligned} \tag{4.28}$$

Since

$$\mathbf{u}_i(t)=\int_0^t\mathbf{v}_i(s)\,ds+\mathbf{u}_0, \qquad \forall t\in[0,T], \tag{4.29}$$

we have

$$\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_V\le\int_0^t\|\mathbf{v}_1(s)-\mathbf{v}_2(s)\|_V\,ds. \tag{4.30}$$

Applying Young's and Hölder's inequalities, (4.28) becomes, via (4.30),

$$\begin{aligned}&\|\Lambda(\boldsymbol{\eta}_1,\chi_1,\phi_1)(t)-\Lambda(\boldsymbol{\eta}_1,\chi_1,\phi_1)(t)\|_{\mathcal{H}\times V'\times L^2(\Omega)}\\ &\quad\le C\Big(\|\beta_1(t)-\beta_2(t)\|_{L^2(\Omega)}+\int_0^t\Big(\|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}\\ &\qquad+\|\mathbf{v}_1(s)-\mathbf{v}_2(s)\|_V+\|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|_V\\ &\qquad+\|\theta_1(s)-\theta_2(s)\|_{L^2(\Omega)}\Big)ds\Big).\end{aligned} \tag{4.31}$$

Taking into account that

$$\boldsymbol{\sigma}_i(t)=\mathcal{A}(\varepsilon(\dot{\mathbf{u}}_i(t)))+\boldsymbol{\eta}_i(t), \qquad \forall t\in[0,T], \tag{4.32}$$

it follows that

$$\begin{aligned}&(\mathcal{A}(\varepsilon(\mathbf{v}_1(s)))-\mathcal{A}(\varepsilon(\mathbf{v}_2(s))),\boldsymbol{\varepsilon}(\mathbf{v}_1(s)-\mathbf{v}_2(s)))_{\mathcal{H}}\\ &\quad\le j(\mathbf{v}_1(s),\mathbf{v}_2(s))+j(\mathbf{v}_2(s),\mathbf{v}_1(s))-j(\mathbf{v}_1(s),\mathbf{v}_1(s))-j(\mathbf{v}_2(s),\mathbf{v}_2(s))\end{aligned} \tag{4.33}$$

So, by using (3.16), (3.33) and (3.23), we deduce that

$$m_{\mathcal{A}}\|\mathbf{v}_1(s)-\mathbf{v}_2(s)\|_V^2 \le C_0^2\|\alpha\|_{L^\infty(\Gamma_3)}\Big(\|\mu\|_{L^\infty(\Gamma_3)}+1\Big)\|\mathbf{v}_1(s)-\mathbf{v}_2(s)\|_V^2 \\ +\|\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)\|_{\mathcal{H}}\|\mathbf{v}_1(s)-\mathbf{v}_2(s)\|_V^2$$

which, by the hypothesis (4.1), implies

$$\|\mathbf{v}_1(s)-\mathbf{v}_2(s)\|_V \le C\|\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)\|_{\mathcal{H}}.$$

Also, by (4.30), we get

$$\int_0^t \|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|_V\,ds \le C\int_0^t\int_0^s \|\boldsymbol{\eta}_1(r)-\boldsymbol{\eta}_2(r)\|_{\mathcal{H}}\,dr\,ds \\ \le \int_0^T \|\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)\|_{\mathcal{H}}\,ds.$$

For the temperature, if we take the substitution $\chi=\chi_1$, $\chi=\chi_2$ in (4.18) and subtracting the two obtained equations, we deduce by choosing $\mathbf{v}=\theta_1-\theta_2$ as test function

$$\|\theta_1(t)-\theta_2(t)\|^2_{L^2(\Omega)}+C_1\int_0^t\|\theta_1(t)-\theta_2(t)\|_V^2 \\ \leqslant \int_0^t \|\chi_1(s)-\chi_2(s)\|_{V'}\|\theta_1(s)-\theta_2(s)\|_V\,ds, \qquad \forall t\in[0,T],$$

Employing Hölder's and Young's inequalities, we deduce that

$$\|\theta_1(t)-\theta_2(t)\|^2_{L^2(\Omega)}+\int_0^t\|\theta_1(s)-\theta_2(s)\|_V^2\,ds \\ \leqslant C\int_0^t\|\chi_1(s)-\chi_2(s)\|_{V'}^2\,ds\,, \qquad \forall t\in[0,T].$$

We use the inclusion $L^2(\Omega)\subset V$, we get

$$\|\theta_1(t)-\theta_2(t)\|^2_{L^2(\Omega)}+\int_0^t\|\theta_1(s)-\theta_2(s)\|_{L^2(\Omega)}^2\,ds \\ \leqslant C\int_0^t\|\chi_1(s)-\chi_2(s)\|_{V'}^2\,ds, \quad \forall t\in[0,T].$$

From this inequality, combined with Gronwall's inequality, we deduce that

$$\|\theta_1(t)-\theta_2(t)\|^2_{L^2(\Omega)}\leqslant C\int_0^t\|\chi_1(s)-\chi_2(s)\|_{V'}^2\,ds. \tag{4.34}$$

For the damage field, from (4.21) we deduce that

$$(\dot\beta_1-\dot\beta_2,\beta_1-\beta_2)_{L^2(\Omega)}+\mathfrak{a}_1(\beta_1-\beta_2,\beta_1-\beta_2)\leqslant(\phi_1-\phi_2,\beta_1-\beta_2)_{L^2(\Omega)}, \quad \text{a.e. } t\in(0,T).$$

Integrating the previous inequality with respect to time, using the initial conditions $\beta_1(0)=\beta_2(0)=\beta_0$ and the inequality $\mathfrak{a}_1(\beta_1-\beta_2,\beta_1-\beta_2)\geqslant 0$ we find

$$\frac{1}{2}\|\beta_1(t)-\beta_2(t)\|^2_{L^2(\Omega)}\leqslant\int_0^t(\phi_1(s)-\phi_2(s),\beta_1(s)-\beta_2(s))_{L^2(\Omega)}\,ds, \tag{4.35}$$

which implies

$$\|\beta_1(t)-\beta_2(t)\|^2_{L^2(\Omega)} \leqslant \int_0^t \|\phi_1(s)-\phi_2(s)\|^2_{L^2(\Omega)}\,ds + \int_0^t \|\beta_1(s)-\beta_2(s)\|^2_{L^2(\Omega)}\,ds.$$

This inequality, combined with Gronwall's inequality, leads to

$$\|\beta_1(t)-\beta_2(t)\|^2_{L^2(\Omega)} \leqslant C\int_0^t \|\phi_1(s)-\phi_2(s)\|^2_{L^2(\Omega)}\,ds, \qquad \forall t\in[0,T]. \tag{4.36}$$

Applying the previous inequalities, the estimates (4.34) and (4.36), and substituting (4.31), we obtain

$$\begin{aligned}&\|\Lambda(\boldsymbol{\eta}_1,\chi_1,\phi_1)(t)-\Lambda(\boldsymbol{\eta}_2,\chi_2,\phi_2)(t)\|^2_{\mathcal{H}\times V'\times L^2(\Omega)}\\ &\qquad\leqslant C\int_0^T \|(\boldsymbol{\eta}_1,\chi_1,\phi_1)(s)-(\boldsymbol{\eta}_2,\chi_2,\phi_2)(s)\|^2_{\mathcal{H}\times V'\times L^2(\Omega)}\,ds.\end{aligned}$$

Thus, for m sufficiently large, Λ^m is a contraction on $\mathcal{C}(0,T;\mathcal{H}\times V'\times L^2(\Omega))$, and so Λ has a unique fixed point in this Banach space. □

Now, we have all the ingredients to prove Theorem 4.1.

Existence

Let $(\boldsymbol{\eta}^*,\chi^*,\phi^*)\in\mathcal{C}(0,T;\mathcal{H}\times V'\times L^2(\Omega))$ be the fixed point of Λ defined by (4.24)–(4.27) and let $\mathbf{g}^*=\mathbf{g}^*_{\eta^*}$ be the fixed point of the operator Λ_{η^*} given by Lemma 4.6. We denote by $\{\mathbf{v},\sigma\}$ the unique solution of Problem $\mathcal{PV}_{\eta^*,\mathbf{g}^*}$ and we define

$$\mathbf{u}(t)=\int_0^t \mathbf{v}(s)\,ds+\mathbf{u}_0.$$

Also, let $\theta=\theta_{\chi^*}$ and $\beta=\beta_{\phi^*}$ be the solutions of Problems $\mathcal{PV}_{\boldsymbol{\phi}^*}$ and respectively, $\mathcal{PV}_{\boldsymbol{\phi}^*}$ obtained in Lemmas 4.6 and 4.7. As $\Lambda_1(\boldsymbol{\eta}^*,\chi^*,\phi^*)=\boldsymbol{\eta}^*$, $\Lambda_2(\boldsymbol{\eta}^*,\chi^*,\phi^*)=\chi^*$ and $\Lambda_3(\boldsymbol{\eta}^*,\chi^*,\phi^*)=\phi^*$, the definitions (4.25)–(4.27) show that (3.34)–(3.38) are satisfied. Next, from Lemmas 4.4, 4.6 and 4.7, the regularity conditions (4.2)–(4.5) follow.

Uniqueness

The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operator Λ defined by (4.24)–(4.27) and the unique solvability of the Problem $\mathcal{PV}_{\boldsymbol{\eta},\mathbf{g}}$, $\mathcal{PV}_\chi$ and $\mathcal{PV}_\phi$ which completes the proof.

References

[1] A. Aissaoui, N. Hemici, A frictional contact problem with damage and adhesion for an electro elastic-viscoplastic body, *Electron. J. Differential Equations* **2014**, No. 11, 1–19. MR3159420

[2] A. Aissaoui, N. Hemici, Bilateral contact problem with adhesion and damage, *Electron. J. Qual. Theory Differ. Equ.* **2014**, No. 18, 1–16. MR3210548.

[3] A. A. Abdelaziz, S. Boutechebak, Analysis of a dynamic thermo-elastic-viscoplastic contact problem, *Electron. J. Qual. Theory Differ. Equ.* **2013**, No. 71, 1–17. MR3151718

[4] V. Barbu, *Optimal control of variational inequalities*, Research Notes in Mathematics Vol. 100, Pitman, Boston, 1984. MR0742624

[5] S. Boutechebak, A dynamic problem of frictionless contact for elastic-thermo-viscoplastic materials with damage, *Int. J. Pure Appl. Math.* **86**(2013), 173–197. url

[6] H. Brezis, Équations et inéquations non linéaires dans les espaces vectoriels en dualité (in French) [Nonlinear equations and inequalities in vector spaces in duality], *Ann. Inst. Fourier (Grenoble)* **18**(1968), 115–175. MR0270222; url

[7] A. Djabi, A. Merouani, Bilateral contact problem with friction and wear for an elastic-viscoplastic materials with damage, *Taiwanese J. Math.*, accepted. url

[8] G. Duvaut, J.-L. Lions, *Les inéquations en mécanique et en physique* (in French) [The inequalities in mechanics and physics], Travaux et Recherches Mathématiques, No. 21, Dunod, Paris, 1972. MR0464857

[9] C. Eck, J. Jarušek, M. Krbec, *Unilateral contact problems: variational methods and existence theorems*, Pure and Applied Mathematics (Boca Raton), Vol. 270, Chapman/CRC Press, Boca Raton, FL, 2005. MR2128865; url

[10] P. Ireman, A. Klarbring, N. Strömberg, A model of damage coupled to wear, *Int. J. Solids Structures* **40**(2003), 2957–2974. MR1988667; url

[11] T. Hadj Ammar, B. Benabderrahmane, S. Drabla, Frictional contact problems for electro-viscoelastic materials with long-term memory, damage, and adhesion, *Electron. J. Differential Equations* **2014**, No. 222, 1–21. MR3273105

[12] J. J. Bikerman, Thermodynamics, adhesion, and sliding friction, *J. Lube. Technol.* **92**(1970), 243–247. url

[13] Z. Lerguet, Z. Zellagui, H. Benseridi, S. Drabla, Variational analysis of an electro viscoelastic contact problem with friction, *J. Assn. Arab Univ. Basic Appl. Sci.* **14**(2013), 93–100. url

[14] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires* (in French) [Some methods of solving problems for nonlinear boundary], Dunod; Gauthier-Villars, Paris, 1969. MR0259693

[15] A. Merouani, F. Messelmi, Dynamic evolution of damage in elastic-thermo-viscoplastic materials, *Electron. J. Differential Equations* **2010**, No. 129, 1–15. MR2685039

[16] A. Merouani, F. Messelmi, Quasi-static transmission problem in thermo-viscoplasticity, *Int. J. Open Problems Compt. Math.* **6**(2013), No. 3, 29–46. url

[17] F. Messelmi, A. Merouani, A monotonicity method in quasistatic processes for viscoplastic materials of the from $\dot{\sigma} = \mathcal{B}\left(\varepsilon\left(\dot{u}\right),\theta\right) + F\left(\sigma,\varepsilon\left(u\right),\chi,\theta\right)$, *Int. J. Open Problems Compt. Math.* **6**(2013), No. 2, 74–83. url

[18] R. D. Mindlin, Polarization gradient in elastic dielectrics, *Int. J. Solids Structures* **4**(1968), 637–663. url

[19] R. D. Mindlin, Elasticity, piezoelectricity and crystal lattice dynamics, *J. Elasticity* **2**(1972), 217–280. url

[20] J. Nečas, I. Hlaváček, *Mathematical theory of elastic and elasto-plastic bodies: an introduction*, Studies in Applied Mechanics, Vol. 3, Elsevier Scientific Publishing Co., Amsterdam–New York, 1980. MR600655

[21] M. Shillor, M. Sofonea, A quasistatic viscoelastic contact problem with friction, *Internat. J. Engrg. Sci.* **38**(2000), 1517–1533. MR1763038; url

[22] N. Strömberg, L. Johansson, A. Klarbring, Derivation and analysis of a generalized standard model for contact friction and wear, *Int. J. Solids Structures* **33**(1996), 1817–1836. MR1392130; url